A DECOMPOSITION OF THE BROWNIAN PATH

by

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1. THE MAIN RESULT

Let us consider a standard Brownian motion process $W = \{W(t);\ t\geq 0\}$ on a suitable probability space $(\Omega,\mathfrak{F},P)$. We define the occupation times

$$\Gamma_+(t) = \text{meas}\{0\leq s\leq t;\ W_s\geq 0\}$$

$$\Gamma_-(t) = \text{meas}\{0\leq s\leq t;\ W_s<0\} = t-\Gamma_+(t),$$

their inverses

$$\Gamma_\pm^{-1}(\tau) = \inf\{t\geq 0;\ \Gamma_\pm(t)>\tau\};\qquad \tau\geq 0,$$

as well as the local time of W at the origin

$$L(t) = \lim_{\epsilon\downarrow 0}\frac{1}{4\epsilon}\,\text{meas}\{0\leq s\leq t;\ |W_s|<\epsilon\}\ ;\qquad t\geq 0.$$

It is fairly well-known (c.f. Ikeda & Watanabe (1981), p. 122) that the processes

$$W_\pm(\tau) = \pm\, W(\Gamma_\pm^{-1}(\tau))\ ;\qquad \tau\geq 0$$

are independent reflecting Brownian motions, with local times

$$L_\pm(\tau) = \lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\,\text{meas}\{0\leq\sigma\leq\tau;\ W_\pm(\sigma)\leq\epsilon\}$$

given by

$$L_\pm(\tau) = L(\Gamma_\pm^{-1}(\tau))\ ;\qquad \tau\geq 0.$$

Besides, the processes

$$B_\pm(\tau) \triangleq L_\pm(\tau) - W_\pm(\tau) \;; \quad \tau\geq 0$$

are independent Brownian motions, with

$$L_\pm(\tau) = \max_{0\leq\sigma\leq\tau} B_\pm(\sigma) \;; \quad \tau\geq 0.$$

We fix now $t>0$ and consider the portion of the Brownian path $\{W(s);\ 0\leq s\leq t\}$ on $[0,t]$. Let us consider the last exit time from zero, before time t:

$$\gamma_t = \sup\{s\leq t;\ s>0,\ W(s) = 0\}.$$

We are now ready to dissect the path $\{W(s);\ 0\leq s\leq t\}$ and re-assemble it in a way that preserves its "Brownian" character.

THEOREM 1: Brownian path decomposition.

The process

$$\begin{aligned} \hat{W}(s) &= B_+(s) &&;\quad 0\leq s<\Gamma_+(\gamma_t) \\ &= B_-(\gamma_t - s) &&;\quad \Gamma_+(\gamma_t)\leq s<\gamma_t \qquad (1.1) \\ &= W(s) &&;\quad \gamma_t\leq s\leq t \end{aligned}$$

is Brownian motion on $[0,t]$. ∎

This result has several interesting implications. In order to discuss them, let us introduce, for each $0 \le u \le t$, the random variables

$$M(u) = \max_{0 \le s \le u} W(s), \quad \theta(u) = \arg \max_{0 \le s \le u} W(s),$$

as well as their counterparts for the process in (1.1),

$$\hat{M}(u) = \max_{0 \le s \le u} \hat{W}(s) \quad , \quad \hat{\theta}(u) = \arg \max_{0 \le s \le u} \hat{W}(s).$$

We recall here that, for Brownian motion, the location θ of the maximum M is a.s. unique. The basic observation is that, P - a.s.,

$$(1.2) \qquad (\hat{W}(t), \hat{\gamma}_t, \hat{M}(\gamma_t), \hat{\theta}(\gamma_t)) = (W(t), \gamma_t, L(t), \Gamma_+(\gamma_t)).$$

The first two identities are obvious; for the third and fourth, let us notice that

$$\hat{W}(\Gamma_+(\gamma_t)-) = B_+(\Gamma_+(\gamma_t)) = L_+(\Gamma_+(\gamma_t)) - W_+(\Gamma_+(\gamma_t))$$
$$= L(\gamma_t) - W(\gamma_t) = L(\gamma_t) = L(t)$$

and

$$\hat{W}(\Gamma_+(\gamma_t)+) = B_-(\Gamma_-(\gamma_t)) = L_-(\Gamma_-(\gamma_t)) - W_-(\Gamma_-(\gamma_t))$$
$$= L(\gamma_t) + W(\gamma_t) = L(\gamma_t)$$

hold with probability one. It follows that $\hat{W}$ is a.s. continuous at $s = \Gamma_+(\gamma_t)$; it is also continuous at $s = \gamma_t$, since

$$\hat{W}(\gamma_t -) = B_-(0) = 0 = \tilde{W}(\gamma_t).$$

On the other hand,

$$\max_{0 \le s \le \Gamma_+(\gamma_t)} B_+(s) = L_+(\Gamma_+(\gamma_t)) = L(\gamma_t) = B_+(\Gamma_+(\gamma_t))$$

and

$$\max_{\Gamma_+(\gamma_t) \le s \le \gamma_t} B_-(\gamma_t - s) = \max_{0 \le s \le \Gamma_-(\gamma_t)} B_-(s) = L_-(\Gamma_-(\gamma_t))$$

$$= L(\gamma_t) = B_-(\gamma_t - \Gamma_+(\gamma_t)).$$

It follows that

$$\hat{M}(\gamma_t) = L(\gamma_t), \quad \hat{\theta}(\gamma_t) = \Gamma_+(\gamma_t)$$

hold a.s. P, and this concludes the verification of (1.2). In particular, this latter gives

$$(1.3) \qquad (\hat{W}(t), \hat{\gamma}_t, \hat{M}(t), \hat{\theta}(t)) = (W(t), \gamma_t, L(t), \Gamma_+(t)); \quad \text{on } \{W(t)<0\},$$

P - a.s. Now it is fairly well-known (P. Lévy (1965), p. 201) that the triple $(W(t), M(t), \theta(t))$ has the density

$$(1.4)\quad P[W(t)\in da;\ M(t)\in db;\ \theta(t)\in du] = \frac{b(b-a)}{\pi u^{3/2}(t-u)^{3/2}} \exp\left\{-\frac{b^2}{2u} - \frac{(b-a)^2}{2(t-u)}\right\} da\,db\,du;\quad b>a,\ b>0,\ 0<u<t.$$

From Theorem 1, (1.3) and (1.4) we conclude that the density of the triple $(W(t),L(t),\Gamma_+(t))$ is given as follows:

$$(1.5)\quad P[W(t)\in da;\ L(t)\in db;\ \Gamma_+(t)\in du] = \begin{cases} \dfrac{b(b-a)}{\pi u^{3/2}(t-u)^{3/2}} \exp\left\{-\dfrac{b^2}{2u} - \dfrac{(b-a)^2}{2(t-u)}\right\} da\,db\,du; & a<0<b,\ 0<u<t \\[2ex] \dfrac{b(b+a)}{\pi u^{3/2}(t-u)^{3/2}} \exp\left\{-\dfrac{b^2}{2(t-u)} - \dfrac{(b+a)^2}{2u}\right\} da\,db\,du; & a>0,\ b>0,\ 0<u<t. \end{cases}$$

The above equality for $a>0$ is obtained from the case $a<0$ and the fact that the triples $(W(t),L(t),\Gamma_+(t))$ and $(-W(t),L(t),t-\Gamma_+(t))$ are equivalent in law.

This trivariate density was derived in our earlier work (Karatzas & Shreve(1984)) using different techniques, and was employed there in the computation of the transition probabilities for a Brownian motion with two-valued drift. It was the striking similarity of the results (1.4) and (1.5) that provided the original motivation for the present investigation.

With the help of (1.4), one can compute the quadrivariate density

$$P[W(t)\in da;\ M(t)\in db;\ \gamma_t \in ds;\ \theta(t)\in d\theta] =$$

(1.6)
$$= \frac{2(-a)b^2}{(2\pi\theta(s-\theta)(t-s))^{3/2}} \exp\left\{-\frac{sb^2}{2\theta(s-\theta)} - \frac{a^2}{2(t-s)}\right\} da\,db\,ds\,d\theta;$$

$$0<\theta<s<t,\quad a<0<b.$$

We present a derivation in section 6 (Appendix). The identity (1.3), and the fact that the process $\hat{W}$ is Brownian motion, account for the result

$$P[W(t)\in da;\ L(t)\in db;\ \gamma_t \in ds;\ \Gamma_+(t)\in d\theta] =$$

(1.7)
$$= \begin{cases} \dfrac{2(-a)b^2}{(2\pi\theta(s-\theta)(t-s))^{3/2}} \exp\left\{-\dfrac{sb^2}{2\theta(s-\theta)} - \dfrac{a^2}{2(t-s)}\right\} da\,db\,ds\,d\theta\,; \\ \qquad 0<\theta<s<t,\quad a<0<b \\ \dfrac{2ab^2}{(2\pi(t-\theta)(\theta+s-t)(t-s))^{3/2}} \exp\left\{-\dfrac{sb^2}{2(t-\theta)(\theta+s-t)} - \dfrac{a^2}{2(t-s)}\right\} da\,db\,ds\,d\theta; \\ \qquad 0<t-\theta<s<t,\quad a>0,\quad b>0. \end{cases}$$

The formula in (1.7) for $a<0$ follows directly from (1.6); the formula valid for $a>0$ is derived from the former and from the fact that, under P, the quadruples $(W(t),L(t),\gamma_t,\Gamma_+(t))$ and $(-W(t),L(t),\gamma_t,t-\Gamma_+(t))$ are equivalent in law.

2. SUMMARY

In section 3 we review basic facts about the Wiener (Brownian motion), Bessel and Brownian Bridge probability measures on $C_{[0,\infty)}$, the space of continuous functions on $[0,\infty)$. Our aim is to discuss the behaviour of these measures under appropriate forms of conditioning. This is accomplished in Theorems 2-4.

The main result (Theorem 1) is proved in section 5, after it has been cast into a more convenient form in terms of a path transformation suggested by the nature of the "welded" process $\hat{W}$ in Theorem 1. Certain properties of this transformation are discussed in sections 4 and 7 (Appendix on measurability). Once the basic motation, definitions and properties have been laid out, it becomes a matter of combining the results of section 3 with a path decomposition result of D. Williams (Theorem 5) to establish Theorem 1.

The derivation of the quadrivariate density (1.6), as well as the proof of Theorem 2, are relegated to section 6 (Appendix).

