AN EXTENSION OF CLARK'S FORMULA

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Abstract: The representation formula of Clark (1970) and Haussmann (1979) is established for Brownian functionals in the space $\mathbf{D}_{1,1}$.

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Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a standard, $\mathcal{R}^d$ - valued Brownian motion $W(t) = (W_1(t), \ldots, W_d(t))$, $0 \le t \le T$ defined on it. We shall denote by $\{\mathcal{F}_t\}$ the P-augmentation of the natural filtration

$$\mathcal{F}_t^W = \sigma(W(s);\ 0 \le s \le t) \wedge \mathcal{N}, \quad 0 \le t \le T ,$$

where $\mathcal{N}$ is the class of P-negligible sets. Let F denote an integrable, $\mathcal{F}_T$-measurable random variable. Then by the well-known martingale representation theorem (see Karatzas and Shreve (1988), pp. 182-184), there is an $\mathcal{R}^d$-valued, progressively measurable process ψ satisfying $\int_0^T ||\psi(s)||^2\, ds < \infty$ almost surely, such that

$$(1) \qquad M(t) \doteq E[F \mid \mathcal{F}_t] = E[F] + \int_0^t \psi^*(s)\, dW(s), \qquad 0 \le t \le T .$$

Assuming that F is Fréchet differentiable, Clark (1970) gave an explicit formula for $\psi(\cdot)$ in terms of the Fréchet derivative of F. Haussmann (1979) extended this formula to the case in which F is a functional of the solution to a stochastic differential equation driven by a Brownian motion. Later it was realized that the Clark-Haussmann formulae were consequences of the adjoint relationshp between the gradient and divergence operators defined on Wiener space, and therefore, they remained true under much more general hypotheses when appropriately formulated; see, for example, Ustunel (1987), Ocone (1984), or Yan (1987). In this note we establish another extension in this direction.

We shall need to recall the definition of the gradient on Wiener space, sometimes called the *Malliavin derivative* (see Nualart & Pardoux (1988), section 2; Ikeda and Watanabe (1989), p. 360; or Ocone (1988)). Denote by $C_b^\infty(\mathcal{R}^m)$ the set of C^∞ functions $f : \mathcal{R}^m \to \mathcal{R}$ which are bounded and have bounded derivatives of all orders. Let $\mathcal{S}$ be the class of *smooth functionals*, i.e., random variables of the form

$$F(\omega) = f(W(t_1, \omega), \ldots, W(t_n, \omega)) ,$$

where $(t_1, \ldots, t_n) \in [0,T]^n$ and the function $f(x^{11}, \ldots, x^{d1}, \ldots, x^{1n}, \ldots, x^{dn})$ belongs to $C_b^\infty(\mathcal{R}^{dn})$. The gradient $DF(\omega)$ of the smooth functional F is defined as the $(L^2([0,T]))^d$

- valued random variable $DF = (D^1F, \ldots, D^dF)$ with components

$$D^iF(\omega)(t) = \sum_{j=1}^{n} \frac{\partial}{\partial x^{ij}} f(W(t_1, \omega), \ldots, W(t_n, \omega)) 1_{[0,t_j]}(t)$$

$i = 1, \ldots, d$. Finally, let $\|\cdot\|$ denote the $L^2([0,T])$ norm; $|\cdot|$ will be reserved for the Euclidean norm on $\mathcal{R}^n, n \geq 1$. For each $p \geq 1$, we introduce the norm

$$\|F\|_{p,1} \doteq (E\{|F|^p + (\sum_{i=1}^{d} \|D^iF\|^2)^{p/2}\})^{1/p}$$

on $\mathcal{S}$, and we denote by $\mathbf{D}_{p,1}$ the Banach space which is the closure of $\mathcal{S}$ under $\|\cdot\|_{p,1}$. Shigekawa (1980), Lemma 2.1 shows that DF is well-defined on $\mathbf{D}_{p,1}$ by closure for any $p \geq 1$. Given $F \in \mathbf{D}_{p,1}$, we can find a measurable process $(t, \omega) \mapsto D_tF(\omega)$ such that for a.e. $\omega \in \Omega$, $D_tF(\omega) = DF(\omega)(t)$ holds for almost all $t \in [0,T]$ (more precisely, $t \mapsto D_tF(\omega)$ is in the equivalence class in $L^2([0,T])$ defined by $DF(\omega)$). $D_tF(\omega)$ is defined uniquely up to sets of measure zero on $[0,T] \times \Omega$. More generally, for a positive integer $k > 1$, indices $j_1, \ldots, j_k$, $1 \leq j_i \leq d$ and a smooth functional F, we define

