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A GENERALIZED CLARK REPRESENTATION FORMULA, WITH APPLICATION TO OPTIMAL PORTFOLIOS

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A modification of J. M. C. Clark's formula is established for the stochastic integral representation of Wiener functionals under an equivalent (Girsanov) change of probability measure. It is shown how this modified Clark formula leads to the representation of optimal portfolios for a variety of situations in the modern theory of financial economics.

KEY WORDS: Clark's formula, Wiener functional derivatives, optimal portfolios, Sobolev spaces on Wiener space.

1. INTRODUCTION

In recent years there has been considerable interest in the applications of stochastic calculus to problems of financial economics. Beginning with the work of Harrison and Pliska [8, 9] which showed that the martingale representation theorem and the Girsanov change of probability measure are the "keys" to understanding option pricing in the celebrated Black and Scholes model, these methodologies have been applied with considerable success to questions of valuation of American options (Bensoussan [2], Karatzas [13]), consumption/investment optimization (Karatzas *et al.* [15], Cox and Huang [5, 6]), equilibrium (Karatzas *et al.* [16]), and term structure of interest rates (Artzner and Delbaen [1], Heath *et al.* [11]), to name only a few. A recent survey of these developments appears in Karatzas [14].

For most stochastic optimization problems, posed in general financial market models, the above-mentioned methodologies are very successful in identifying closed-form expressions for quantities like the optimal consumption and terminal wealth levels, but are able to ascertain only the *existence* of the associated portfolio strategies. The purpose of this paper is to derive general representation formulae for the optimal portfolios associated with option pricing, maximizing utility from terminal wealth, and maximizing utility from consumption (formulae (3.10), (4.13) and (5.9), respectively. The case of utility from both consumption and terminal wealth can then be handled by superposition, as in Section 6 of Karatzas *et al.* [15].

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Instrumental in obtaining these representations is an extension of the familiar Clark formula (Clark [3], Haussmann [10], Ocone [21]):

$$F = E(F) + \int_0^T E[(D_t F)^* | \mathscr{F}_t] \, dW(t) \tag{2.8}$$

under an *equivalent change of probability meaasure*; here $W(\cdot)$ is a multidimensional Wiener process on $[0, T]$, $\mathscr{F}_t = \sigma(W(s); 0 \leqq s \leqq t)$, D is the Malliavin–Fréchet functional derivative on Wiener space, and F is an $\mathscr{F}_T$-measurable Wiener functional in the Sobolev space $\mathbf{D}_{2,1}$. This extension, which we believe to be of sufficient independent interest, is carried out in Section 2, culminating with Theorem 2.5. In particular, the representation (2.8) is first extended to Wiener functionals in the space $\mathbf{D}_{1,1}$ (Proposition 2.1), and is then re-expressed in the form

$$F = E(FZ(T)) + \int_0^T \tilde{E}\left[\left(D_t F - F \int_t^T D_t \theta(u) \, d\tilde{W}(u)\right)^* \middle| \mathscr{F}_t\right] d\tilde{W}(t) \tag{2.20}$$

where $\tilde{W}(t) \doteq W(t) + \int_0^t \theta(s) \, ds$, $0 \leqq t \leqq T$ is Brownian motion under the probability measure $\tilde{P}(A) \doteq E[Z(T) 1_A]$ on $\mathscr{F}_T$, under appropriate conditions on the random variable F and the bounded, $\{\mathscr{F}_t\}$-adapted process $\theta(\cdot)$.

In the particular contexts of option pricing, and of utility maximization from investment and/or consumption, the formula (2.20) leads directly to representations of optimal portfolios for these tasks; such developments are carried out in Sections 3, 4 and 5, respectively, and lead to the representation formulae (3.10), (4.13) and (5.9). Much like (2.20), these expressions are fairly general but also quite hard to manipulate further, as they involve functional derivatives of the Malliavin type, stochastic integrals, and conditional expectations under the auxiliary probability measure $\tilde{P}$ mentioned above. When specialized to the case of logarithmic utility functions, or to a financial market with *deterministic coefficients*, the formulae (4.20), (4.32) provide very explicit expressions for the optimal portfolios, in feedback form on the current level of wealth. This task is carried out in Section 6, and extends results of Karatzas *et al.* [15, Section 7]. It would be interesting to try to extract more useful information from these formulae in situations with random, possibly Markovian, coefficients.

A version of formula (3.12), in a more specialized context, was derived by Colwell *et al.* [4].

2. THE CLARK FORMULA UNDER AN EQUIVALENT CHANGE OF MEASURE

Consider a complete probability space $(\Omega, \mathscr{F}, P)$ and a standard, $\mathscr{R}^d$-valued Brownian motion $W(t) = (W_1(t), \ldots, W_d(t))^*$, $0 \leqq t \leqq T$ defined on it. We shall denote by $\{\mathscr{F}_t\}$ the P-augmentation of the natural filtration

$$\mathscr{F}_t^W = \sigma(W(s); 0 \leqq s \leqq t), \qquad 0 \leqq t \leqq T.$$

As is well known, $\{\mathscr{F}_t\}$ satisfies the "usual conditions" of right-continuity and completion (of $\mathscr{F}_0$) *by* P-negligible sets (Karatzas and Shreve [17, Section 2.7]).

Let now $\theta(t) = (\theta_1(t), \ldots, \theta_d(t))^*$, $0 \leqq t \leqq T$ be an $\mathscr{R}^d$-valued, bounded and $\{\mathscr{F}_t\}$-progressively measurable process, and consider the associated exponential martingale

$$Z(t) = \exp\left\{-\int_0^t \theta^*(s)\,dW(s) - \frac{1}{2}\int_0^t |\theta(s)|^2\,ds\right\}, \mathscr{F}_t; \qquad 0 \leqq t \leqq T \tag{2.1}$$

(where juxtaposition $x^*y = \sum_{i=1}^d x_i y_i$ denotes inner product in $\mathscr{R}^d$, and $|x|^2 = x^*x$). Then the process

$$\tilde{W}(t) \doteq W(t) + \int_0^t \theta(s)\,ds, \mathscr{F}_t; \qquad 0 \leqq t \leqq T \tag{2.2}$$

is a standard, $\mathscr{R}^d$-valued Brownian motion under the new probability measure

$$\tilde{P}(A) \doteq \int_A Z(T)\,dP, \qquad \forall A \in \mathscr{F}_T \tag{2.3}$$

by virtue of the Girsanov Theorem (ibid, Section 3.5). Note also that, because of the boundedness of $\theta(\cdot)$, we have

$$E(Z(t))^q < \infty; \qquad \forall q \in \mathscr{R}, \quad 0 \leqq t \leqq T. \tag{2.4}$$

We shall need to recall the definition of the Malliavin derivative (Nualart and Pardoux [20, Section 2]; see also Ikeda and Watanabe [12, p. 360] and Ocone [22]). Denote by $C_b^\infty(\mathscr{R}^m)$ the set of C^∞ functions $f: \mathscr{R}^m \to \mathscr{R}$ which are bounded and have bounded derivatives of all orders. Let $\mathscr{S}$ be the class of *smooth functionals*, i.e., random variables of the form

$$F(\omega) = f(W(t_1, \omega), \ldots, W(t_n, \omega)), \tag{2.5}$$

where $(t_1, \ldots, t_n) \in [0, T]^n$ and the function $f(x^{11}, \ldots, x^{d1}, \ldots, x^{1n}, \ldots, x^{dn})$ belongs to $C_b^\infty(\mathscr{R}^{dn})$. The gradient $DF(\omega)$ of the smooth functional F is defined as the $(L^2([0, T]))^d$-valued random variable $DF = (D^1F, \ldots, D^dF)$ with components

$$D^iF(\omega)(t) = \sum_{j=1}^n \frac{\partial}{\partial x^{ij}} f(W(t_1, \omega), \ldots, W(t_n, \omega)) 1_{[0, t_j]}(t) \qquad i = 1, \ldots, d. \tag{2.6}$$

Finally, let $\|\cdot\|$ denote the $L^2([0, T])$ norm; $|\cdot|$ will be reserved for the Euclidean norm on $\mathscr{R}^n$, $n \geqq 1$. For each $p \geqq 1$, we introduce the norm

$$\|F\|_{p,1} \doteq \left(E\left\{|F|^p + \left(\sum_{i=1}^{d} \|D^i F\|^2\right)^{p/2}\right\}\right)^{1/p} \tag{2.7}$$

on $\mathscr{S}$, and we denote by $\mathbf{D}_{p,1}$ the Banach space which is the closure of $\mathscr{S}$ under $\|\cdot\|_{p,1}$. Shigekawa [24, Lemma 2.1] shows that DF is well-defined on $\mathbf{D}_{p,1}$ by closure for any $p \geqq 1$. Given $F \in \mathbf{D}_{p,1}$, we can find a measurable process $(t, \omega) \mapsto D_t F(\omega)$ such that for a.e. $\omega \in \Omega$, $D_t F(\omega) = DF(\omega)(t)$ holds for almost all $t \in [0, T]$ (more precisely, $t \mapsto D_t F(\omega)$ is in the equivalence class in $L^2([0, T])$ defined by $DF(\omega)$). $D_t F(\omega)$ is defined uniquely up to sets of measure zero on $[0, T] \times \Omega$. (In general, if $X: \Omega \to L^2([0, T])$ is measurable, there exists a $\mathscr{B}([0, T]) \otimes \mathscr{F}$ measurable random variable, $\{\tilde{X}(t, \omega); (t, \omega) \in [0, T] \times \Omega\}$, such that $\tilde{X}(\cdot, \omega) = X(\omega)$ holds almost surely. In the remainder of the paper we shall identify $X(\omega)(t)$ with $\tilde{X}(t, \omega)$ without further comment.)