3. BROWNIAN, BRIDGE AND BESSEL MEASURES

From now on we take as our sample space Ω, the set $C_{[0,\infty)}$ of continuous, real-valued functions $\omega = \{\omega(t);\ t\geq 0\}$ on $[0,\infty)$. Ω is thus a Polish (complete, separable metric) space under the topology of uniform convergence on compact subsets of $[0,\infty)$. Let $\mathfrak{F}(\mathfrak{F}_t)$ be the σ-field of Borel subsets of $C_{[0,\infty)}$ (respectively, $C_{[0,t]}$). $\mathfrak{F}(\mathfrak{F}_t)$ coincides with the smallest σ-field on $C_{[0,\infty)}$ (resp., $C_{[0,t]}$) such that, for every $s\geq 0$ (resp., $0\leq s\leq t$), the map $\omega \to \omega(s)$: $C_{[0,\infty)} \to \mathbb{R}$ is $\mathfrak{F}/\mathcal{B}$ - measurable (resp., $\mathfrak{F}_t/\mathcal{B}$ - measurable). We have denoted by $\mathcal{B}$ the Borel σ-field $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$. For a detailed study of the space $C_{[0,1]}$, the reader is referred to Parthasarathy (1967), Chapter VII.

In this section we recall three standard probability measures on $(\Omega,\mathfrak{F})$, and we discuss certain relations that they bear to one another.

3.a: Wiener (Brownian Motion) measure P_x; $x\in\mathbb{R}$.

This measure satisfies $P_x[\omega(0) = x] = 1$, and it has finite-dimensional distributions given by

$$P_x[\omega(t_1)\in dx_1;\ldots;\omega(t_n)\in dx_n] = p(t_1;x,x_1)dx_1$$
$$p(t_2-t_1;\ x_1,x_2)dx_2\ \ldots\ p(t_n-t_{n-1};\ x_{n-1},x_n)dx_n$$

for $0<t_1<\ldots<t_n$, $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $p(t;x,y)\triangleq\frac{1}{\sqrt{2\pi t}}\exp\left\{-\frac{(x-y)^2}{2t}\right\}$.

3.b: Brownian Bridge measure $B^{\tau}_{a\to b}$; $\tau>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$.

With given $\tau>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$, this measure satisfies

$$B^{\tau}_{a\to b}[\omega(0) = a;\ \omega(s) = b:\ \forall\ s\geq\tau] = 1,$$

and it has finite-dimensional distributions given by

$$(3.1)\qquad B^{\tau}_{a\to b}[\omega(t_i)\in dx_i:\ 1\leq i\leq n] \triangleq P_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n]\cdot \frac{p(\tau-t;\ x_n,b)}{p(\tau;\ a,b)}$$

for $0<t_1<\ldots<t_n = t<\tau$, $(x_1,\ldots,x_n)\in\mathbb{R}^n$. This assignment corresponds to the intuitive notion of "Brownian motion conditioned (tied-down) by $\omega(\tau) = b$", since the expression in (3.1) coincides with

$$P_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n|\omega(\tau) = b];\quad \text{for a.e.}\quad b\in\mathbb{R}.$$

In particular, for every $0<t<\tau$, we have

$$(3.2)\qquad B^{\tau}_{a\to b}[A] = E_a\left[1_A\,\frac{p(\tau-t;\ \omega(t),b)}{p(\tau;\ a,b)}\right];\quad \forall\ A\in\mathfrak{F}_t.$$

To simplify notation, the measure $B^{\tau}_{a\to a}$ will be denoted simply by B^{τ}_a.

For later convenience, let us agree that $s>0$ is an entrance time to b, $b\in\mathbb{R}$, for the path $\omega\in\Omega$, iff $\omega(s) = b$ and there exists a strictly increasing sequence $\{s_n\}_{n=1}^{\infty}$ with $\lim_{n\to\infty} s_n = s$, such that $\omega(s_n) \neq b$ for every $n\geq 1$. With fixed $t>0$, $b\in\mathbb{R}$ define:

$$(3.3) \qquad \begin{aligned} \gamma_t(\omega) &\triangleq \sup\{s\leq t;\ s>0 \text{ is an entrance time to zero, for } \omega\} \\ &\triangleq 0\ ;\ \text{if } \{\ldots\} = \emptyset, \end{aligned}$$

$$(3.4) \qquad \begin{aligned} \sigma_b(\omega) &\triangleq \sup\{s<\infty;\ s>0 \text{ is an entrance time to } b, \text{ for } \omega\} \\ &\triangleq 0\ ;\ \text{if } \{\ldots\} = \emptyset. \end{aligned}$$

Under most measures we consider, γ_t is a.s. equal to the last exit time from zero before t; a notable exception is Brownian Bridge measure B_0^{τ} with $t>\tau$. Under Bessel measure Q_x defined in §3.c, σ_b is a.s. the last exit time from b.

The following result is intuitively very plausible (and undoubtedly quite well-known, although we have been unable to cite any particular reference). Its proof is given in the Appendix (section 6).

THEOREM 2: Brownian motion P_0 conditioned by the value $\gamma_t = \tau$ of its last entrance to zero prior to t, is Brownian Bridge on $[0,\tau]$. In other words, with fixed $0<s<t$, we have for any $A\in\mathfrak{F}_s$:

$$(3.5) \qquad P_0[A|\gamma_t=\tau] = B_0^\tau[A]; \quad \text{for a.e. } \tau\in(s,t).$$

□

3.c: Bessel measure Q_x; $x>0$.

The Bessel process $\xi(t) = \sqrt{W_1^2(t) + W_2^2(t) + W_3^2(t)}$; $t\geq 0$ is the radial part of Brownian motion $\underset{\sim}{W}(t) = (W_1(t), W_2(t), W_3(t))$ in three-dimensional space. As is well-known (Mc Kean (1969), Beneš (1975)), this process satisfies the stochastic differential equation:

$$d\xi(t) = \frac{dt}{\xi(t)} + db(t); \quad t>0$$

$$\xi(0) = x>0,$$

where $\{b(t);\ t\geq 0\}$ is a one-dimensional Brownian motion, and it induces a measure Q_x on $C_{[0,\infty)}$, henceforth called Bessel measure, which, when restricted to any $\mathfrak{F}_t$, is absolutely continuous with respect to Wiener measure P_x. The corresponding Radon-Nikodým derivative is

$$\left.\frac{dQ_x}{dP_x}\right|_{\mathfrak{F}_t} = \exp\left\{\int_0^t \frac{d\omega(s)}{\omega(s)} - \frac{1}{2}\int_0^t \frac{ds}{\omega^2(s)}\right\}\cdot 1_{\{T_0>t\}},$$

where T_0 is the first passage time at the origin. Now Itô's rule shows that the exponent can be written as $\log \frac{\omega(t)}{x}$, and thus

(3.6) $$Q_x[A] = E_x\left[\frac{\omega(t)}{x} 1_{A\cap\{T_0>t\}}\right]; \quad \forall\ x>0,\ t>0,\ A\in\mathcal{F}_t.$$

In particular, the transition probabilities for this measure are given by (3.6) as

(3.7) $$Q_x[\omega(t)\in dy] = \frac{y}{x} P_x[\omega(t)\in dy;\ T_0>t] = \frac{y}{x}\, q(t;\ x,y)dy\ ,$$

for $t>0$, $x>0$, $y>0$, where: $q(t;\ x,y) \triangleq (p(t;\ x,y)-p(t;\ x,-y))^+$. Besides, with $x>0$, $z>y>0$ we have

$$Q_x[\sigma_y\le t;\ \omega(t)\in dz] = Q_x[\omega(t)\in dz].Q_z[\min_{s\ge 0}\omega(s)>y]$$

$$= \frac{z}{x}\, q(t;\ x,z)dz.\ \frac{z-y}{z}\ ,$$

because the Q_z - distribution of $\min_{s\ge 0}\omega(s)$ is uniform on $(0,z)$; c.f. Williams (1974). Integrating out z over (y,∞) in the above expression, we obtain

(3.8) $$xQ_x[\sigma_y\le t] = t\,q(t;x,y)+(x-y).\Phi\left(\frac{x-y}{\sqrt{t}}\right)+(x+y).\left[1-\Phi\left(\frac{x+y}{\sqrt{t}}\right)\right]$$

and

(3.9) $$Q_x[\sigma_y\in dt] = \frac{dt}{2x}\ .\ q(t;\ x,y);\quad t>0.$$

This last equation appears in Williams (1974), p. 751, without proof. Here and in the sequel, $\Phi(.)$ denotes the standard normal distribution function $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\{-\frac{1}{2}z^2\}dz;\ x\in R.$

3.d: Local Time and Maximum for Brownian Bridge.

The distribution under Wiener measure, and further properties of standard local time at the origin

(3.10) $$L(t,\omega) \triangleq \overline{\lim_{\epsilon\downarrow 0}} \frac{1}{4\epsilon} \operatorname{meas}\{0\le s\le t;\ |\omega(s)|\le\epsilon\};\ t\ge 0,\ \omega\in\Omega$$

have been studied extensively. In particular, with

(3.11) $$h(t,x) = \frac{x}{\sqrt{2\pi t^3}} e^{-\frac{x^2}{2t}};\ x\in\mathbb{R},\ t>0,$$

we have from p.45 of Itô - Mc Kean (1975), as well as from relation (4.6) in Karatzas & Shreve (1984):

(3.12) $$P_x[\omega(t)\in da;\ L(t)\in db] = 2h(t;\ 2b+|x|+|a|)da\,db;\ a\in\mathbb{R},\ b>0,$$

whence

(3.13) $$P_x[L(t)\le m] = 2\Phi\left(\frac{2m+|x|}{\sqrt{t}}\right) - 1;\ m\ge 0,\ t>0,\ x\in\mathbb{R}\ .$$

Besides, with

(3.14) $$M(t,\omega) \triangleq \sup_{0\le s\le t} \omega(s),\ m(t,\omega) \triangleq \inf_{0\le s\le t} \omega(s),\ m(\infty,\omega) \triangleq \inf_{s\ge 0} \omega(s)\ ,$$

we have

(3.15) $$P_0[\omega(t)\in da;\ M(t)\in db] = 2h(t;\ 2b-a)da\,db,$$

with $t>0$, $b>0$, $a<b$. Now let us consider (3.15), and (3.12) with $x = 0$; upon multiplying by $\sqrt{\frac{\tau}{\tau-t}}\exp\left[-\frac{a^2}{2(\tau-t)}\right]$, we

obtain the joint densities of $(\omega(t), L(t))$ and $(\omega(t), M(t))$ under Brownian Bridge measure B_0^τ, with $\tau>t$; recall (3.2). In the resulting expressions we can integrate out a (over $\mathbb{R}$ and $(-\infty,b)$, respectively), and then let $t\uparrow\tau$, to obtain:

$$(3.16) \qquad B_0^\tau[L(\tau)\in db] = B_0^\tau[M(\tau)\in db] = \frac{4b}{\tau}\, e^{-\frac{2b^2}{\tau}}\, db; \quad b>0.$$

3.e: Brownian Bridge B_0^τ conditioned to accumulate m units of local time.