$$\begin{aligned}(D^k)^{j_1,\ldots,j_k}_{s_1,\ldots,s_k} F(\omega) &= \sum_{i_1,\ldots,i_k}^{n} \frac{\partial^k f}{\partial x^{j_1 i_1} \cdots \partial x^{j_k i_k}} (W(t_1), \ldots, W(t_n)) 1_{[0,t_{i_1}]}(s_1) \cdots 1_{[0,t_{i_k}]}(s_k) \\ &= D^{j_1}_{s_1} \cdots D^{j_k}_{s_k} F.\end{aligned}$$

Then $\mathbf{D}_{p,k}$ denotes the Banach space which is the closure of $\mathcal{S}$ under the norm $\|\cdot\|_{p,k}$ defined by

$$\|F\|_{p,k} \doteq (E\{|F|^p + (\sum_{j_1,\ldots,j_k} \|(D^k)^{j_1,\ldots,j_k} F\|^2)^{p/2}\})^{1/p},$$

where $\|(D^k)^{j_1,\ldots,j_k}\|$ denotes the $L^2([0,T]^k)$ norm of the higher-order derivative. From these Sobolev spaces we construct also the projective limit

$$\mathbf{D}_\infty \doteq \cap_{p>1,k\geq 1} \mathbf{D}_{p,k}$$

and its topological dual

$$\mathbf{D}_{-\infty} \doteq \cup_{p>1,k\geq 1} (\mathbf{D}_{p,k})',$$

where $(\mathbf{D}_{p,k})'$ denotes the dual of $\mathbf{D}_{p,k}$. These spaces are the setting for a theory of distributions over Wiener space; see the book by Ikeda and Watanabe (1989).

Let us now return to Clark's formula. If $F \in \mathbf{D}_{2,1}$ then it can be shown that

$$F = E(F) + \int_0^T E[(D_t F)^* | \mathcal{F}_t] dW(t) \; . \tag{2}$$

For a proof see, for example, Ocone (1984),(1988) or Nualart & Pardoux (1988), Appendix A. Recognizing that the esential point behind (2) is an adjoint relationship between stochastic integration and D, Ustunel (1987) showed how to extend this formula to any distribution $F \in \mathbf{D}_{-\infty}$ by defining an integral for distribution-valued processes. His results were further expounded upon and simplified by Yan (1987).

It is the purpose of this note to show that (2) is also valid for functionals $F \in \mathbf{D}_{1,1}$. We have found this extension useful in an application to optimal portfolio representation in Ocone & Karatzas (1989), because it simplifies the technical hypotheses one needs to impose.

Remark: The space $\mathbf{D}_\infty$ is naturally embedded in its dual $\mathbf{D}_{-\infty}$, by associating to each $G \in \mathbf{D}_\infty$ the distribution defined by the duality pairing

$$\langle G, F \rangle = E(GF) \qquad \forall\, F \in \mathbf{D}_\infty. \tag{3}$$

$\mathbf{D}_{1,1}$ is not contained in $\mathbf{D}_{-\infty}$, in the sense that this embedding does not extend to $\mathbf{D}_{1,1}$. To see this, let $T = 1$ and consider the simple functional $G = \phi(W_1(1))$ where $\phi(x) = (1+x^4)^{-1} e^{x^2/2}$. A calculation shows that

$$E\big[\phi(W(1)) + (\int_0^T |\phi'(W(1)) D_t W(1)|^2 \, dt)^{1/2}\big] < \infty$$