We now state an extension of Clark's representation formula for Brownian martingales. For its proof, see the article by Karatzas, Ocone and Li [18].

Proposition 2.1 *For every $F \in \mathbf{D}_{1,1}$ we have*

$$F = E(F) + \int_0^T E[(D_t F)^* | \mathscr{F}_t] \, dW(t). \tag{2.8}$$

Remark 2.2 i) From (2.8) it follows also that

$$E[F | \mathscr{F}_t] = E(F) + \int_0^t E[(D_s F)^* | \mathscr{F}_s] \, dW(s), \qquad 0 \leqq t \leqq T. \tag{2.9}$$

ii) We need the extension of Clark's formula in (2.8) from $\mathbf{D}_{2,1}$ to $\mathbf{D}_{1,1}$ in order to give ourselves extra room in Theorem 2.5. In that theorem we shall want to represent $\tilde{P}$-martingales $\tilde{E}(F | \mathscr{F}_t)$ as stochastic integrals with respect to the process $\tilde{W}$ of (2.2), by using the Bayes formula $\tilde{E}(F | \mathscr{F}_t) = (Z(t))^{-1} E[FZ(T) | \mathscr{F}_t]$ and then applying the Clark representation to $FZ(T)$. The extension of (2.8) to $\mathbf{D}_{1,1}$ is therefore useful for avoiding unnecessarily restrictive moment bounds on F and DF. For example, if $F \in L^2(\tilde{P})$, it does not follow that $FZ(T) \in L^2(P)$. However,

$$E|FZ(T)|^p = \tilde{E}(|F|^p Z(T)^{p-1}) \leqq (\tilde{E}F^2)^{p/2} (\tilde{E}Z(T)^{2(p-1)/(2-p)})^{1-(p/2)} < \infty$$

if $1 \leqq p < 2$. ∎

Let $\mathbf{L}^a_{1,1}$ denote the set of $\mathscr{R}^d$-valued progressively measurable processes $\{u(s, \omega); 0 \leqq s \leqq T, \omega \in \Omega\}$ such that

i) For a.e. $s \in [0, T]$, $u(s, \cdot) \in (\mathbf{D}_{1,1})^d$;

ii) $(s, \omega) \mapsto Du(s, \omega) \in (L^2([0, T]))^{d^2}$ admits a progressively measurable version; and

iii) $|||u|||^a_{1,1} \doteq$

$$E\left[\left(\int_0^T |u(s)|^2\,ds\right)^{1/2}+\left(\int_0^T \|Du(s)\|^2\,ds\right)^{1/2}\right]<\infty. \tag{2.10}$$

Observe that for each (s,ω), $Du(s,\omega)=[D^i u_j(s,\omega)]$ is a $d\times d$ matrix of elements in $L^2([0,T])$. Thus, in (2.10), $\|Du(s)\|^2$ should really be written as

$$\sum_{i,j=1}^{d}\|D^i u_j(s)\|^2.$$

For notational convenience, we prefer to write simply $\|Du(s)\|^2$. For $u\in\mathbf{L}^a_{1,1}$, there exist processes $D^i_t u_j(s,\omega)$, $1\leqq i,j\leqq d$ which are progressively measurable in (t,s,ω), and for which

$$D^i u_j(s)(t)=D^i_t u_j(s,\omega)\quad\text{for a.e.}\quad t\in[0,T]$$

holds for $ds\otimes dP$-a.e. (s,ω). Here, progressive measurability of a process $f(t,s,\omega)$ means that for each $r>0$, the map $(t,s,\omega)\in[0,T]\times[0,r]\times\Omega\mapsto f(t,s,\omega)$ is $\mathscr{B}([0,T])\otimes\mathscr{B}([0,r])\otimes\mathscr{F}_r$-measurable. Clearly, $D_t u(s,\omega)$ is defined uniquely up to sets of $dt\otimes ds\otimes dP$ measure zero. Conversely, if we assume that $u(s,\cdot)\in(\mathbf{D}_{1,1})^d$ for a.e. s, and that $(t,s,\omega)\mapsto D_t u(s,\omega)$ admits a progressively measurable version satisfying

$$E\left\{\left(\int_0^T |u(s)|^2\,ds\right)^{1/2}+\left(\int_0^T\int_0^T\sum_{i,j}|D^i_t u_j(s)|^2\,dt\,ds\right)^{1/2}\right\}<\infty, \tag{2.11}$$

then $u\in\mathbf{L}^a_{1,1}$ and $Du(s,\omega)(\cdot)=D_{.}u(s,\omega)$ for $ds\otimes dP$-a.e. (s,ω). Finally, notice that, because u is progressively measurable, if $u\in\mathbf{L}^a_{1,1}$, then

$$D_t u(s,\omega)=0\qquad\text{for all } t\in(s,T)$$

holds for $ds\otimes dP$-a.e. (s,ω); see Corollary A.7 in the Appendix.

In what follows we shall work with processes of the form

$$Y(t,\omega)=\int_0^T D_t u(s,\omega)\,dW(s) \tag{2.12}$$

for $u\in\mathbf{L}^a_{1,1}$. $Y(t,\omega)$ is a parametrized stochastic integral, and it will be necessary to have a version of it measurable in (t,ω). Rather than proving the existence of a measurable version by appealing to some general theorem, we give $Y(t,\omega)$ the following precise interpretation. Let $X:\Omega\to(L^2([0,T]))^d$ be the random vector

$$X(\omega)=\int_0^T (Du(s))\,dW(s). \tag{2.13}$$

This stochastic integral is well defined because, from (2.10), $P(\int_0^T \|Du(s)\|^2\, ds < \infty) = 1$. From the comments preceding Proposition 2.1, we can assume that $X(\omega)(t)$ is measurable in (t, ω). Corollary A.5 of the Appendix implies that, for almost every $t \in [0, T]$, $X(\omega)(t) = Y(t, \omega)$ holds almost surely. *We shall identify* $Y(t, \omega)$ *with* $X(\omega)(t)$, because $X(\omega)(t)$ supplies the desired measurable version.

In Proposition A.6 of the Appendix we establish the following result.

PROPOSITION 2.3 *Let* $u \in \mathbf{L}^a_{1,1}$. *Then* $\int_0^T u^*(s)\, dW(s) \in \mathbf{D}_{1,1}$, *and*

$$D_t \int_0^T u^*(s)\, dW(s) = \int_0^T D_t u(s)\, dW(s) + u(t). \tag{2.14}$$

Remark 2.4 i) The Burkholder–Davis–Gundy inequalities imply that there is a constant c, independent of the process u, such that

$$E\left\{\max_{0 \leq t \leq T} \left| \int_0^t u(s)\, dW(s) \right| \right\} \leq cE\left\{ \left(\int_0^T |u(s)|^2\, ds \right)^{1/2} \right\} \tag{2.15}$$

and

$$E\left\{\max_{0 \leq t \leq T} \left\| \int_0^t Du(s)\, dW(s) \right\| \right\} \leq cE\left\{ \left(\int_0^T \|Du(s)\|^2\, ds \right)^{1/2} \right\}. \tag{2.16}$$