Corresponding to any given numbers $\tau>0$, $m>0$, we wish to construct a probability measure $B_0^{\tau;m}$ on $(\Omega, \mathfrak{F})$ with

(3.17) $$B_0^{\tau;m}[\omega(0) = 0;\ \omega(s) = 0;\ \forall\ s\geq\tau] = 1$$

corresponding to the intuitive notion of "Brownian Bridge conditioned to accumulate m units of local time on $[0,\tau]$". In other words, we want that for every $0<t<\tau$, $A\in\mathfrak{F}_t$, we have

(3.18) $$B_0^\tau[A|L(\tau) = m] = B_0^{\tau;m}[A]; \quad \text{for a.e. } m>0.$$

We start by selecting numbers:

$$0 = t_0<t_1<\ldots<t_n = t<\tau<\infty$$

$$\underset{\sim}{x} = (x_1,\ldots,x_n)\in\mathbb{R}^n,\ x_n = y,\ x_0 = 0$$

$$0 = a_0<a_1<\ldots<a_{n-1}<a_n = b<m,$$

and introducing the notation

(3.19) $$D(m) \triangleq \{\underset{\sim}{a} = (a_1,\ldots,a_n)\in\mathbb{R}^n\ |0<a_1<\ldots<a_n<m\}.$$

With $0<\epsilon<\frac{\tau-t}{2}$, we have, by virtue of (3.12) and of the Markovian character of Brownian motion:

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n;\ \omega(\tau-\epsilon)\in dz;\ L(\tau-\epsilon)\in dm]$$

$$= \int\cdots\int_{D(m)} \prod_{i=1}^{n} P_{x_{i-1}}[\omega(t_i-t_{i-1})\in dx_i;\ L(t_i-t_{i-1})\in da_i-a_{i-1}]\ .$$

$$.\ P_y[\omega(\tau-t-\epsilon)\in dz;\ L(\tau-t-\epsilon)\in dm-b]$$

$$=2\int\cdots\int_{D(m)} H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ h(\tau-t-\epsilon;\ 2(m-b)+|y|+|z|)d\underset{\sim}{a}.d\underset{\sim}{x}dzdm,$$

where

(3.20) $$H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a}) \triangleq 2^n.\prod_{i=1}^{n} h(t_i-t_{i-1};\ 2(a_i-a_{i-1})+|x_i|+|x_{i-1}|).$$

The corresponding Brownian Bridge computation can now be obtained thanks to (3.2), upon multiplying by the factor $\sqrt{\frac{\tau}{\epsilon}}\exp\{-\frac{z^2}{2\epsilon}\}$:

$$B_0^\tau[\omega(t_i)\in dx_i:\ 1\le i\le n;\ \omega(\tau-\epsilon)\in dz;\ L(\tau-\epsilon)\in dm] =$$

$$2\int\cdots\int_{D(m)} H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a}).h(\tau-t-\epsilon;\ 2(m-b)+|y|+|z|)\sqrt{\frac{\tau}{\epsilon}}\,e^{-z^2/2\epsilon}$$

$$d\underset{\sim}{a}.d\underset{\sim}{x}dzdm\ .$$

The terms involving z in the above expression can be written, with $\beta = 2(m-b)+|y|>0$, $\mu = \frac{\epsilon}{\tau-t}\beta$ and $\sigma^2 = \frac{\epsilon(\tau-t-\epsilon)}{\tau-t}$:

$$h(\tau-t-\epsilon;\ \beta+|z|)\sqrt{\frac{\tau}{\epsilon}}\exp\{-\frac{z^2}{2\epsilon}\} = \frac{|z|+\beta}{\sigma(\tau-t-\epsilon)\sqrt{2\pi}}\sqrt{\frac{\tau}{\tau-t}}\ .$$

$$.\ \exp\{-\frac{(|z|+\mu)^2}{2\sigma^2} - \frac{\beta^2}{2(\tau-t)}\}\ .$$

The integral of this expression with respect to z, over $\mathbb{R}$, is given by: $\sqrt{\frac{\tau}{\tau-t}}\, e^{-\frac{\beta^2}{2(\tau-t)}}$.

$$\cdot \left[\frac{(\frac{2\epsilon}{\pi})^{1/2}}{\sqrt{(\tau-t)(\tau-t-\epsilon)}} e^{-\frac{\epsilon\beta^2}{2(\tau-t)(\tau-t-\epsilon)}} + \frac{2\beta}{\tau-t}\left\{1-\Phi\left(\sqrt{\frac{\epsilon}{\tau-t}}\frac{\beta}{\sqrt{\tau-t-\epsilon}}\right)\right\}\right],$$

and thus: $B_0^\tau[\omega(t_i)\in dx_i: 1\le i\le n;\ L(\tau-\epsilon)\in dm]$

$$= 2\int_{D(m)}\cdots\int H(\underset{\sim}{t};\underset{\sim}{x},\underset{\sim}{a})\sqrt{\frac{\tau}{\tau-t}}\exp\left\{-\frac{(2(m-b)+|y|)^2}{2(\tau-t)}\right\} d\underset{\sim}{a}.d\underset{\sim}{x}\ dm$$

$$\cdot \left[\frac{(\frac{2\epsilon}{\pi})^{1/2}}{\sqrt{(\tau-t)(\tau-t-\epsilon)}} e^{-\frac{\epsilon(2(m-b)+|y|)^2}{2(\tau-t)(\tau-t-\epsilon)}} + \right.$$

$$\left. + \frac{2(2(m-b)+|y|)}{\tau-t}\left\{1-\Phi\left(\sqrt{\frac{\epsilon}{\tau-t}}\frac{2(m-b)+|y|}{\sqrt{\tau-t-\epsilon}}\right)\right\}\right].$$

Letting now $\epsilon\downarrow 0$, we obtain by the Dominated Convergence Theorem:

$$B_0^\tau[\omega(t_i)\in dx_i: 1\le i\le n;\ L(\tau)\in dm] =$$

$$= 2\int_{D(m)}\cdots\int H(\underset{\sim}{t};\underset{\sim}{x},\underset{\sim}{a})\sqrt{2\pi\tau}\,h(\tau-t;2(m-b)+|x_n|)\,d\underset{\sim}{a}.d\underset{\sim}{x}dm.$$

But from (3.16): $B_0^\tau[L(\tau)\in dm] = 2\sqrt{2\pi\tau}\,h(\tau;2m)dm$, for $m>0$. Therefore, for a.e. $m>0$, we have

$$B_0^{\tau}[\omega(t_i)\in dx_i:\ 1\le i\le n \mid L(\tau) = m] =$$

$$= \int_{D(m)}\cdots\int H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ \frac{h(\tau-t;\ 2(m-b)+|x_n|)}{h(\tau;\ 2m)}\ d\underset{\sim}{a}.d\underset{\sim}{x}\ .$$

The expression on the right-hand side is defined and continuous for every $m>0$. Thus, in accordance with (3.18), we decree that the sought probability measure $B_0^{\tau;m}$ has the finite-dimensional distributions

(3.21) $$B_0^{\tau;m}[\omega(t_i)\in dx_i:\ 1\le i\le n] \triangleq$$

$$\triangleq \int_{D(m)}\cdots\int H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ \frac{h(\tau-t;2(m-b)+|x_n|)}{h(\tau;\ 2m)}\ d\underset{\sim}{a}d\underset{\sim}{x}$$

as long as $0<t_1<\ldots<t_n = t<\tau$. We regard $B_0^{\tau;m}$ as a measure on $C_{[0,\infty)}$ rather than just on $C_{[0,\tau)}$, by stipulating (3.17).

3.f: Brownian motion P_0 conditioned to first attain m units of local time at the instant τ.

For any given numbers $m>0$, $\tau>0$, we want now to construct a probability measure $P_0^{m;\tau}$ on $(\Omega, \mathcal{F})$ with

$$(3.22) \qquad P_0^{m;\tau}[\omega(0) = 0;\ \omega(s) = 0,\ \forall\ s\geq\tau] = 1,$$

in accordance with the intuitive notion of "Brownian motion conditioned by $L^{-1}(m) = \tau$"; i.e., such that for any $t>0$, $m>0$, $A\in\mathcal{F}_t$, we have

$$(3.23) \qquad P_0[A|L^{-1}(m) = \tau] = P_0^{m;\tau}[A]; \quad \text{for a.e.} \quad \tau>t.$$

With the notation of (and just preceding) (3.19), we have:

$$P_0[\omega(t_i)\in dx_i:\ 1\leq i\leq n; L^{-1}(m)\geq\tau] =$$

$$P_0[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ L(\tau)\leq m] =$$

$$\int\cdots\int_{D(m)} P_0[\bigcap_{i=1}^{n}\{\omega(t_i)\in dx_i;\ L(t_i)\in da_i\};\ L(\tau)\leq m] =$$

$$\int\cdots\int_{D(m)} \prod_{i=1}^{n} P_{x_{i-1}}[\omega(t_i-t_{i-1})\in dx_i;\ L(t_i-t_{i-1})\in da_i-a_{i-1}]\ \cdot$$

$$\cdot\ P_y[L(\tau-t)\leq m-b] =$$

$$\int\cdots\int_{D(m)} H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ \left[2\Phi\left(\frac{2(m-b)+|y|}{\sqrt{\tau-t}}\right) - 1\right] d\underset{\sim}{a}.d\underset{\sim}{x},$$

by virtue of formulae (3.12), (3.13) and (3.20). Differentiating the above expression with respect to τ, we obtain

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n;\ L^{-1}(m)\in d\tau] =$$

$$= \int_{D(m)}\cdots\int H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ .\ h(\tau-t;\ 2(m-b)+|x_n|)d\underset{\sim}{a}.d\underset{\sim}{x}\,d\tau.$$

But $L^{-1}(m)$ is identical in law, under P_0, to the first passage time T_{2m} at level 2m (Itô - Mc Kean (1974), p.25); in particular,

$$P_0[L^{-1}(m)\in d\tau] = P_0[T_{2m}\in d\tau] = h(\tau;2m)d\tau;\quad \tau>0.$$

Therefore, for a.e. $\tau>t$, we have

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n|L^{-1}(m) = \tau] =$$

$$= \int_{D(m)}\cdots\int H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ \frac{h(\tau-t;\ 2(m-b)+|x_n|)}{h(\tau;2m)}\,d\underset{\sim}{a}.d\underset{\sim}{x}.$$

As before, the expression on the right-hand side is defined and continuous for all $\tau>t$; so, in accordance with (3.23), we require that the sought probability measure $P_0^{m;\tau}$ have the finite-dimensional distributions

$$(3.24)\qquad P_0^{m;\tau}[\omega(t_i)\in dx_i:\ 1\le i\le n] \triangleq$$

$$\triangleq \int\cdots\int_{D(m)} H(\underset{\sim}{t};\ \underset{\sim}{x},\underset{\sim}{a})\ \frac{h(\tau-t;\ 2(m-b)+|x_n|)}{h(\tau;\ 2m)}\ d\underset{\sim}{a}.d\underset{\sim}{x}\ ,$$

as long as $0<t_1<\ldots<t_n = t<\tau$. We regard $P_0^{m;\tau}$ as a measure on $C_{[0,\infty)}$ rather than merely on $C_{[0,\tau)}$, by stipulating (3.22).