because

$$E(\int_0^T |\phi'(W(1)) D_t W(1)|^2 \, dt)^{1/2} = E|\phi(W(1))| < \infty.$$

It follows from Lemma A.1 in Ocone & Karatzas (1989) that $G \in \mathbf{D}_{1,1}$ and

$$D_t\phi(W(1)) = \phi'(W(1))1_{[0,1]}(t).$$

In order to extend the embedding of (3) to $\mathbf{D}_{1,1}$, we would need that $E(GF)$ be defined and finite for every $F \in \mathbf{D}_\infty$, and, in particular, for every polynomial function of $W_1(1)$, since all polynomial functionals of $W(\cdot)$ are in the space of test functions $\mathbf{D}_\infty$. However $(1 + W_1^6(1))G$ is not integrable, and so the extension fails.

Theorem: For every $F \in \mathbf{D}_{1,1}$ we have

$$F = E(F) + \int_0^T E[(D_tF)^*|\mathcal{F}_t]dW(t) \; . \tag{4}$$

Proof. For $F \in \mathbf{D}_{1,1}$, consider a sequence $\{F_n\}_{n=1}^\infty \subseteq \mathcal{S}$ such that $\lim_{n\to\infty} ||F_n - F||_{1,1} = 0$. The martingales $M(t) \doteq E[F|\mathcal{F}_t]$ and $M_n(t) \doteq E[F_n|\mathcal{F}_t]$ admit the respective representations

$$M(t) = E(F) + \int_0^t \psi^*(s)dW(s)$$

and

$$M_n(t) = E(F_n) + \int_0^t \psi_n^*(s)dW(s) \; ,$$

where ψ is an $\mathcal{R}^d$ - valued, progressively measurable process satisfying $\int_0^T ||\psi(s)||^2 ds < \infty$ almost surely. By (2), $\psi_n(t) = E[D_tF_n|\mathcal{F}_t]$. The submartingale inequality (Karatzas and Shreve (1988), p. 13) yields

$$P(\max_{0\le t\le T} |M_n(t) - M(t)| > \epsilon) \le \frac{1}{\epsilon}\, E|M_n(T) - M(T)| = \frac{1}{\epsilon}\, E|F_n - F| \to 0 \tag{5}$$

as $n \to \infty$, for every $\epsilon > 0$. The "good λ - inequality" of Burkholder-Gundy (cf. Rogers and Williams (1987), pp. 94-95), namely

$$P[< N >_T > 4\lambda^2, \max_{0\le t\le T} |N_t| \le \delta\lambda] \le \delta^2 P[< N >_T > \lambda^2]$$

for all $\lambda > 0, \delta \in (0,1)$, valid for local martingales N with continuous paths, leads to

$$(6) \qquad P[\int_0^T |\psi_n(t) - \psi(t)|^2 dt > 4\lambda^2] \leq \delta^2 + P[\max_{0\leq t\leq T} |M_n(t) - M(t)| > \delta\lambda]$$

for all $\lambda > 0, \delta \in (0,1)$. It follows from (5) and (6) that

$$(7) \qquad \int_0^T |\psi_n(t) - \psi(t)|^2 dt \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

However, at the same time, the Cauchy-Schwarz inequality implies

$$(8) \qquad \begin{aligned} E\int_0^T |\psi_n(s) - E[D_sF|\mathcal{F}_s]|ds &\leq E\int_0^T |D_s(F_n - F)|ds \\ &\leq T^{1/2}E[(\int_0^T |D_s(F_n - F)|^2 ds)^{1/2}] \\ &\leq T^{1/2}||F_n - F||_{1,1} \to 0 \end{aligned}$$

as $n \to \infty$. It follows from (7) and (8) that $E[D_tF|\mathcal{F}_t] = \psi(t)$, $dt \otimes dP$ - almost surely.

Remark: Notice that the Theorem implies

$$\int_0^T |E[D_sF|\mathcal{F}_s]|^2 ds < \infty \qquad \text{almost surely},$$

for every $F \in \mathbf{D}_{1,1}$. It does not seem possible to argue this fact using only that

$$E||DF|| = E\{(\int_0^T |D_sF|^2 ds)^{1/2}\} < \infty \, .$$

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