These inequalities are not generally stated in the literature for the vector-valued case. However, suppose that K is a separable Hilbert space and $f: [0, T] \times \Omega \to K^d$ is a measurable process such that $P(\int_0^T \sum_1^d \|f_i(s)\|_K^2\, ds < \infty) = 1$. Then the processes $\int_0^t f(s)\, dW(s)$, $\|\int_0^t f(s)\, dW(s)\|^2 - \int_0^t \sum \|f_i(s)\|_K^2\, ds$ are continuous local martingales. This fact is enough to derive the Burkholder–Davis–Gundy inequality

$$E\left\{\max_{0 \leq t \leq T} \left\| \int_0^t f(s)\, dW(s) \right\|_K \right\} \leq cE\left\{ \left(\int_0^T \sum_{i=1}^d \|f_i(s)\|_K^2\, ds \right)^{1/2} \right\}$$

by use of the "good λ-inequality"; see Rogers and Williams [23, pp. 94–95].

ii) From Proposition 2.3 and the inequalities (2.15), (2.16) it follows that

$$\left\| \int_0^T u(s)\, dW(s) \right\|_{1,1} \leq cE\left\{ 2\left(\int_0^T |u(s)|^2\, ds \right)^{1/2} + \left(\int_0^T \|Du(s)\|^2\, ds \right)^{1/2} \right\} < \infty$$

for $u \in \mathbf{L}^a_{1,1}$. ∎

We can present now the basic result of this section; it states conditions under which we can give a precise formula for the integrand in the representation of an $\mathscr{F}_T$-measurable random variable F as a stochastic integral with respect to $\tilde{W}$. Throughout, we shall let $\|DF\|^2$ denote $\sum_1^d \|D^i F\|^2$ for notational convenience.

THEOREM 2.5 *Let $\{\theta(t); 0 \leqq t \leqq T\}$ be a bounded process such that $\theta \in \mathbf{L}^a_{1,1}$. Consider also a random variable F, such that $F \in \mathbf{D}_{1,1}$ and*

$$E[|F|Z(T)] < \infty \tag{2.17}$$

$$E[Z(T)\|DF\|] < \infty \tag{2.18}$$

$$E\left[\,|F|Z(T)\left\|\int_0^T D\theta(s)\,dW(s) + \int_0^T D\theta(s)\cdot\theta(s)\,ds\right\|\right] < \infty. \tag{2.19}$$

Then $FZ(T) \in \mathbf{D}_{1,1}$, and we have the stochastic integral representation

$$F = E(FZ(T)) + \int_0^T \left[\tilde{E}(D_tF|\mathscr{F}_t) - \tilde{E}\left(F\int_t^T D_t\theta(u)\,d\tilde{W}(u)\middle|\mathscr{F}_t\right)\right]^* d\tilde{W}(t). \tag{2.20}$$

Proof We write $Z(T) = e^G$ where $G = -\int_0^T \theta^*(s)\,dW(s) - (1/2)\int_0^T |\theta(s)|^2\,ds$. Proposition 2.3 implies that $\int_0^T \theta^*(s)\,dW(s) \in \mathbf{D}_{1,1}$. On the other hand, $\int_0^T |\theta(s)|^2\,ds$ is also in $\mathbf{D}_{1,1}$ by the following argument: thanks to Lemma A.2 of the Appendix we may approximate θ in $\mathbf{L}^a_{1,1}$ by a uniformly bounded sequence of simple processes $\{\theta^n(\cdot)\}$. For each $s \in [0, T]$, $|\theta^n(s)|^2 \in \mathbf{D}_{1,1}$ by Lemma A.1, using the boundedness of $\theta^n(s)$. It is then simple to see that $\int_0^T |\theta^n(s)|^2\,ds \in \mathbf{D}_{1,1}$ and

$$D\int_0^T |\theta^n(s)|^2\,ds = 2\int_0^T D\theta^n(s)\cdot\theta^n(s)\,ds.$$

From the uniform bound on $\{\theta^n\}$ we obtain

$$\begin{aligned}
&E\left\|\int_0^T [D\theta^n(s)\cdot\theta^n(s) - D\theta(s)\cdot\theta(s)]\,ds\right\| \\
&\quad\leqq E\left[\int_0^T |\theta^n(s)|\,\|D\theta^n(s) - D\theta(s)\|\,ds + \int_0^T |\theta(s) - \theta^n(s)|\,\|D\theta(s)\|\,ds\right] \\
&\quad\leqq cE\left(\int_0^T \|D\theta^n(s) - D\theta(s)\|^2\,ds\right)^{1/2} \\
&\qquad + E\left[\left(\int_0^T |\theta(s) - \theta^n(s)|^2\,ds\right)^{1/2}\left(\int_0^T \|D\theta(s)\|^2\,ds\right)^{1/2}\right].
\end{aligned}$$

The first term tends to zero as $n \to \infty$, since $\||\theta^n - \theta\||^a_{1,1} \to 0$ as $n \to \infty$; the second term tends to zero along a subsequence for which $\theta^{n_k}(s, \omega) \to \theta(s, \omega)$ for

$ds \otimes dP$-almost every (s,ω), by dominated convergence. It follows that $\int_0^T |\theta(s)|^2\,ds \in \mathbf{D}_{1,1}$ and $D\int_0^T |\theta(s)|^2\,ds = 2\int_0^T D\theta(s)\cdot\theta(s)\,ds$.

We have thus proved that $G \in \mathbf{D}_{1,1}$. Lemma A.1 shows that if $E|Fe^G|$, $E[e^G\|DF\|]$, $E[|F|e^G\|DG\|]$ are finite, then $Fe^G \in \mathbf{D}_{1,1}$. But these hypotheses are precisely those stated in (2.17), (2.18), (2.19); strictly speaking, we should have imposed

$$\infty > E[|F|e^G\|DG\|] = E\left[|F|Z(T)\left\|\int_0^T D\theta(s)\,dW(s) + \theta(\cdot) + \int_0^T D\theta(s)\cdot\theta(s)\,ds\right\|\right]$$

instead of (2.19). However, since θ is bounded, this condition is implied by (2.19) and (2.17). We conclude that $FZ(T) \in \mathbf{D}_{1,1}$ and that, by Lemma A.1,

$$D_t(FZ(T)) = Z(T)\left[D_tF - F\left\{\theta(t) + \int_t^T D_t\theta(u)\,d\tilde{W}(u)\right\}\right]. \tag{2.21}$$

From the Bayes formula and Proposition 2.1, we have

$$\tilde{E}[F|\mathscr{F}_t] = \frac{E[FZ(T)|\mathscr{F}_t]}{Z(t)}$$

$$= \Lambda(t)\left[E[FZ(T)] + \int_0^t (E[D_s(FZ(T))|\mathscr{F}_s])^*\,dW(s)\right] \tag{2.22}$$

where

$$\Lambda(t) \doteq \frac{1}{Z(t)} = \exp\left\{\int_0^t \theta^*(s)\,d\tilde{W}(s) - \frac{1}{2}\int_0^t |\theta(s)|^2\,ds\right\}.$$

Because $d\Lambda(t) = \Lambda(t)\theta^*(t)\,d\tilde{W}(t)$, an application of Itô's rule yields

$$d_t(\tilde{E}[F|\mathscr{F}_t]) = \left[\frac{1}{Z(t)}E[D_t(FZ(T))|\mathscr{F}_t] + \tilde{E}[F|\mathscr{F}_t]\theta(t)\right]^* d\tilde{W}(t). \tag{2.23}$$

But from (2.21) we obtain

$$\frac{1}{Z(t)}E[D_t(FZ(T))|\mathscr{F}_t] = \frac{1}{Z(t)}E[Z(T)D_tF|\mathscr{F}_t] - \frac{1}{Z(t)}\theta(t)E[FZ(T)|\mathscr{F}_t]$$

$$-\frac{1}{Z(t)}E\left[FZ(T)\int_t^T D_t\theta(u)\,d\tilde{W}(u)\middle|\mathscr{F}_t\right]$$

$$=\tilde{E}[D_tF|\mathscr{F}_t]-\theta(t)\tilde{E}[F|\mathscr{F}_t]-\tilde{E}\left[F\int_t^T D_t\theta(u)\,d\tilde{W}(u)|\mathscr{F}_t\right].$$

By substituting back into (2.23) and integrating from 0 to T, we obtain the representation formula (2.20). ∎

For the purposes of applying Theorem 2.5 we want to specify hypotheses on F and θ separately.

COROLLARY 2.6 *Let $\{\theta(t);\ 0\leqq t\leqq T\}$ be a bounded process such that $\theta\in\mathbf{L}^a_{1,1}$ and*

$$E\left(\int_0^T\|D\theta(s)\|^2\,ds\right)^{r/2}<\infty$$

for some $r>1$. Consider a random variable F such that $F\in\mathbf{D}_{1,1}$ and $E[\|DF\|^p]<\infty$ for some $p>1$, and suppose that $F\in L^q(P)$ for some q such that $(1/q)+(1/r)<1$. Then $FZ(T)\in\mathbf{D}_{1,1}$ and the representation formula (2.20) is valid.