The identity of the two expressions in definitions (3.20) and (3.22) yields the following characterization:

<u>THEOREM 3</u>: <u>Brownian motion P_0 conditioned to first attain m units of local time at the instant τ, is equivalent to Brownian Bridge $[0,\tau]$ conditioned to accumulate m units of local time.</u> More precisely, for all $m>0$, $\tau>0$, $A\in\mathfrak{F}$ we have

$$(3.25)\qquad P_0^{m;\tau}[A] = B_0^{\tau;m}[A].$$

□

<u>Remark</u>: Despite its value as an intuitive interpretation, the first sentence of Theorem 3 has actually no mathematical meaning. It appears to state that for $A\in\mathfrak{F}_t$ and $\tau>t$,

$$(3.26)\qquad P_0[A|L^{-1}(m) = \tau] = B_0^{\tau}[A|L(\tau) = m].$$

However, the expression on the left-hand side of (3.26) is a function of τ defined only up to almost everywhere equivalence, but the expression on the right is a function of m, also defined up to almost everywhere equivalence. Thus, the two expressions cannot be meaningfully equated. The content of Theorem 3 appears in the second sentence.

3.g: Brownian Bridge $B^{\tau}_{a\to b}$, conditioned by the value $m(\infty)\equiv m(\tau) = m$ of its minimum.

The objective here is to construct a probability measure $\hat{B}^{\tau;m}_{a\to b}$ on $(\Omega, \mathfrak{F})$, such that for any given $\tau>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$, $m<a\wedge b$, we have

$$\hat{B}^{\tau;m}_{a\to b}\left[\omega(0)=a;\ \omega(s)=b:\ \forall s\geq\tau;\ \min_{0\leq s\leq\tau}\omega(s)=m\right] = 1.$$

This measure will play the rôle of "Brownian Bridge conditioned by the value m of its minimum"; i.e., with $0<t<\tau$, $\forall\ A\in\mathfrak{F}_t$,

$$B^{\tau}_{a\to b}\left[A\,|\,m(\tau)=m\right] = \hat{B}^{\tau;m}_{a\to b}[A];\quad \text{for a.e. } m\in(-\infty,\ a\wedge b). \tag{3.27}$$

For the construction, one takes $0 = t_0<t_1<\ldots<t_n = t<\tau-\epsilon<\tau$ and real numbers $a = x_0, x_1, \ldots, x_n = x, z, b, m,$ such that $\min(\min_{0\leq i\leq n} x_i, z, b) > m$. From (3.2) one then has

$$B^{\tau}_{a\to b}[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ \omega(\tau-\epsilon)\in dz;\ m(\tau-\epsilon) > m] =$$

$$= P_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ \omega(\tau-\epsilon)\in dz;\ m(\tau-\epsilon) > m]\ \frac{p(\epsilon;\ z,b)}{p(\tau;\ a,b)}$$

$$= \frac{1}{p(\tau;a,b)}\prod_{i=1}^{n} q\,(t_i-t_{i-1};x_{i-1}-m, x_i-m)dx_i\ .$$

$$\frac{q\,(\tau-\epsilon-t;\ x-m, z-m)}{\sqrt{2\pi\epsilon}}\ e^{-\frac{(z-b)^2}{2\epsilon}}\ dz.$$

Integrating out $z\in(m,\infty)$ in the above expression and using the notation $\sigma^2 = \frac{\epsilon(\tau-t-\epsilon)}{\tau-t}$, $\mu = \frac{\epsilon x+(\tau-t-\epsilon)b}{\tau-t}$ and $\nu = \frac{\epsilon(2m-x)+(\tau-t-\epsilon)b}{\tau-t}$, we obtain

$$p(\tau;\ a,b)\ .\ B^{\tau}_{a\to b}[\omega(t_i)\epsilon dx_i:\ 1\le i\le n;\ m(\tau-\epsilon) > m] =$$

$$= \frac{\prod_{i=1}^{n} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)dx_i}{\sqrt{2\pi(\tau-t)}}$$

$$\left[e^{-\frac{(x-b)^2}{2(\tau-t)}}\Phi\left(\frac{\mu-m}{\sigma}\right)-e^{-\frac{(2m-x-b)^2}{2(\tau-t)}}\Phi\left(\frac{\nu-m}{\sigma}\right)\right],$$

and in the limit as $\epsilon\downarrow 0$:

$$p(\tau;\ a,b)\ .\ B^{\tau}_{a\to b}\left[\omega(t_i)\epsilon dx_i:\ 1\le i\le n;\ m(\tau) > m\right] =$$

(3.28)
$$= \prod_{i=1}^{n} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)dx_i\ .\ q(\tau-t;\ x-m, b-m).$$

Similarly, one obtains

(3.29) $$p(\tau;\ a,b)\ B^{\tau}_{a\to b}[m(\tau)>m] = q(\tau;\ a-m, b-m),$$

and the differential forms of (3.28), (3.29) point out that, for a.e. $m\epsilon(-\infty, \min(\min_{0\le i\le n} x_i, b))$, we have

$$B^{\tau}_{a\to b}[\omega(t_i)\in dx_i:\ 1\le i\le n \mid m(\tau) = m] =$$

(3.30)

$$= \frac{\frac{\partial}{\partial m}\left[\prod_{i=1}^{n} q(t_i - t_{i-1};\ x_{i-1}-m, x_i - m)\, q(\tau - t;\ x_n - m, b-m)\right]}{\frac{\partial}{\partial m} q(\tau;\ a-m, b-m)}\, dx_1 \cdots dx_n .$$

Now again, the expression on the right-hand side of (3.30) is defined and continuous for every real $m < \min(\min_{0\le i\le n} x_i, b)$. Therefore, given any $a\in\mathbb{R}$, $b\in\mathbb{R}$, $\tau>0$ and $m<a\wedge b$, we can decree in accordance with (3.27) that the sought probability measure $\hat{B}^{\tau;m}_{a\to b}$ has finite - dimensional distributions

(3.31) $\quad \hat{B}^{\tau;m}_{a\to b}[\omega(t_i)\in dx_i:\ 1\le i\le n] \triangleq$ RHS of (3.30)

for $0<t_1<\ldots<t_n = t<\tau$, $(x_1,\ldots,x_n)\in\mathbb{R}^n$, $\min_{1\le i\le n} x_i > m$.

Remark: The symmetry of Brownian motion allows us now to compute easily, for any given $\tau>0$, $a\in\mathbb{R}$, $b\in\mathbb{R}$, $m>a\vee b$, a probability measure $\check{B}^{\tau;m}_{a\to b}$ on $(\Omega,\mathfrak{F})$ with

$$\check{B}^{\tau;m}_{a\to b}[\omega(0)=a;\ \omega(s)=b:\ \forall\ s\ge\tau;\ \max_{0\le s\le\tau}\omega(s) = m] = 1 ,$$

which corresponds to the intuitive notion of "Brownian Bridge conditioned by the value of its maximum"; i.e., with $0<t<\tau$, $\forall\ A\in\mathfrak{F}_t$:

$$B^{\tau}_{a\to b}[A|M(\tau)=m] = \check{B}^{\tau;m}_{a\to b}[A]; \quad \text{for a.e.} \quad m\in(a\vee b,\infty). \tag{3.32}$$

Indeed,

$$\check{B}^{\tau;m}_{a\to b}[A] = \hat{B}^{\tau;-m}_{-a\to -b}[-A]. \tag{3.33}$$

3.h: Bessel process Q_a conditioned by the value $m(\infty)=m$ of its minimum.

Let us consider real numbers $0 = t_0<t_1<\ldots<t_n = t<\tau = t_{n+1}$, $0<m\leq b<z$, as well as $x_0 = a, x_1, \ldots, x_n = x, x_{n+1} = z$ with $\min\limits_{0\leq i\leq n+1} x_i>m>0$. We have

$$Q_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ \omega(\tau)\in dz;\ \sigma_b\leq\tau;\ m(\infty)>m] =$$

$$Q_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ \omega(\tau)\in dz;\ m(\tau)>m]\ .\ Q_z[m(\infty)>b] =$$

$$\frac{z}{a}\, P_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ \omega(\tau)\in dz;\ m(\tau)>m]\ .\ \frac{z-b}{z}\ ,$$

by virtue of the Markovian property, formula (3.6), and the fact that the Q_z - distribution of $m(\infty)$ is uniform on $(0,z)$. It follows that

$$(3.34)\qquad a.Q_a[\omega(t_i)\in dx_i:\ 1\leq i\leq n;\ \omega(\tau)\in dz;\ \sigma_b\leq\tau;\ m(\infty)>m] =$$

$$(z-b)\prod_{i=1}^{n} P_{x_i}[\omega(t_i-t_{i-1})\in dx_i;\ T_m>t_i-t_{i-1}] =$$

$$\prod_{i=1}^{n} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)dx_i\ .\ (z-b)\ q(\tau-t;\ x-m, z-m)dz.$$

With $b=m$, we obtain, since from the definition (3.4) we have $\sigma_m = 0$ on $\{m(\infty) > m\}$:

$$a.Q_a[\omega(t_i)\in dx_i:\ 1\le i\le n+1;\ m(\infty)>m] =$$

$$= (x_{n+1}-m) \prod_{i=1}^{n+1} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)dx_i.$$

But now: $Q_a[m(\infty)\in dm] = \frac{dm}{a}$; $0<m<a$, and thus, for almost every m in the interval $(0, \min\limits_{1\le i\le n+1} x_i)$:

$$Q_a[\omega(t_i)\in dx_i:\ 1\le i\le n+1 | m(\infty) = m] =$$

(3.35)

$$= -\frac{\partial}{\partial m}[(x_{n+1}-m) \prod_{i=1}^{n+1} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)]dx_1\ldots dx_{n+1}.$$

The right-hand side of (3.35) is well-defined and continuous at any $m\in(0, \min\limits_{1\le i\le n+1} x_i)$. So, we can fix $0<m<a$ and define a measure $\hat{Q}_a^m$ on $(\Omega, \mathfrak{F})$ with

$$\hat{Q}_a^m[\omega(0) = a;\ \min_{s\ge 0} \omega(s) = m] = 1$$

and finite-dimensional distributions given by

$$\hat{Q}_a^m[\omega(t_i)\in dx_i:\ 1\le i\le n+1] =$$

(3.36)

$$= -\frac{\partial}{\partial m}[(x_{n+1}-m) \prod_{i=1}^{n+1} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)]dx_1\ldots dx_{n+1}$$

for any numbers $(x_1, \ldots, x_{n+1})\in\mathbb{R}^{n+1}$: $\min\limits_{1\le i\le n+1} x_i>m$. According to (3.35), this new measure corresponds to "Bessel process Q_a

conditioned by the value $m(\infty) = m$ of its minimum": for every $A \in \mathfrak{F}$,

$$(3.35)' \quad Q_a[A | m(\infty) = m] = \hat{Q}_a^m[A]; \quad \text{for a.e. } m \in (0,a).$$

Now let us integrate out z in expression (3.34) over the interval (b,∞), using the equality

$$\int_b^\infty (z-b) q(\tau-t;\ x-m, z-m) dz = (\tau-t) q(\tau-t;\ x-m, b-m) +$$

$$+ (x-b)\ \Phi\left(\frac{x-b}{\sqrt{\tau-t}}\right) + (b+x-2m)\ \left\{1-\Phi\left(\frac{b+x-2m}{\sqrt{\tau-t}}\right)\right\},$$

to obtain, after differentiating with respect to τ:

$$2a.Q_a[\omega(t_i) \in dx_i:\ 1 \le i \le n;\ \sigma_b \in d\tau;\ m(\infty) > m] =$$

$$= \prod_{i=1}^n q(t_i - t_{i-1};\ x_{i-1}-m, x_i-m) dx_i \cdot q(\tau-t;\ x-m, b-m) d\tau.$$

Therefore, with $0 = t_0 < t_1 < \ldots < t_n = t < \tau$, $a = x_0, b > m > 0$ and $\min_{0 \le i \le n} x_i > m$, we have

$$2\hat{Q}_a^m[\omega(t_i) \in dx_i:\ 1 \le i \le n;\ \sigma_b \in d\tau] =$$

$$= -\frac{\partial}{\partial m}\left[\prod_{i=1}^n q(t_i - t_{i-1};\ x_{i-1}-m, x_i-m) dx_i \cdot q(\tau-t; x-m, b-m)\right] d\tau,$$

and in a similar fashion we can obtain

$$2\hat{Q}_a^m[\sigma_b \in d\tau] = -\frac{\partial}{\partial m} q(\tau-t;\ a-m, b-m)d\tau.$$

From the last two identities we conclude that, for a.e. $\tau>t$, we have

$$(3.37)\qquad \hat{Q}_a^m[\omega(t_i)\in dx_i:\ 1\leq i\leq n\,|\,\sigma_b=\tau] =$$

$$= \frac{\frac{\partial}{\partial m}\left[\prod_{i=1}^{n} q(t_i-t_{i-1};\ x_{i-1}-m, x_i-m)\ q(\tau-t;\ x_n-m, b-m)\right]}{\frac{\partial}{\partial m} q(\tau;\ a-m, b-m)}\, dx_1\ldots dx_n.$$

From the identity of the expressions in (3.31), (3.37), we deduce our third characterization of conditional processes:

THEOREM 4: Bessel process Q_a, $a>0$, conditioned by the value m of its minimum and by the last entrance time τ in level b, is equivalent on $[0,\tau]$ to Brownian Bridge from a to b, conditioned to attain a minimum m. More precisely, for any $a>0$, $b>0$, $0<m<a\wedge b$, $t>0$ and any $A\in\mathfrak{F}_t$, we have

$$(3.38)\qquad \hat{Q}_a^m[A\,|\,\sigma_b = \tau] = \hat{B}^{\tau;m}_{a\to b}[A];\quad \text{for a.e. } \tau>t.$$

□

4: A PATH TRANSFORMATION

In this section we introduce a path transformation, which will help us recast our main result, Theorem 1, into a more convenient form. The study of this transformation will require a brief digression, involving the space D of right-continuous, left-limited functions on $\mathbb{R}^+ = [0,\infty)$.

For $0 \le t_1 < t_2 \le \infty$, let $D[t_1, t_2)$ be the set of all real-valued, right-continuous functions $\omega = \{\omega(t);\ t \ge 0\}$ which have left-limits everywhere on $(t_1, t_2] \cap (0,\infty)$. When t_2 is finite, we endow $D[t_1,t_2)$ with the Skorohod metric $d_{t_1,t_2}(\omega_1,\omega_2)$; see Parthasarathy (1967), p. 234. On $D[t_1,\infty)$, the metric $d_{t_1,\infty}(\omega_1,\omega_2)$ is defined by $\sum_{n=1}^{\infty} 2^{-n}[d_{t_1,t_1+n}(\omega_1,\omega_2) \wedge 1]$. The topology associated with the Skorohod metric generates the smallest σ-field with respect to which the projections $\omega \to \omega(t)$ are measurable, for every $t \in [t_1,t_2)$; c.f. Parthasarathy (1967), p. 249. We denote this σ-field by $\mathcal{G}_{[t_1,t_2)}$. We shall use the notation $D \equiv D_{[0,\infty)}$, $\mathcal{G} = \mathcal{G}_{[0,\infty)}$.

For fixed $t>0$, D can be regarded as the Cartesian product $D[0,t) \times D[t,\infty)$, and $\mathcal{G} = \mathcal{G}_{[0,t)} \otimes \mathcal{G}_{[t,\infty)}$; we denote by $\mathcal{G}_t$ the σ-field of subsets of D, of the form $A \times D_{[t,\infty)}$; $A \in \mathcal{G}_{[0,t)}$.

The space $\Omega = C_{[0,\infty)} \subseteq D$ of <u>continuous</u> functions ω on $\mathbb{R}^+$ is given the topology of uniform convergence on compact subsets of $\mathbb{R}^+$, which coincides with the relative topology of Ω as a sub-space of D (Parthasarathy (1967), p. 248), and under which Ω is complete. Therefore, Ω is a closed, hence measurable, subset of D.

Motivated by the considerations of section 1, let us define the following mappings from $\mathbb{R}^+ \times \Omega$ into $\mathbb{R}$: the occupation times

$$(4.1) \qquad \Gamma_+(t,\omega) = \text{meas}\{0 \le s \le t;\ \omega(s) \ge 0\}$$

$$(4.2) \qquad \Gamma_-(t,\omega) = \text{meas}\{0 \le s \le t;\ \omega(s) < 0\} = t - \Gamma_+(t,\omega)$$

and their inverses

$$(4.3) \qquad \Gamma_\pm^{-1}(\tau,\omega) = \inf\{t \ge 0;\ \Gamma_\pm(t,\omega) > \tau\},$$

the standard local time at the origin

$$(3.10) \qquad L(t,\omega) = \overline{\lim_{\epsilon \downarrow 0}} \frac{1}{4\epsilon} \text{meas}\{0 \le s \le t;\ |\omega(s)| \le \epsilon\},$$

as well as the mappings

$$(4.4) \qquad W_\pm(\tau,\omega) = \pm\omega(\Gamma_\pm^{-1}(\tau,\omega))$$

$$(4.5) \qquad L_\pm(\tau,\omega) = L(\Gamma_\pm^{-1}(\tau,\omega),\omega)$$

$$(4.6) \qquad B_\pm(\tau,\omega) = L_\pm(\tau,\omega) - W_\pm(\tau,\omega) + (\omega(0))^\pm$$

$$(4.7) \qquad M_\pm(t,\omega) = \sup_{0 \le s \le t} B_\pm(s,\omega), \quad m_\pm(t,\omega) = \inf_{0 \le s \le t} B_\pm(s,\omega).$$

We recall (c.f. Ikeda & Watanabe (1981), pp. 122-123) that under Wiener measure P_x, $W_\pm = \{W_\pm(\tau);\ \tau \ge 0\}$ are independent reflecting Brownian motions with $W_\pm(0) = x^\pm$, almost surely, and that

$B_{\pm} = \{B_{\pm}(\tau);\ \tau \geq 0\}$ are independent Brownian motions with $B_{\pm}(0) = 0$, almost surely.

With the help of the above mappings we can introduce now our basic path transformation: for any $t \in \mathbb{R}^+$, we consider the mapping $\varphi_t\colon \Omega \to D$ given by

$$(4.8)\qquad \begin{aligned} \varphi_t(\omega)(s) &= B_+(s,\omega) \quad &;\ 0 \leq s < \Gamma_+(t,\omega) \\ &= B_-(t-s,\omega) \quad &;\ \Gamma_+(t,\omega) \leq s < t \\ &= \omega(s) \quad &;\ s \geq t. \end{aligned}$$

The motivation for the introduction of this mapping comes of course from Theorem 1, which will be proved in section 5 by a method that employs the properties of φ_t and the results of section 3.

In section 7 (Appendix) we discuss the measurability of the mappings (4.1)-(4.8). In particular, we establish the following result:

<u>PROPOSITION 1</u>: For any fixed $t>0$, the mapping $\varphi_t\colon \Omega \to D$ is $\mathcal{F}_t/\mathcal{G}_t$ - measurable. Besides, the mapping $\varphi\colon \mathbb{R}^+ \times \Omega \to D$ is $\mathcal{B}^+ \otimes \mathcal{F}/\mathcal{G}$ - measurable. □

Here and in the sequel, we denote by $\mathcal{B}(\mathcal{B}^+)$ the σ-field of Borel sets in $\mathbb{R}$ (respectively, $\mathbb{R}^+$).

Now let us fix $x \geq 0$, $m>0$ with $x \neq m$, and define

$$(4.9)\qquad \begin{aligned} T_\pm(m,\omega) &= \inf\{s\geq 0;\ M_\pm(s,\omega) \geq m-x\}; \quad \text{if} \quad m>x \\ &= \inf\{s\geq 0;\ m_\pm(s,\omega) \leq m-x\}; \quad \text{if} \quad m<x \end{aligned}$$

with the convention $\inf \emptyset = +\infty$, set

$$(4.10)\qquad T(m,\omega) = T_+(m,\omega) + T_-(m,\omega),$$

and consider the "auxiliary" path transformation $\psi_m: \Omega \to D$ given by

$$(4.11)\qquad \begin{aligned} \psi_m(\omega)(s) &= x + B_+(s,\omega) &&;\ 0\leq s<T_+(m,\omega) \\ &= x + B_-(T(m,\omega)-s,\omega); &&\ T_+(m,\omega)\leq s<T(m,\omega) \\ &= \omega(s) &&;\ s\geq T(m,\omega). \end{aligned}$$

The measurability of this mapping is discussed in section 7; in particular, it is shown:

PROPOSITION 2: The mapping $\psi_m: \Omega \to D$ is $\mathfrak{F}/\mathcal{G}$ - measurable. □

Under Wiener measure P_x; $x\geq 0$, the processes L, $L_\pm$ and $B_\pm$ are almost surely continuous, and $T_\pm(m)$ are then passage times to level $m-x$ for the two independent Brownian motions $B_\pm$. Besides, with $x = 0$ and $m>0$, we note that

$$L_\pm(\tau) = M_\pm(\tau)\ ;\quad \forall\ \tau\geq 0$$

$$T_\pm(m) = L_\pm^{-1}(m)$$

$$T(m) = L_+^{-1}(m)+L_-^{-1}(m) = \Gamma_+(L^{-1}(m))+\Gamma_-(L^{-1}(m)) = L^{-1}(m)$$

hold P_0 - almost surely. It follows then from (4.8), (4.11) that

(4.12) $$\psi_m(\omega) = \varphi_{L^{-1}(m,\omega)}(\omega), \quad \text{for } P_0 \text{ - a.e. } \omega\in\Omega.$$