Proof We exploit the fact that $Z(T)\in L^v(P)$ for all $v>1$ (cf. (2.4)). It follows that $FZ(T)\in L^{q'}(P)$ for all q' with $1\leqq q'<q$. Let $s>1$ satisfy $(1/s)+(1/p)=1$. Then, from Hölder's inequality,

$$E(Z(T)\|DF\|)\leqq E^{1/p}\|DF\|^p\cdot E^{1/s}[Z^s(T)]<\infty.$$

Let q' satisfy $(1/q')+(1/r)=1$. Then by Hölder's inequality and the Burkholder–Davis–Gundy inequality,

$$E\left[Z(T)|F|\left\|\int_0^T D\theta(s)\,dW(s)+\int_0^T D\theta(s)\cdot\theta(s)\,ds\right\|\right]$$

$$\leqq K\cdot E^{1/q'}(Z(T)|F|)^{q'}\cdot E^{1/r}\left[\left\|\int_0^T D\theta(s)\,dW(s)\right\|^r+\left(\int_0^T\|D\theta(s)\|^2\,ds\right)^{r/2}\right]$$

$$\leqq K\cdot E^{1/q'}(Z(T)|F|)^{q'}\cdot E^{1/r}\left(\int_0^T\|D\theta(s)\|^2\,ds\right)^{r/2}<\infty.\quad ∎$$

Finally, we remark that under the hypotheses of either Proposition 2.5 or Corollary 2.6, we obtain from the proof of Proposition 2.5 that

$$\tilde{E}[F|\mathscr{F}_t]=E[FZ(T)]+\int_0^t\left[\tilde{E}(D_sF|\mathscr{F}_s)-\tilde{E}\left(F\int_s^T D_s\theta(u)\,d\tilde{W}(u)\middle|\mathscr{F}_s\right)\right]^*d\tilde{W}(s),\qquad(2.25)$$

for $0\leqq t\leqq T$.

3. A FINANCIAL MARKET MODEL

Consider now a financial market that consists of a *bank account*, where deposited money accrues interest at the rate $r(\cdot)$, and of d *stocks* with prices-per-share $P_i(\cdot)$, $1 \leqq i \leqq d$ governed by the stochastic equations

$$P_i(t) = P_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t) \right], \qquad 0 \leqq t \leqq T. \tag{3.1}$$

The *interest rate* $r(t)$, the vector of stock *appreciation rates* $b(t) = (b_1(t), \ldots, b_d(t))^*$, and the matrix $\sigma(t) = \{\sigma_{ij}(t)\}_{1 \leqq i, j \leqq d}$ of stock *volatilities*, will be referred to collectively as the "coefficients" of the model; these are bounded random processes, progressively measurable with respect to $\{\mathscr{F}_t\}$ (the augmentation of the natural filtration $\{\mathscr{F}_t^W\}$ generated by the Brownian motion $W = (W_1, \ldots, W_d)^*$). The following nondegeneracy condition will be assumed throughout: there exists a real number $\varepsilon > 0$ such that

$$\xi^* \sigma(t, \omega) \sigma^*(t, \omega) \xi \geqq \varepsilon \|\xi\|^2; \quad \forall\, \xi \in \mathscr{R}^d, \quad (t, \omega) \in [0, T] \times \Omega. \tag{3.2}$$

This is the Bensoussan [2] model, further expounded upon in Karatzas *et al.* [15], Karatzas [13, 14].

Let us introduce now a "small investor" (i.e., an economic agent whose decisions cannot affect the prices), and at time $t \in [0, T]$ denote by $X(t)$ his wealth, by $\pi_i(t)$ the amount he invests in the ith stock, and by $c(t) \geqq 0$ the rate at which he withdraws money for consumption. The resulting *portfolio* $\{\pi(t) = (\pi_1(t), \ldots, \pi_d(t))^*, 0 \leqq t \leqq T\}$ and *consumption* $\{c(t), 0 \leqq t \leqq T\}$ processes are assumed to be adapted to $\{\mathscr{F}_t\}$, to take values in $\mathscr{R}^d$ and $[0, \infty)$, respectively, and to satisfy the integrability constraint

$$\int_0^T \{\|\pi(t)\|^2 + c(t)\}\,dt < \infty, \qquad \text{a.s.} \tag{3.3}$$

The wealth process $X(\cdot)$ corresponding to such a pair (π, c) satisfies the equation

$$dX(t) = \sum_{i=1}^{d} \pi_i(t)\left[b_i(t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t) \right] + \left(X(t) - \sum_{i=1}^{d} \pi_i(t) \right) r(t)\,dt - c(t)\,dt,$$

or in vector form

$$\begin{aligned} dX(t) &= [r(t)X(t) - c(t)]\,dt + \pi^*(t)[(b(t) - r(t)\mathbf{1})\,dt + \sigma(t)\,dW(t)] \\ &= [r(t)X(t) - c(t)]\,dt + \pi^*(t)\sigma(t)\,d\tilde{W}(t), \qquad 0 \leqq t \leqq T. \end{aligned} \tag{3.4}$$

Here $\mathbf{1}$ is the vector in $\mathscr{R}^d$ with all entries equal to 1, and $\tilde{W}$ is the process of (2.2) with

$$\theta(t) \doteq (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}]. \tag{3.5}$$

In terms of the "discount process"

$$\beta(t) \doteq \exp\left\{-\int_0^t r(s)\,ds\right\}, \tag{3.6}$$

the solution of (3.4), corresponding to a pair (π, c) as above and to a given initial capital $x > 0$, is

$$\beta(t)X(t) = x - \int_0^t \beta(s)c(s)\,ds + \int_0^t \beta(s)\pi^*(s)\sigma(s)\,d\tilde{W}(s), \qquad 0 \leqq t \leqq T. \tag{3.7}$$

We call the pair (π, c) *admissible* for x (and write $(\pi, c) \in \mathscr{A}(x)$), if

$$X(t) \geqq 0, \qquad \forall\, 0 \leqq t \leqq T$$

holds almost surely.

3.1. *The Portfolio that Attains a Given Level of Terminal Wealth*

Consider a non-negative, $\mathscr{F}_T$-measurable random variable B with $\tilde{E}[B\beta(T)] = E[B\beta(T)Z(T)] = x$. From Proposition 4.7 in Karatzas [14], there exists a unique (up to equivalence) pair $(\pi, c) \in \mathscr{A}(x)$ with $c \equiv 0$, such that the corresponding wealth process $X(\cdot)$ of (3.7) satisfies $X(0) = x$ and $X(T) = B$ almost surely; this wealth process is given by

$$\beta(t)X(t) = \tilde{E}[B\beta(T)\,|\,\mathscr{F}_t], \qquad 0 \leqq t \leqq T. \tag{3.8}$$

In other words, the portfolio π attains the level $X(T) = B$ of terminal wealth, starting with an initial capital $X(0) = x$.

From (3.8) and (3.7) with $c = 0$, one obtains

$$\tilde{E}[B\beta(T)\,|\,\mathscr{F}_t] = E(B\beta(T)Z(T)) + \int_0^t \beta(s)\pi^*(s)\sigma(s)\,d\tilde{W}(s). \tag{3.9}$$

But now, if one imposes the assumptions of Corollary 2.6 on the process $\theta(\cdot)$ and the random variable $F \doteq B\beta(T)$, one obtains the portfolio $\pi(\cdot)$ as

$$\pi(t) = \frac{1}{\beta(t)}(\sigma^*(t))^{-1}\tilde{E}\left[D_t(B\beta(T)) - B\beta(T)\int_t^T D_t\theta(u)\,d\tilde{W}(u)\,\middle|\,\mathscr{F}_t\right] \tag{3.10}$$

for $0 \leqq t \leqq T$, by comparing (3.9) and (2.25).

In the case of deterministic coefficients $\theta(\cdot)$ and $r(\cdot)$, the only conditions of

Corollary 2.6 that are non-vacuous amount to $F \in \mathbf{D}_{1,1}$ and $E\|DF\|^p < \infty$, for some $1 < p < \infty$. Under these conditions, the portfolio $\pi(\cdot)$ of (3.10) takes the form

$$\pi(t) = e^{-\int_t^T r(s)\,ds}(\sigma^*(t))^{-1}\tilde{E}[D_t B|\mathscr{F}_t], \tag{3.11}$$

provided that $\theta(\cdot)$ and $r(\cdot)$ are deterministic.

In particular, if for a.e. $t \in [0, T]$ the right-hand side of (3.10) (or (3.11), in the case of deterministic coefficients) is in $[0, \infty)^d$ almost surely, then the portfolio that steers the initial capital $X(0) = x$ into the terminal wealth $X(T) = B$ does so *without short-selling of any stock.*

Example 3.2 (Deterministic Coefficients) Consider a random variable B of the form $B = \psi(\mathbf{P}(T))$, where $\psi:(0, \infty)^d \to [0, \infty)$ is a function of class C^1 and $\mathbf{P}(t) = (P_1(t), \ldots, P_d(t))^*$ is the vector of stock prices

$$P_i(t) = P_i(0) \cdot \exp\left\{\sum_{j=1}^d \int_0^t \sigma_{ij}(s)\,d\tilde{W}_j(s) - \frac{1}{2}\sum_{j=1}^d \int_0^t \sigma_{ij}^2(s)\,ds - \int_0^t r(s)\,ds\right\}.$$

If ψ and its partial derivatives satisfy polynomial growth conditions, then $B \in \mathbf{D}_{p,1}$ for every $p \in (1, \infty)$, and a simple computation shows that $D_t B = \sum_{i=1}^d P_i(T)(\partial\psi/\partial p_i)(\mathbf{P}(T))\sigma_i(t)$, where $\sigma_i(t)$ denotes the ith row vector of the matrix $\sigma(t)$. Then the portfolio $\pi(\cdot)$ of (3.11) becomes

$$\pi(t) = e^{-\int_t^T r(s)\,ds}(\sigma^*(t))^{-1}\sum_{i=1}^d \alpha_i(t, \mathbf{P}(t))\sigma_i(t), \tag{3.12}$$

where the function $\alpha_i(t, \mathbf{p}):[0, T] \times (0, \infty)^d \to \mathscr{R}$ is defined by

$$\alpha_i(t, \mathbf{p}) \doteq \tilde{E}\left[P_i(T)\frac{\partial\psi}{\partial p_i}(\mathbf{P}(T))\middle|\mathbf{P}(t) = \mathbf{p}\right]$$

for every $i = 1, \ldots, d$. In particular, if for every $(t, \mathbf{p}) \in [0, T] \times (0, \infty)^d$ the vector $(\sigma^*(t))^{-1}\sum_{i=1}^d \alpha_i(t, \mathbf{p})\sigma_i(t)$ is in $[0, \infty)^d$, the portfolio $\pi(\cdot)$ of (3.12) avoids any short-selling.

4. OPTIMAL PORTFOLIO FOR INVESTMENT

Let $U:(0, \infty) \to \mathscr{R}$ be a strictly increasing, strictly concave function of class C^2, with $U(0+) \doteq \lim_{c \downarrow 0} U(c) \in [-\infty, \infty)$ and $U'(\infty) \doteq \lim_{c \to \infty} U'(c) = 0$. We shall refer to such a function as a *utility function*, and denote by $I:[0, U'(0+)] \to [0, \infty]$ the inverse of its derivative $U'(\cdot)$. If $U'(0+) < \infty$, we set $I(y) \equiv 0$ for $y \geqq U'(0+)$.

In the setting of Section 3, an important question in financial economics can then be formulated as follows: *to maximize the expected utility*

$$E(U(X(T))) \tag{4.1}$$

from terminal wealth, over all pairs $(\pi, c) \in \mathscr{A}(x)$. Here, $X(\cdot) \equiv X^{x,(\pi,c)}(\cdot)$ is the wealth process corresponding to the initial capital $x > 0$ and an admissible pair $(\pi, c) \in \mathscr{A}(x)$, as in (3.7).

This problem was discussed in Karatzas *et al.* [15] (hereafter abbreviated KLS) and in the review article Karatzas [14, Section 9]. According to the theory of KLS [15, Section 5], there exists an optimal pair $(\hat{\pi}, \hat{c}) \in \mathscr{A}(x)$ for this problem, with $\hat{c} \equiv 0$ (naturally), corresponding optimal level of terminal wealth

$$\hat{X}(T) \doteq I(\mathscr{Y}(x)\zeta(T)), \tag{4.2}$$

and corresponding wealth process $\hat{X}(\cdot)$ given by

$$\begin{aligned} \beta(t)\hat{X}(t) &= \tilde{E}[\beta(T)\hat{X}(T) \mid \mathscr{F}_t] \\ &= x + \int_0^t \beta(s)\hat{\pi}^*(s)\sigma(s)\, d\tilde{W}(s), \qquad 0 \leqq t \leqq T. \end{aligned} \tag{4.3}$$

In (4.2) we use the notation

$$\zeta(t) \doteq \beta(t)Z(t) \tag{4.4}$$

(where $\beta(\cdot)$ is the process of (3.6) and $Z(\cdot)$ is the exponential martingale of (2.1) with $\theta(\cdot)$ as in (3.5)) and denote by $\mathscr{Y}(\cdot)$ the inverse of the continuous, decreasing function

$$\mathscr{X}(y) \doteq \tilde{E}[\beta(T)I(y\zeta(T))] = E[\zeta(T)I(y\zeta(T))]; \qquad 0 < y < \infty \tag{4.5}$$

which we assume maps $(0, \infty)$ into $[0, \infty)$. Furthermore, under additional technical conditions on the utility function U, it is shown in KLS that the value function

$$V(x) \doteq \inf_{(\pi,c) \in \mathscr{A}(x)} EU(X(T)) \tag{4.6}$$

of this problem can be computed as

$$V(x) = G(\mathscr{Y}(x)), \tag{4.7}$$

where

$$G(y) \doteq EU(I(y\zeta(T)); \qquad 0 < y < \infty. \tag{4.8}$$

The theory that we have just outlined describes very explicitly the optimal level of terminal wealth and the value function (in (4.2) and (4.7), respectively), but fails in general to ascertain anyting more than existence for the optimal portfolio $\hat{\pi}$.

The goal of this section is to obtain a representation for $\hat{\pi}$ (formula (4.13) below), by applying Theorem 2.5 to the random variable

$$F \doteq \beta(T) I(\mathscr{Y}(x)\zeta(T)) \tag{4.9}$$

suggested by (4.3) and (4.2). In order to do that, it will be necessary to impose additional conditions on the function $I(\cdot)$ and on the interest rate process $r(\cdot)$.

We shall use extensively the notation

$$\phi(y) \doteq y I(y), \qquad 0 < y < \infty. \tag{4.10}$$

THEOREM 4.2 (Representation of Optimal Portfolio for Investment) *Suppose that* $U'(0+) = \infty$, *so that* $I \in C^1(0, \infty)$, *and*

$$I(y) + |I'(y)| \leqq K(y^{\alpha} + y^{-\beta}), \qquad 0 < y < \infty \tag{4.11}$$

holds for some real, positive constants α, β, K. *Assume also that the bounded processes* θ *and* r *belong to* $\mathbf{L}^a_{1,1}$, *and that for some* $p > 1$ *we have*

$$E\left[\left(\int_0^T \|D\theta(s)\|^2 \, ds\right)^{p/2}\right] < \infty, \quad E\left[\left(\int_0^T \|Dr(s)\|^2 \, ds\right)^{p/2}\right] < \infty. \tag{4.12}$$

Then the optimal portfolio (4.3) admits the representation

$$\hat{\pi}(t) = -\frac{1}{\beta(t)} (\sigma^*(t))^{-1} \Big\{ \theta(t) \tilde{E}[\beta(T) \mathscr{Y}(x)\zeta(T) I'(\mathscr{Y}(x)\zeta(T)) \,|\, \mathscr{F}_t]$$

$$+ \tilde{E}\left[\beta(T) \phi'(\mathscr{Y}(x)\zeta(T)) \left\{ \int_t^T D_t r(u) \, du + \int_t^T D_t \theta(u) \, d\tilde{W}(u) \right\} \Big| \mathscr{F}_t \right] \Big\}. \tag{4.13}$$

Proof Given F as defined in (4.9), it suffices to verify the hypotheses of Corollary 2.6 and to compute DF. First, because r and hence β are bounded, assumption (4.11) yields in conjunction with (2.4):

$$E[F^q] \leqq K E(Z^{\alpha}(T) + Z^{-\beta}(T))^q < \infty, \qquad \forall\, q \in \mathscr{R}^+.$$

(Throughout, K is a generic positive constant, possibly different from equation to equation.) Therefore $F \in L^q(P)$, for all $q > 1$.