Remark: For fixed $m>0$ we have $P_0^{m;\tau}[L^{-1}(m,\omega) = \tau] = 1$, for every $\tau\in G(m)$, where the set $G(m) \subseteq (0, \infty)$ satisfies $P_0[L^{-1}(m)\in G(m)]=1$. Now $\{L^{-1}(m,\omega) = \tau\} \subseteq \{L(\tau,\omega) = m\}$, so Theorem 3 implies

$$P_0^{m;\tau}[L^{-1}(m,\omega) = \tau] = B_0^{\tau;m}[L(\tau,\omega) = m] = 1; \quad \tau\in G(m).$$

Thus, for $\tau\in G(m)$:

$$\Gamma_\pm(\tau,\omega) = L_\pm^{-1}(m,\omega) = T_\pm(m,\omega)$$

hold for $B_0^{\tau;m}$ and $P_0^{m;\tau}$ - almost every $\omega\in\Omega$. On the other hand,

$$\omega(s) = 0, \ \forall \ s\geq\tau; \quad \text{for } B_0^{\tau;m}, \ P_0^{m;\tau} \text{ - a.e. } \omega\in\Omega,$$

and thus

(4.13) $$\psi_m(\omega) = \varphi_\tau(\omega); \quad \text{for } B_0^{\tau;m}, P_0^{m;\tau} \text{ - a.e. } \omega\in\Omega, \ \tau\in G(m).$$

□

In terms of the path transformation ψ_m in (4.11), one can provide a concise formulation of a path decomposition result due to D. Williams (1974; Theorem 3.5):

THEOREM 5: D. Williams (1974)

Welding back-to-back the pieces of two independent Brownian motions started at $x>0$ and run until their respective first passage times at level $m(0<m<x)$ yields the piece of a Bessel process started at x and run until its last entrance time to this level, conditioned by $\{m(\infty) = m\}$. More precisely, for any $\Lambda\in\mathfrak{F}$ and a.e. $m\in(0,x)$:

$$(4.14)\qquad \hat{Q}_x^m[\{\omega(t\wedge\sigma_x);\ t\geq 0\}\in\Lambda] = P_x[\{\psi_m(\omega)(t\wedge T(m));\ t\geq 0\}\in\Lambda].$$

5: RESULTS

We are now in a position to put the various results together.

THEOREM 6: Welding back-to-back the pieces of two independent Brownian motions started at the origin and run until their respective passage times at level $m>0$, and conditioning on the sum of these two independent passage times being equal to $\tau>0$, yields a Brownian Bridge on $[0,\tau]$ conditioned to achieve a maximum m. Specifically, for any $t>0$, $A\in\mathfrak{F}_t$ and for a.e. $m>0$, we have

$$(5.1)\qquad P_0[\psi_m^{-1}A\,|\,L^{-1}(m) = \tau] = \check{B}_0^{\tau;m}[A];\quad \text{for a.e. } \tau>t.$$

Proof: First, let us notice that $\psi_m^{-1}A\in\mathfrak{F}$, according to Pro-prosition 2.

From Theorem 5, we have for a.e. $m>0$, with $x = 2m$:
$P_{2m}[\psi_m^{-1}A|T(m) = \tau] = \hat{Q}_{2m}^{m}[A|\sigma_{2m} = \tau]$; for a.e. $\tau>t$.

Theorem 4, on the other hand, provides a characterization of the right-hand side in the above identity, in terms of a "conditional" Brownian Bridge:

$$\hat{Q}_{2m}^{m}[A|\sigma_{2m} = \tau] = \hat{B}_{2m}^{\tau;m}[A]; \quad \text{for a.e.} \quad \tau>t.$$

It follows that

$$P_{2m}[\psi_m^{-1}A|T(m) = \tau] = \hat{B}_{2m}^{\tau;m}[A]; \quad \text{for a.e.} \quad \tau>t,$$

and by symmetry:

$$P_0[\psi_m^{-1}A|T(m)=\tau]= \check{B}_0^{\tau;m}[A]; \quad \text{for a.e.} \quad \tau>t.$$

But under P_0, $T(m) = L^{-1}(m)$ holds almost surely, and this yields (5.1).

□

<u>THEOREM 7</u>: <u>Brownian Bridge Decomposition</u>.

For any fixed $\tau>0$ and every $A\in\mathfrak{F}_\tau$, we have

$$(5.2) \qquad \mathcal{B}_0^s[A] = B_0^s[\varphi_s^{-1} A]; \quad \text{for a.e.} \quad s>\tau.$$

<u>Proof</u>: We start by noticing that $\varphi_s^{-1}A\in\mathfrak{F}_s$, for every $s\geq\tau$, by virtue of Proposition 1. Now from Theorem 3 we have, for every $m>0$:

$$P_0[\psi_m^{-1}A|L^{-1}(m) = s] = P_0^{m;s}[\psi_m^{-1}A] = B_0^{s;m}[\psi_m^{-1}A],$$

for a.e. $s>\tau$. Comparing with (5.1), Theorem 6, we obtain

$$\check{B}_0^{s;m}[A] = B_0^{s;m}[\psi_m^{-1}A]; \quad \forall A\in\mathfrak{F}_\tau$$

for a.e. $s>\tau$ and a.e. $m>0$. But, under $B_0^{s;m}$, we have $\psi_m(\omega) = \varphi_s(\omega)$, for $s\in G(m)$ and a.e. $\omega\in\Omega$ (relation (4.13)). Therefore, for every $A\in\mathfrak{F}_\tau$, we have for a.e. $m>0$:

$$\check{B}_0^{s;m}[A] = B_0^{s;m}[\varphi_s^{-1}A]; \quad \text{for a.e. } s\in G(m)\cap(\tau,\infty).$$

and thus by definition

$$B_0^s[A|M(s) = m] = B_0^s[\varphi_s^{-1}A|L(s) = m]; \quad \text{for a.e. } m>0.$$

Integrating both sides over $(0,\infty)$ with respect to the common density (3.16), we obtain (5.2).

□

We discuss now the relation between the pre- and post - γ_t processes, under P_0. To that effect, we choose $0 = t_0<t_1<\dots<t_n\le t$ and $x_0 = 0$, $(x_1,\dots,x_n)\in\mathbb{R}^n$. Let $t_k<\tau<t_{k+1}$, and compute

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n;\ \gamma_t\in d\tau] = \frac{\partial}{\partial\tau} P_0[\omega(t_i)\in dx_i:\ 1\le i\le n;\ \gamma_t\le\tau]d\tau$$

$$= \frac{\partial}{\partial\tau}\int_{z=-\infty}^{\infty} P_0[\omega(t_i)\in dx_i:\ 1\le i\le k]\ .\ P_{x_k}[\omega(\tau-t_k)\in dz]\ .$$

$$(5.3) \qquad .\ P_z[\omega(t_i-\tau)\in dx_i:\ k+1\le i\le n;\ T_0>t-\tau]d\tau$$

$$= \prod_{i=0}^{k-1} p(t_{i+1}-t_i;\ x_i,x_{i+1})\cdot \prod_{i=k+1}^{n-1} q(t_{i+1}-t_i;\ x_i,x_i+1).$$

$$\cdot \frac{\partial}{\partial\tau} \; I(t_k,\tau,t_{k+1};\; x_k,x_{k+1})dx_1\ldots dx_n\; d\tau,$$

where:

$$I(s,\tau,t;\; x,y) \triangleq \int_{-\infty}^{\infty} p(\tau-s;\; x,z)\;\; q(t-\tau;\; z,y)dz\;.$$

Computations analogous to those employed in §6.a (Appendix) yield

$$\frac{\partial}{\partial\tau} I(s,\tau,t;\; x,y) = \frac{|y|}{t-\tau}\, p(\tau-s;\; x,0)\;\; p(t-\tau;\; 0,y), \tag{5.4}$$

and substituting back into (5.3) we obtain

$$\begin{aligned} P_0[\omega(t_i)\in dx_i:\; 1\le i\le n;\; \gamma_t\in d\tau] &= \prod_{i=0}^{k-1} p(t_{i+1}-t_i;\; x_i,x_{i+1})\cdot \\ &\cdot \prod_{i=k+1}^{n-1} q(t_{i+1}-t_i;\; x_i,x_{i+1})\cdot p(\tau-t_k;x_k,0)\,\frac{|x_{k+1}|}{t_{k+1}-\tau} \\ &\qquad p(t_{k+1}-\tau;\; 0,x_{k+1})dx_1\ldots dx_n\; d\tau. \end{aligned} \tag{5.5}$$

In particular,

$$\begin{aligned} P_0[\omega(t_i)\in dx_i:\; 1\le i\le k;\; \gamma_t\in d\tau] &= \\ &= \int_{-\infty}^{\infty} P_0[\omega(t_i)\in dx_i:\; 1\le i\le k;\; \omega(t)\in dz;\; \gamma_t\in d\tau] \\ &= \sqrt{\frac{2}{\pi(t-\tau)}}\; p(\tau-t_k;\; x_k,0)\prod_{i=0}^{k-1} p(t_{i+1}-t_i;\; x_i,x_{i+1})\cdot dx_1\ldots dx_k\; d\tau. \end{aligned} \tag{5.6}$$

From (5.5), (5.6) we obtain now, for a.e. $\tau\in(t_k,t_{k+1})$ and a.e. $(x_1,\ldots,x_k)\in\mathbf{R}^k$:

$$P_0[\omega(t_i)\in dx_i:\ k+1\le i\le n \mid \omega(t_i) = x_i:\ 1\le i\le k;\ \gamma_t = \tau] =$$

(5.7)

$$= \frac{|x_{k+1}|\sqrt{2\pi(t-\tau)}}{2(t_{k+1}-\tau)}\, p(t_{k+1}-\tau;\ 0,x_{k+1})\ .$$

$$\cdot \prod_{i=k+1}^{n-1} q(t_{i+1}-t_i;\ x_i,x_{i+1})dx_{k+1}\cdots dx_n .$$

Remark: If we multiply the expression in (5.7) by $P_0[\gamma_t \in d\tau] = \frac{d\tau}{\pi\sqrt{\tau(t-\tau)}}$, we obtain Chung's formula (1976, Eq. 3.1) for the distribution of a Brownian meander. Whereas Chung computed the unconditional distribution, we have conditioned on the pre - γ_t process.

THEOREM 8: Under P_0, the processes $\{\omega(s\wedge\gamma_t);\ s\ge 0\}$ and $\{\omega(s\vee\gamma_t);\ s\ge 0\}$ are independent when conditioned on γ_t.