To complete the proof, we shall show that $F \in \mathbf{D}_{1,1}$ and $E\|DF\|^{p'} < \infty$ for $1 \leqq p' < p$. Let G be any positive random variable in $\mathbf{D}_{1,1}$ such that $G \in L^q(P)$ for all $q \in \mathscr{R}$, and $E[\|DG\|^r] < \infty$ for some $r > 1$. Then, using (4.11), the fact that I is decreasing, and an approximation argument similar to that in the proof of Lemma A.1, one can show that

$$I(G),\ I'(G) \in L^q(P), \qquad \text{for all } q > 1, \tag{4.14}$$

and

$$I(G)\in \mathbf{D}_{1,1},\quad DI(G)=I'(G)DG,\quad \text{and} \tag{4.15}$$

$$E\|DI(G)\|^{p'} \leqq (E(\|DG\|^{r}))^{p'/r}(E(|I'(G)|^{[p'r/(r-p')]}))^{[(r-p')/r]}<\infty, \quad \text{for } 1\leqq p'<r. \tag{4.16}$$

Let $G=\mathscr{Y}(x)\beta(T)Z(T)$. Clearly $G\in L^q(P)$ for all $q\in\mathscr{R}$, and by Lemma A.1 in the Appendix we obtain that $G\in\mathbf{D}_{1,1}$, with

$$DG=-G\left[\int_0^T Dr(s)\,ds+\int_0^T D\theta(s)\,d\tilde{W}(s)+\theta(\cdot)\right]. \tag{4.17}$$

From (4.12), Hölder's inequality, and the Burkholder–Davis–Gundy inequality, we see that $E[\|DG\|^r]<\infty$, for any $1\leqq r<p$. Thus by (4.14)–(4.16), $I(\mathscr{Y}(x)\beta(T)Z(T))\in\mathbf{D}_{1,1}$, and

$$E[\|DI(\mathscr{Y}(x)\zeta(T))\|^{p'}]<\infty \quad \text{for any } 1\leqq p'<p.$$

Finally, we can apply Lemma A.1 and (4.17) to conclude that $F=\beta(T)I(\mathscr{Y}(x)\zeta(T))\in\mathbf{D}_{1,1}$ and

$$DF=-F\int_0^T Dr(s)\,ds-\beta(T)\mathscr{Y}(x)\zeta(T)I'(\mathscr{Y}(x)\zeta(T)) \times\left[\int_0^T Dr(s)\,ds+\int_0^T D\theta(s)\,d\tilde{W}(s)+\theta(\cdot)\right]. \tag{4.18}$$

Another application of (4.12) and Hölder's inequality proves that

$$E[\|DF\|^{p'}]<\infty \quad \text{for all } 1\leqq p'<p.$$

Therefore, the hypotheses of Corollary 2.6 are satisfied, and we may apply formula (2.20) to F. A comparison of (2.20) with (4.3) then yields

$$\beta(t)\sigma^*(t)\hat{\pi}(t)=\tilde{E}(D_tF\,|\,\mathscr{F}_t)-E\left[F\cdot\int_t^T D_t\theta(u)^*\,d\tilde{W}(u)\,\middle|\,\mathscr{F}_t\right]. \tag{4.19}$$

By substituting $F=\beta(T)I(\mathscr{Y}(x)\zeta(T))$ and the expression (4.18) for DF into (4.19) we obtain the representation (4.13). ∎

Example 4.3 $U(x)=\log x$. In this case $I(y)=1/y$, and so $\phi(y)=1$, $-yI'(y)=I(y)$. The expressions (4.13), (4.3) yield

$$\hat{\pi}(t)=\frac{1}{\beta(t)}(\sigma^*(t))^{-1}\theta(t)\tilde{E}[\beta(T)I(\mathcal{Y}(x)\zeta(T))\,|\,\mathcal{F}_t]$$

$$=(\sigma^*(t))^{-1}\theta(t)\hat{X}(t)$$

$$=(\sigma(t)\sigma^*(t))^{-1}[b(t)-r(t)\mathbf{1}]\hat{X}(t). \tag{4.20}$$

In other words, the entries of the vector $(\sigma(t)\sigma^*(t))^{-1}[b(t)-r(t)\mathbf{1}]$ provide, at every time $t\in[0,T]$, the proportions of wealth that are to be invested in the individual stocks. In this special case, we do not need to assume that θ, r belong to $\mathbf{L}^a_{1,1}$; the result is true without regularity assumptions on θ and r (c.f. Section 9.3 in Karatzas [14]).

Example 4.4 $U(x)=(1/\delta)x^\delta$, $\delta<1$, $\delta\neq 0$. In this case $I(y)=y^{-1/(1-\delta)}$, $\phi(y)=y^{-\delta/(1-\delta)}$ and $-yI'(y)=[I(y)/(1-\delta)]$. It follows from (4.13), (4.3):

$$\beta(t)\sigma^*(t)\hat{\pi}(t)=\frac{1}{1-\delta}\tilde{E}[\beta(T)I(\mathcal{Y}(x)\zeta(T))\,|\,\mathcal{F}_t]\cdot\theta(t)$$

$$+\frac{\delta}{1-\delta}\tilde{E}\left[\beta(T)I(\mathcal{Y}(x)\zeta(T))\left\{\int_t^T D_t r(u)\,du\right.\right.$$

$$\left.\left.+\int_t^T (D_t\theta(u))\,d\tilde{W}(u)\right\}\right|\mathcal{F}_t\Bigg],$$

or equivalently

$$\hat{\pi}(t)=\frac{(\sigma^*(t))^{-1}}{1-\delta}\left[\theta(t)\hat{X}(t)+\frac{\delta}{\beta(t)}\tilde{E}\left[\beta(T)\hat{X}(T)\left\{\int_t^T D_t r(u)\,du\right.\right.\right.$$

$$\left.\left.\left.+\int_t^T (D_t\theta(u))\,d\tilde{W}(u)\right\}\right|\mathcal{F}_t\right]\Bigg]. \tag{4.21}$$

In particular, if the coefficients $r(\cdot)$, $\theta(\cdot)$ are deterministic, the expression of (4.21) becomes

$$\hat{\pi}(t)=(\sigma(t)\sigma^*(t))^{-1}[b(t)-r(t)\mathbf{1}]\frac{\hat{X}(t)}{1-\delta}. \tag{4.22}$$

Example 4.4 (Deterministic Coefficients) In this case the hypotheses on θ and r of Theorem 4.2 are satisfied trivially; under the remaining assumptions $U'(0+)=\infty$ and (4.11), we have from (4.13):

$$\hat{\pi}(t) = -(\sigma^*(t))^{-1}\theta(t)\frac{\beta(T)}{\beta(t)}\tilde{E}[\mathscr{Y}(x)\zeta(T)I'(\mathscr{Y}(x)\zeta(T))|\mathscr{F}_t]. \tag{4.23}$$

On the other hand, (4.3) yields

$$\hat{X}(t) = \frac{\beta(T)}{\beta(t)}\tilde{E}[I(\mathscr{Y}(x)\zeta(T))|\mathscr{F}_t]. \tag{4.24}$$

We would like to obtain $\hat{\pi}(t)$ as a function of $\hat{X}(t)$, for every $t \in [0, T]$; in other words, to express at every time t the optimal portfolio in "feedback form" on the current level of wealth.

In order to do this, introduce the notation

$$\alpha(t) \doteq \int_t^T \|\theta(s)\|^2\,ds,\ Y(t) \doteq -\int_0^t \theta^*(s)\,d\tilde{W}(s); \qquad 0 \leqq t \leqq T \tag{4.25}$$

$$\gamma \doteq \mathscr{Y}(x)\beta(T)\,e^{\alpha(0)/2} \tag{4.26}$$

and assume $\alpha(t) > 0$, $\forall\, 0 \leqq t < T$. With this notation,

$$\mathscr{Y}(x)\zeta(T) = \mathscr{Y}(x)\beta(T)\exp\left\{-\int_0^T \theta^*(s)\,d\tilde{W}(s) + \frac{1}{2}\int_0^T |\theta(s)|^2\,ds\right\}$$

$$= \gamma\, e^{Y(T)}. \tag{4.27}$$

On the other hand, the conditional $\tilde{P}$ law of $Y(T)$, given $\mathscr{F}_t$, is normal with mean $Y(t)$ and variance $\alpha(t)$ (this is because

$$Y(T) - Y(t) = -\int_t^T \theta^*(s)\,d\tilde{W}(s) \tag{4.27'}$$

is independent of $\mathscr{F}_t$, and is a zero-mean normal random variable with variance $\alpha(t) = \int_t^T \|\theta(s)\|^2\,ds$). Therefore, with

$$K(t, x) \doteq \int_{\mathscr{R}} I(\gamma\, e^{x+y})\frac{e^{-y^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\,dy, \tag{4.28}$$

the expression (4.24) becomes

$$\hat{X}(t) = \frac{\beta(T)}{\beta(t)}K(t, Y(t)). \tag{4.29}$$

The function $K(t,\cdot)$ of (4.28) is continuous and strictly decreasing; differentiating *formally* under the integral sign, we obtain

$$K_x(t,x)=\int_{\mathscr{R}}\gamma\,e^{x+y}I'(\gamma\,e^{x+y})\frac{e^{-y^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\,dy \tag{4.30}$$

for its derivative. In conjunction with (4.27), this expression lets us rewrite the optimal portfolio $\hat{\pi}$ of (4.23) in the form

$$\hat{\pi}(t)=-(\sigma^*(t))^{-1}\theta(t)\frac{\beta(T)}{\beta(t)}K_x(t,Y(t)). \tag{4.31}$$

In other words, the process $Y(\cdot)$ of (4.25) is a *sufficient statistic* for the computation of both the optimal portfolio and wealth processes, via (4.31) and (4.29), respectively.

Even more to the point is the following observation: let $\Lambda(t,\cdot)$ be the inverse function of $K(t,\cdot)$, i.e. $K(t,\Lambda(t,\xi))=\xi$. Then from (4.29) we have $Y(t)=\Lambda(t,(\beta(t)\hat{X}(t)/\beta(T)))$, and substitution of this expression into (4.31) leads to

$$\hat{\pi}(t)=-(\sigma^*(t))^{-1}\theta(t)\frac{\beta(T)}{\beta(t)}\frac{1}{\Lambda_\xi(t,(\beta(t)/\beta(T))\hat{X}(t))}, \tag{4.32}$$

a formula that provides the optimal portfolio as a (deterministic) function of the current level of wealth, as desired.

In Section 6 we shall show that these heuristic considerations can be made precise, under very weak conditions; in particular, none of $U'(0+)=\infty$, (4.11) will be necessary.

Remark 4.5 In this remark we answer a question about optimal portfolio selection posed to us by Héctor Sussmann. From (4.32) it is clear that in the case of deterministic coefficients, the ratio

$$\frac{\hat{\pi}^i(t)}{\hat{\pi}^j(t)}=\frac{((\sigma^*(t))^{-1}\theta(t))^i}{((\sigma^*(t))^{-1}\theta(t))^j};\qquad 1\leqq i\neq j\leqq d,$$

of the optimal portfolios in any two different assets, is *independent of the utility function U*. Is this true more generally? The answer, in the case of general, random coefficients, is no. For instance, in the context of Example 4.4 we obtain from (4.21) and (4.2)–(4.5):

$$\hat{\pi}^i_\delta(t)=\frac{-x(\sigma^*(t))^{-1}}{(1-\delta)\alpha_\delta\beta(t)}\tilde{E}\left[Q_\delta(T)\left\{\theta(t)+\delta\int_t^T D_t r(u)\,du\right.\right.$$

$$+\delta\int_t^T (D_t\theta(u))\,d\tilde{W}(u)\Big\}\Big|\mathscr{F}_t\Big] \tag{4.33}$$

for $1 \leqq i,\ j \leqq d$, with $Q_\delta(T) \doteq (\beta^\delta(T)Z(T))^{1/(\delta-1)}$ and $\alpha_\delta \doteq E(\beta T)Z(T))^{\delta/(\delta-1)}$. Putting $\delta=0$ in (4.33), we recover the optimal portfolio for the logarithmic case of Example 4.3. It is not hard to see that, in general, the ratio

$$\frac{\hat{\pi}_\delta^i(t)}{\hat{\pi}_\delta^j(t)} \quad \text{is not independent of} \quad \delta \in [0, 1) \tag{4.34}$$

for $i \neq j$. Indeed, take $\sigma \equiv I_d$ and $\theta(\cdot)$ deterministic; the negation of (4.34) leads to the relation

$$\theta^i(t)\rho_\delta^j(t) = \theta^j(t)\rho_\delta^i(t); \quad \forall\, t \in [0, T), \quad \delta \in (0, 1) \tag{4.35}$$

for all $1 \leqq i, j \leqq d$, where

$$\rho_\delta^i(t) \doteq \delta\, \frac{\tilde{E}[Q_\delta(T)\int_t^T D_t^i r(u)\,du\,|\,\mathscr{F}_t]}{\tilde{E}[Q_\delta(T)\,|\,\mathscr{F}_t]}.$$

It is not hard to find interest rate processes $r(\cdot)$ and $\mathscr{R}^d$-valued functions $\theta(\cdot)$, for which (4.35) is violated for some $i \neq j$.

5. OPTIMAL PORTFOLIO FOR CONSUMPTION

Let us now turn our attention to the problem of *maximizing expected utility*

$$E\int_0^T U(t, c(t))\,dt \tag{5.1}$$

from consumption during the finite interval $[0, T]$, over all admissible pairs $(\pi, c) \in \mathscr{A}(x)$. We retain the setting and notation of Section 3. In (5.1) the function $U(t, c)$: $[0, T] \times (0, \infty) \to \mathscr{R}$ is of class $C^{0,1}$ and such that, for every $t \in [0, T]$, $U(t, \cdot)$ satisfies the properties of a utility function set forth in Section 4; $I(t, \cdot)$ will denote the inverse of $U'(t, \cdot)$ on $[0, U'(t, 0+)]$, and $I(t, y) \equiv 0$ for $y \geqq U'(t, 0+)$.

This problem was addressed in KLS [15, Section 4] (see also Karatzas [14, Section 8]). It follows from the theory of this article that there exists an optimal pair $(\hat{\pi}, \hat{c}) \in \mathscr{A}(x)$ for this problem, with

$$\hat{c}(t) = I(t, \mathscr{Y}(x)\zeta(t)), \tag{5.2}$$

and corresponding wealth process $\hat{X}$ given by

$$\beta(t)\hat{X}(t)+\int_0^t \beta(s)\hat{c}(s)\,ds=\tilde{\mathbf{E}}\left[\int_0^T \beta(s)\hat{c}(s)\,ds\,\middle|\,\mathscr{F}_t\right]$$

$$=x+\int_0^t \beta(s)\hat{\pi}^*(s)\sigma(s)\,d\tilde{W}(s) \tag{5.3}$$

for $0\leqq t\leqq T$. Here again, $x>0$ is the initial capital and $\mathscr{Y}(\cdot)$ is the inverse of the continuous, decreasing function

$$\mathscr{X}(y)\doteq\tilde{E}\int_0^T \beta(s)I(s,y\zeta(s))\,ds$$

$$=E\int_0^T \zeta(s)I(s,y\zeta(s))\,ds,\qquad 0<y<\infty \tag{5.4}$$

which maps $(0,\infty)$ into $[0,\infty)$, and the value function

$$V(x)\doteq\inf_{(\pi,c)\in\mathscr{A}(x)} E\int_0^T U(t,c(t))\,dt$$

of this problem can be expressed as

$$V(x)=G(\mathscr{Y}(x)) \tag{5.5}$$

where

$$G(y)\doteq E\int_0^T U(t,I(t,y\zeta(t)))\,dt;\qquad 0<y<\infty$$

(under additional technical conditions on the function $U(t,c)$).

By analogy with Theorem 4.2, we have the following representation for the optimal portfolio $\hat{\pi}$ of (5.3).

THEOREM 5.1 (Representation of Optimal Portfolio for Consumption) *Suppose that* $U'(t,0+)=\infty$ *for all* $t\in[0,T]$, $I(t,y)\in C^{0,1}([0,T]\times(0,\infty))$, *and let*

$$I(t,y)+|I'(t,t,y)|\leqq K(y^{\alpha}+y^{-\beta}),\qquad \forall (t,y)\in[0,T]\times(0,\infty) \tag{5.6}$$

hold for some positive constants α, β, K. *Furthermore, let* $\{\theta(t),0\leqq t\leqq T\}$ *and* $\{r(t),0\leqq t\leqq T\}$ *be bounded processes such that* $\theta\in\mathbf{L}^a_{1,1}$, $r\in\mathbf{L}^a_{1,1}$ *and, for some* $p>1$,

$$E\left(\int_0^T \|Dr(u)\|^2\,du\right)^{p/2}<\infty, \tag{5.7}$$

and

$$E\left(\int_0^T \|D\theta(u)\|^2\,du\right)^{p/2}<\infty. \tag{5.8}$$

Then the optimal portfolio $\hat{\pi}$ *of* (5.3) *is given by*

$$\hat{\pi}(t)=-\frac{1}{\beta(t)}(\sigma^*(t))^{-1}\left[\theta(t)\cdot\tilde{E}\left\{\int_t^T \beta(s)\mathscr{Y}(x)\zeta(s)I'(s,\mathscr{Y}(x)\zeta(s)\,ds\,\middle|\,\mathscr{F}_t\right\}\right.$$

$$+\tilde{E}\left\{\int_t^T \beta(s)\phi'(s,\mathscr{Y}(x)\zeta(s))\left[\int_t^s D_t r(u)\,du\right.\right.$$

$$\left.\left.\left.+\int_t^s D_t\theta(u)\,d\tilde{W}(u)\right]ds\,\middle|\,\mathscr{F}_t\right\}\right], \tag{5.9}$$

where $\phi(t,y)\doteq yI(t,y)$ *and* $I'(t,y)\doteq(\partial/\partial y)I(t,y)$, $\phi'(t,y)\doteq(\partial/\partial y)\phi(t,y)$.