Proof: The conditional distribution in (5.7) does not depend on $(x_1,\ldots,x_k)$.

□

For other results of this type, in the context of general Markov Process theory, the interested reader is referred to the survey article by Millar (1977), and to the papers cited there. We have preferred to derive Theorem 8 from first principles.

Remark: Let $\mathcal{F}_{\le\gamma_t} \triangleq \sigma(\omega(s\wedge\gamma_t):\ s\ge 0)$ and $\mathcal{F}_{\ge\gamma_t} \triangleq \sigma(\omega(s\vee\gamma_t):\ s\ge 0)$. Then γ_t is both $\mathcal{F}_{\le\gamma_t}$ and $\mathcal{F}_{\ge\gamma_t}$ - measurable, and since

$$\varphi_{\gamma_t}(\omega)(u) = \varphi_{\gamma_t}(\omega(\cdot\wedge\gamma_t))(u) \quad ; \text{ for } u\le\gamma_t$$

$$= \varphi_{\gamma_t}(\omega(\cdot\vee\gamma_t))(u) = \omega(u) \quad ; \text{ for } u\ge\gamma_t,$$

we see from Proposition 1 that the process $\{\varphi_{\gamma_t}(\omega)(u\wedge\gamma_t);\ u\ge 0\}$ is $\mathfrak{F}_{\le\gamma_t}$ - measurable, while the process $\{\varphi_{\gamma_t}(\omega)(u\vee\gamma_t);\ u\ge 0\}$ is $\mathfrak{F}_{\ge\gamma_t}$ - measurable. Theorems 8 and 2 imply now that, if $0<t_1<t_2<\ldots<t_n\le t$ and $(x_1,\ldots,x_n)\in\mathbb{R}^n$, then for almost every $\tau\in(t_k,t_{k+1})$ we have

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n\,|\,\gamma_t = \tau] =$$

$$(5.8)\qquad = P_0[\omega(t_i)\in dx_i:\ 1\le i\le k\,|\,\gamma_t = \tau].\ P_0[\omega(t_i)\in dx_i:\ k+1\le i\le n\,|\,\gamma_t=\tau]$$

$$= B_0^{\tau}[\omega(t_i)\in dx_i:\ 1\le i\le k].\ P_0[\omega(t_i)\in dx_i:\ k+1\le i\le n\,|\,\gamma_t = \tau]$$

and

$$P_0[\varphi_{\gamma_t}(\omega)(t_i)\in dx_i:\ 1\le i\le n\,|\,\gamma_t = \tau] =$$

$$(5.9)\qquad = P_0[\varphi_{\gamma_t}(\omega)(t_i)\in dx_i:\ 1\le i\le k\,|\,\gamma_t=\tau].\ P_0[\omega(t_i)\in dx_i:\ k+1\le i\le n\,|\,\gamma_t=\tau]$$

$$= B_0^{\tau}[\varphi_{\gamma_t}(\omega)(t_i)\in dx_i:\ 1\le i\le k].\ P_0[\omega(t_i)\in dx_i:\ k+1\le i\le n\,|\,\gamma_t=\tau].$$

<u>THEOREM 1</u>: <u>Brownian Path Decomposition</u>.

For fixed $t>0$, we define γ_t by (3.3). We have:

$$(5.10)\qquad P_0[A] = P_0[\varphi_{\gamma_t}^{-1}A];\quad \forall\ A\in\mathfrak{F}_t.$$

Proof: Theorem 7 implies that the last expression in (5.8) agrees with the last expression in (5.9), for almost every $\tau\epsilon(t_k, t_{k+1})$. Integrating with respect to $P_0[\gamma_t \epsilon d\tau]$, and summing up over k, we obtain the desired result:

$$P_0[\omega(t_i)\epsilon dx_i;\ 1\leq i\leq n] = P_0[\varphi_{\gamma_t}(\omega)(t_i)\epsilon dx_i;\ 1\leq i\leq n].$$

□

6. APPENDIX

6.a: Derivation of the quadrivariate density (1.6).

With $x<0$, we compute:

$$P_0[\omega(t)\in da;\ M(t)\in db;\ \gamma_t\leq s;\ \omega(s)\in dx;\ \theta(t)\in d\theta] =$$

$$= P_0[\omega(s)\in dx;\ M(s)\in db;\ \theta(s)\in d\theta].\ P_x[\omega(t-s)\in da;\ T_0>t-s]$$

$$= \frac{b(b-x)}{\pi\theta^{3/2}(s-\theta)^{3/2}}\, e^{-\frac{b^2}{2\theta}-\frac{(b-x)^2}{2(s-\theta)}}\, db\,d\theta\,dx\ .$$

$$\cdot \frac{da}{\sqrt{2\pi(t-s)}}\left[e^{-\frac{(x-a)^2}{2(t-s)}} - e^{-\frac{(x+a)^2}{2(t-s)}}\right]$$

by virtue of (1.5) and the reflection principle. Integrating out x, we obtain

$$P_0[\omega(t)\in da;\ M(t)\in db;\ \gamma_t\leq s;\ \theta(t)\in d\theta =$$

$$= \frac{2be^{-\frac{b^2}{2\theta}}\ .\ db\,d\theta\,da}{(2\pi)^{3/2}\theta^{3/2}(s-\theta)^{3/2}(t-s)^{1/2}}\ .$$

(6.1)

$$\cdot \int_0^\infty (b+x)e^{-\frac{(b+x)^2}{2(s-\theta)}}\left[e^{-\frac{(x+a)^2}{2(t-s)}} - e^{-\frac{(x-a)^2}{2(t-s)}}\right]dx.$$

A bit of algebra gives

(6.2) $$\frac{(b+x)^2}{s-\theta} + \frac{(x\pm a)^2}{t-s} = \frac{(x+\mu_\pm)^2}{\sigma^2} + \frac{(b\mp a)^2}{t-s}\ ,$$

where: $\mu_{\pm} = \dfrac{b(t-s) \pm a(s-\theta)}{t-\theta}$, $\sigma^2 = \dfrac{(t-s)(s-\theta)}{t-\theta}$.

The integral in (6.1) is thus written as

$$e^{-\frac{(b-a)^2}{2(t-\theta)}} \int_0^\infty (b+x) e^{-\frac{(x+\mu_+)^2}{2\sigma^2}} dx - e^{-\frac{(b+a)^2}{2(t-\theta)}} \int_0^\infty (b+x) e^{-\frac{(x+\mu_-)^2}{2\sigma^2}} dx,$$

and these integrals are easily evaluated:

$$\int_0^\infty (b+x) e^{-\frac{(x+\mu_\pm)^2}{2\sigma^2}} dx = \sigma^2 e^{-\frac{\mu_\pm^2}{2\sigma^2}} + \frac{s-\theta}{t-\theta}(b\mp a)\sigma\sqrt{2\pi}\cdot\left[1-\Phi\left(\frac{1}{\sigma}\mu_\pm\right)\right]. \tag{6.3}$$

Finally, the observation

$$\frac{\mu_\pm^2}{\sigma^2} + \frac{(b\mp a)^2}{t-\theta} = \frac{a^2}{t-s} + \frac{b^2}{s-\theta} \tag{6.4}$$

yields, when the computations (6.3) are substituted back into (6.1):

$$P_0[\omega(t)\in da;\ M(t)\in db;\ \gamma_t \leq s;\ \theta(t)\in d\theta] = \frac{b e^{-\frac{b^2}{2\theta}}\, da\, db\, d\theta}{\pi\theta^{3/2}(t-\theta)^{3/2}} \cdot \tag{6.5}$$
$$\left\{(b-a)e^{-\frac{(b-a)^2}{2(t-\theta)}}\,\Phi\left(-\frac{1}{\sigma}\mu_+\right) - (b+a)e^{-\frac{(b+a)^2}{2(t-\theta)}}\,\Phi\left(-\frac{1}{\sigma}\mu_-\right)\right\}.$$

Differentiation with respect to s yields the desired formula (1.6) for the quadrivariate density, after a bit of algebra.

It is helpful to note, in carrying out this programme, the observation (6.4) and the computation

$$\frac{\partial}{\partial s}\left(\frac{1}{\sigma}\mu_{\pm}\right) = \frac{-(t-\theta)^{1/2}}{2(t-s)^{1/2}(s-\theta)^{1/2}}\left[\frac{b}{s-\theta} \mp \frac{a}{t-s}\right] .$$

□

6.b: Proof of Theorem 2.

With $0<t_1<t_2<\ldots<t_n = s<\tau<t$, and real numbers $x_1, x_2, \ldots, x_n = x, a$, we have

$$P_0[\omega(t_i)\in dx_i : 1\le i\le n;\ \gamma_t>\tau;\ \omega(\tau)\in da] =$$

$$= P_0[\omega(t_i)\in dx_i : 1\le i\le n].\ P_x[\omega(\tau-s)\in da].\ P_a[T_0<t-\tau].$$

Therefore, integrating out a over $(-\infty,\infty)$, we obtain

$$\frac{P_0[\omega(t_i)\in dx_i : 1\le i\le n;\ \gamma_t>\tau]}{P_0[\omega(t_i)\in dx_i : 1\le i\le n]} =$$

$$= \frac{2}{\sqrt{2\pi(\tau-s)}}\int_{-\infty}^{\infty} e^{-\frac{(x-a)^2}{2(\tau-s)}}\left[1 - \Phi\left(\frac{|a|}{\sqrt{t-\tau}}\right)\right]da.$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_{-\infty}^{\frac{x}{\sqrt{\tau-s}}} e^{-\frac{y^2}{2}}\left\{1-\Phi\left(\frac{x-y(\tau-s)^{1/2}}{(t-\tau)^{1/2}}\right)\right\} dy +\right.$$

$$+ \int_{\frac{x}{\sqrt{\tau-s}}}^{\infty} e^{-\frac{y^2}{2}} \Phi\left(\frac{x-y(\tau-s)^{1/2}}{(t-\tau)^{1/2}}\right) dy\Big] .$$

We differentiate now with respect to τ, and observe that

$$\frac{d}{d\tau}\left[\frac{x-y\sqrt{\tau-s}}{\sqrt{t-\tau}}\right] = - \frac{y-\mu}{2\sigma^2} \frac{1}{\sqrt{(t-\tau)(\tau-s)}} \text{, where } \mu \triangleq x\left(\frac{\tau-s}{t-s}\right)^{1/2},$$

$\sigma^2 \triangleq \frac{t-\tau}{t-s}$. We obtain, using the identity

$$y^2 + \frac{(x-y\sqrt{\tau-s})^2}{t-\tau} = \frac{(y-\mu)^2}{\sigma^2} + \frac{x^2}{t-s} :$$

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n;\ \gamma_t \in d\tau] =$$

$$= \frac{d\tau}{\pi\sqrt{(t-\tau)(\tau-s)}} e^{-\frac{x^2}{2(\tau-s)}} . P_0[\omega(t_i)\in dx_i:\ 1\le i\le n] .$$

Finally, the arc-sine law: $P_0[\gamma_t \in d\tau] = \frac{d\tau}{\pi\sqrt{\tau(t-\tau)}}$ is employed

to show that, for a.e. $\tau\in(s,t)$:

$$P_0[\omega(t_i)\in dx_i:\ 1\le i\le n \mid \gamma_t = \tau] =$$

$$= \sqrt{\frac{\tau}{\tau-s}}\, e^{-\frac{x^2}{2(\tau-s)}} . P_0[\omega(t_i)\in dx_i:\ 1\le i\le n]$$

$$= B_0^{\tau}[\omega(t_i)\in dx_i:\ 1\le i\le n],$$

by virtue of (3.1). Relation (3.5) is established. □

7. APPENDIX

We devote this section to certain questions of measurability. Let us begin by considering a continuous function $\rho: \mathbb{R} \to [0,\infty)$, with support in $[0,1]$ and with $\int \rho(s)ds = 1$. We define the mapping $\theta: D \to \Omega$ by

$$\theta(\omega)(t) \triangleq \int_0^\infty \omega(t+s)\ \rho(s)ds = \int_0^\infty \omega(s)\ \rho(s-t)ds; \quad t\geq 0.$$

It is not hard to verify that θ is actually a <u>continuous</u> mapping from D, equipped with the topology induced by the Skorohod metric, into Ω, equipped with the topology of uniform convergence on compact subsets of $\mathbb{R}^+$.

<u>Lemma 1</u>: The evaluation mapping $e: \mathbb{R}^+ xD \to \mathbb{R}$ defined by $e(t,\omega) = \omega(t);\ t\geq 0,\ \omega\in D,$ is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable.

<u>Proof:</u> For each positive integer n, define the continuous mapping $\theta_n: D \to \Omega$ by

$$\theta_n(\omega)(t) \triangleq \int_0^\infty \omega(t+s)\ \rho_n(s)ds; \quad t\geq 0,$$

where $\rho_n: \mathbb{R} \to [0,\infty)$ is continuous, with support on $[0,\frac{1}{n}]$ and with $\int \rho_n(s)ds = 1$.

Consider also the mapping $\hat{e}: \mathbb{R}^+ \times \Omega \to \mathbb{R}$, defined by $\hat{e}(t,\omega) = \omega(t)$. It is easily checked that $\hat{e}$ is continuous, and so the mapping $e_n(t,\omega) \triangleq \theta_n(\omega)(t)$ from $\mathbb{R}^+ \times D$ into $\mathbb{R}$, being the composition of $(t,\omega) \longmapsto (t,\theta_n(\omega))$ and $\hat{e}$, is also continuous. Since we have:

$$e(t,\omega) = \lim_{n\to\infty} e_n(t,\omega); \quad \forall\ t\geq 0, \quad \omega\in D,$$

e is measurable.

□

The following result is also easy to verify:

Lemma 2: Let $f: \mathbb{R} \to \mathbb{R}$ be a bounded and Borel-measurable function, and define $F: \mathbb{R}^+ \times D \to \mathbb{R}$ by

$$F(t,\omega) \triangleq \int_0^t f(\omega(s))ds; \quad t\geq 0, \quad \omega\in D.$$

Then F is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable.

□

Lemma 3: Let $\varphi: D \to D$ be $\mathcal{G}/\mathcal{G}$ - measurable and assume that for some fixed $t_1>0$, $t_2>0$, $\varphi(\omega)$ and $\varphi(\omega')$ restricted to $[0,t_2)$ agree, whenever ω and ω' restricted to $[0,t_1)$ agree. Then φ is $\mathcal{G}_{t_1}/\mathcal{G}_{t_2}$ - measurable.

Proof: We write $D = D_{[0,t_1)} \times D_{[t_1,\infty)}$, so that $\omega\in D$ has the form $\omega = (\omega_1,\omega_2)$ with $\omega_1\in D_{[0,t_1)}$, $\omega_2\in D_{[t_1,\infty)}$. For any fixed $t\in[0,t_2)$ and $B\in\mathcal{B}$, we know that the set $E \triangleq \{(\omega_1,\omega_2) \mid \varphi(\omega_1,\omega_2)(t)\in B\}$ is a member of $\mathcal{G}_{[0,t_1)} \otimes \mathcal{G}_{[t_1,\infty)}$, and we must show that E is of the form $A \cap D_{[t_1,\infty)}$, where $A\in\mathcal{G}_{[0,t_1)}$. By assumption, however, $\varphi(\omega_1,\omega_2)(t) \in B$ if and only if $\varphi(\omega_1,\omega_2')(t)\in B$, for all

$\omega_2' \in D_{[t_1,\infty)}$. It follows that $E = A \cap D_{[t_1,\infty)}$ where, with arbitrary but fixed $\bar{\omega}_2 \in D_{[t_1,\infty)}$:

$$A = \{\omega_1 \in D_{[0,t_1)} \mid (\omega_1,\bar{\omega}_2) \in E\} \in \mathcal{G}_{[0,t_1)} . \qquad \square$$

We are now in a position to discuss the measurability properties of the various mappings in section 4. The joint measurability of $\Gamma_\pm(t,\omega)$ on $\mathbb{R}^+ \times D$ follows directly from Lemma 2. In order to prove that the mappings $\Gamma_\pm^{-1}(\tau,\omega)$: $\mathbb{R}^+ \times D \to \mathbb{R}$ in (4.3) are $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable, fix $t>0$, observe that $\Gamma_\pm(t,.)$ is $\mathcal{G}/\mathcal{B}$ - measurable, and argue that

$$\{(\tau,\omega) \mid \Gamma_\pm^{-1}(\tau,\omega)<t\} = \{(\tau,\omega) \mid \Gamma_\pm(t,\omega)-\tau>0\} \in \mathcal{B}^+ \otimes \mathcal{G}.$$

In order to establish the measurability of local time in (3.10), we appeal again to Lemma 2 for the measurability of

$$K_\varepsilon(t,\omega) \triangleq \frac{1}{4\varepsilon} \int_0^\infty 1_{[-\varepsilon,\varepsilon]}(\omega(s))ds; \quad t\geq 0, \ \omega \in D,$$

for every $\varepsilon>0$. For fixed (t,ω), $K_\varepsilon(t,\omega)$ is right-continuous in ε, and so

$$K_0(t,\omega) \triangleq \overline{\lim_{\varepsilon\downarrow 0}} K_\varepsilon(t,\omega) = \lim_{n\to\infty} \sup_{\substack{0<\delta<\frac{1}{n} \\ \delta \text{ rational}}} K_\delta(t,\omega).$$

It follows that K_0: $\mathbb{R}^+ \times D \to \mathbb{R}$ is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable, and non-decreasing in t; in order to select a right-continuous version as in (3.10), we recall the sequence of functions $\{\rho_n\}_{n=1}^{\infty}$ used in the Proof of Lemma 1, and set

$$L(t,\omega) = \lim_{n\to\infty} \int_0^{\infty} K_0(t+s,\omega)\, \rho_n(s)ds.$$

Then $L(t,\omega) = K_0(t+,\omega)$, and Fubini's theorem implies that $L: \mathbb{R}^+ \times D \to \mathbb{R}$ is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable.

By composition, one can show the $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurability of the mappings in (4.4) - (4.6).

<u>PROOF OF PROPOSITION 1</u>: Because Ω is a measurable subset of D, it suffices to show that

(i) for any fixed $t>0$, the mapping $\varphi_t: D \to D$ is $\mathcal{G}_t/\mathcal{G}_t$ - measurable, and that

(ii) the mapping $\varphi: \mathbb{R}^+ \times D \to D$ is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{G}$ - measurable.

From the preceding discussion it follows that, as a function of (t,s,ω) into $\mathbb{R}$, $\varphi_t(\omega)(s)$ in (4.8) is $\mathcal{B}^+ \otimes \mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable. As a function of (t,ω), φ maps $\mathbb{R}^+ \times D$ into D, and since for each fixed $s\geq 0$, $B\in\mathcal{B}$:

$$\{(t,\omega)\,|\,\varphi_t(\omega)(s)\in B\}\in \mathcal{B}^+ \otimes \mathcal{G};$$

we see that the function $(t,\omega) \longmapsto \varphi_t(\omega)(\cdot)$ is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{G}$ - measurable. For fixed $t>0$, the function $\omega \longmapsto \varphi_t(\omega)(\cdot)$ from D into D is thus $\mathcal{G}/\mathcal{G}$ - measurable, and since $\varphi_t(\omega)$ restricted to $[0,t)$ agrees with $\varphi_t(\omega')$ restricted to $[0,t)$

whenever ω and ω' restricted to $[0,t)$ agree, we see from Lemma 3 that the mapping $\omega \longmapsto \varphi_t(\omega)$ is $\mathcal{G}_t/\mathcal{G}_t$ - measurable.

□

In order to discuss the measurability of $M_\pm$ in (4.7), observe that

$$M^0_\pm(t,\omega) \triangleq \sup_{\substack{q\in\mathbb{R}^+ \\ q \text{ rational}}} 1_{(q,\infty)}(t)B_\pm(q,\omega); \quad t\geq 0, \ \omega\in D$$

is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable, and that it is the left-continuous version of $M_\pm$: $M^0_\pm(t,\omega) = \sup_{0\leq s<t} B_\pm(s,\omega)$.

Therefore,

$$M_\pm(t,\omega) = \lim_{n\to\infty} \int_0^\infty M^0_\pm(t+s,\omega)\rho_n(s)ds; \quad t\geq 0, \ \omega\in D$$

where again $\{\rho_n\}_{n=1}^\infty$ is the sequence of functions used in the proof of Lemma 1. It follows that $M_\pm$ is $\mathcal{B}^+ \otimes \mathcal{G}/\mathcal{B}$ - measurable. A similar argument establishes the measurability of $m_\pm$, and it follows that the mapping $T_\pm(m,.)$: $D \to \mathbb{R}^+$ in (4.9) is $\mathcal{G}/\mathcal{B}^+$ - measurable.

<u>PROOF OF PROPOSITION 2</u>: Again, it suffices to show that, for fixed $m>0$, $x\geq 0$, $x\neq m$, the mapping ψ_m: $D \to D$ defined in (4.11) is $\mathcal{G}/\mathcal{G}$ - measurable.

Since all functions involved are measurable, the mapping $(s,\omega) \longmapsto \psi_m(\omega)(s)$ is $\mathcal{B}^+ \oplus \mathcal{G}/\mathcal{B}$ - measurable. Consequently, for fixed $s\in\mathbb{R}^+$, the mapping $\omega \longmapsto \psi_m(\omega)(s)$ is $\mathcal{G}/\mathcal{B}$ - measurable, and so the conclusion follows.

□

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