Proof Let $F=\int_0^T \beta(s)I(s,\mathscr{Y}(x)\zeta(s))\,ds$. To prove Theorem 5.1, it is sufficient to verify that the hypotheses of Corollary 2.6 hold for F. Let $\gamma(s)=\beta(s)I(s,\mathscr{Y}(x)\zeta(s))$. We shall prove

$$\sup_{[0,T]}\gamma(s)\in L^q(P)\quad\text{for all } q>1,\quad\text{and} \tag{5.10}$$

$$\gamma\in \mathbf{L}^a_{1,1}. \tag{5.11}$$

It follows from (5.10) and (5.11) that $F\in\mathbf{D}_{1,1}$ and

$$DF=\int_0^T D[\gamma(s)I(s,\mathscr{Y}(x)\zeta(s))]\,ds.$$

To prove (5.10), observe that the boundedness of the process $r(\cdot)$ and assumption (5.6) imply

$$\sup_{[0,T]}\gamma(s)\leq K\left[\left(\max_{[0,T]}Z(t)\right)^{\alpha}+\left(\max_{[0,T]}Z(t)\right)^{-\beta}\right]. \tag{5.12}$$

(Again, K is a generic positive constant, and the K in (5.12) may differ from that in (5.6).) Because $\{Z(t);0\leq t\leq T\}$ is a martingale with $Z(T)\in L^q(P)$ for all $q>1$, we also have $\max_{[0,T]}Z(t)\in L^q(P)$, for all $q>1$. Likewise, because $\{V(t)=Z^{-1}(t)\exp[-\int_0^t|\theta(s)|^2\,ds],\ 0\leq t\leq T\}$ is a martingale with $V(T)\in L^q(P)$ for all $q>1$,

$\max_{[0,T]} Z^{-1}(t) \in L^q(P)$, for all $q>1$. The claim (5.10) follows immediately. By the same argument, assumption (5.6) also yields

$$\sup_{[0,T]} |\beta(s)\zeta(s)I'(s, \mathscr{Y}(x)\zeta(s))| \in L^q(P) \qquad \text{for all } q>1. \tag{5.13}$$

To demonstrate (5.11), first note that the proof of Theorem 4.2 shows that for every s, $\gamma(s) \in \mathbf{D}_{1,1}$ and

$$D\gamma(s) = -\gamma(s)\int_0^s Dr(u)\,du - \beta(s)\mathscr{Y}(x)\zeta(s)I'(s, \mathscr{Y}(x)\zeta(s))$$

$$\times\left[\int_0^s D\theta(u)\,d\tilde{W}(u) + \theta(\cdot)1_{[0,s]}(\cdot) + \int_0^s Dr(u)\,du\right]. \tag{5.14}$$

Therefore, because of (5.12), (5.13) and the boundedness of θ,

$$\|D\gamma(s)\| \le X\left(1 + \int_0^s \|Dr(u)\|\,du + \left\|\int_0^s D\theta(u)\,dW(u)\right\| + \int_0^s \|D\theta(u)\|\,du\right), \tag{5.15}$$

where X is a random variable satisfying $X \in L^q(P)$ for all $q>1$. If $1 \le p' < p$, we obtain

$$E\left(\int_0^T \|D\gamma(s)\|^2\,ds\right)^{p'/2} \le KE^{1/q}[X^{qp'}]\left(1 + E^{p'/p}\left(\int_0^T\left(\int_0^T \|Dr(u)\|^2\,du\right)^2 ds\right)^{p/2}\right.$$

$$+ E^{p'/p}\left(\int_0^T\left(\int_0^T \|D\theta(u)\|\,du\right)^2 ds\right)^{p/2}$$

$$\left. + E^{p'/p}\left(\int_0^T \sup_{[0,T]}\left\|\int_0^s D\theta(u)\,dW(u)\right\|^2 dv\right)^{p/2}\right), \tag{5.16}$$

where $(1/q)+(p'/p)=1$. On the other hand, we have

$$E\left(\int_0^T\left(\int_0^T \|Dr(u)\|\,du\right)^2 ds\right)^{p/2} \le KE\left(\int_0^T \|Dr(u)\|^2\,du\right)^{p/2},$$

and similarly with $D\theta(u)$; moreover, since

$$E\left(\int_0^T \sup_{[0,T]}\left\|\int_0^s D\theta(u)\,dW(u)\right\|^2 dr\right)^{p/2} \le KE\left[\sup_{[0,T]}\left\|\int_0^s D\theta(u)\,dW(u)\right\|^p\right]$$

$$\leqq KE\left(\int_0^T \|D\theta(u)\|^2\,du\right)^{p/2}$$

by the Burkholder–Davis–Gundy inequalities, it follows from (5.16) that

$$E\left(\int_0^T \|D\gamma(s)\|^2\,ds\right)^{p'/2} < \infty, \tag{5.17}$$

for $1 \leqq p' < p$. This implies $\gamma \in \mathbf{L}_{1,1}^a$. Also, (5.17) implies that

$$E[\|DF\|^{p'}] \leqq E\left(\int_0^T \|D\gamma(s)\|\,ds\right)^{p'} < \infty. \tag{5.18}$$

The inequality (5.18) verifies the remaining hypothesis on F in Corollary 2.6 and so completes the proof. ▮

Example 5.2 $U(t,c) = e^{-\int_0^t \mu(s)ds} \log c$, where $\mu:[0,T] \to \mathscr{R}$ is a bounded, measurable function (a deterministic discount factor). In this case $I(t,y) = e^{-\int_0^t \mu(s)ds} \cdot (1/y)$ and $-yI'(t,y) = I(t,y)$, so (5.11) and (5.3) give

$$\hat{\pi}(t) = \frac{1}{\beta(t)} (\sigma^*(t))^{-1} \theta(t) \cdot \tilde{E}\left[\int_t^T \beta(s) I(s, \mathscr{Y}(x)\zeta(s))\,ds \,\middle|\, \mathscr{F}_t\right]$$

$$= (\sigma^*(t))^{-1} \theta(t) \hat{X}(t) = (\sigma(t)\sigma^*(t))^{-1} [b(t) - r(t)\mathbf{1}] \hat{X}(t),$$

just as in Example 4.3.

Example 5.3 $U(t,c) = e^{-\int_0^t \mu(s)ds} (1/\delta) c^\delta$, where $\mu(\cdot)$ is as in Example 5.2 and $\delta < 1$, $\delta \neq 0$. Then $I(t,y) = e^{-1/(1-\delta)\int_0^t \mu(s)ds} y^{1/(1-\delta)}$, $\phi(t,y) = e^{-1/(1-\delta)\int_0^t \mu(s)ds} y^{-\delta/(1-\delta)}$ and $-yI'(t,y) = (I(t,y)/(1-\delta))$; it follows from (5.9), (5.3) that

$$\beta(t)\sigma^*(t)\hat{\pi}(t) = \frac{\theta(t)}{1-\delta} \tilde{E}\left[\int_t^T \beta(s) I(s, \mathscr{Y}(x)\zeta(s))\,ds \,\middle|\, \mathscr{F}_t\right]$$

$$+ \frac{\delta}{1-\delta} \tilde{E}\left[\int_t^T \beta(s) I(s, \mathscr{Y}(x)\zeta(s)) \left\{\int_t^s D_t r(u)\,du \right.\right.$$

$$\left.\left. + \int_t^s (D_t\theta(u))\,d\tilde{W}(u)\right\} ds \,\middle|\, \mathscr{F}_t\right],$$

whence

$$\hat{\pi}(t)=\frac{(\sigma^*(t))^{-1}}{1-\delta}\Bigg[\theta(t)\hat{X}(t)+\frac{\delta}{\beta(t)}\tilde{E}\Bigg(\int_t^T\beta(s)\hat{c}(s)\Bigg\{\int_t^s D_t r(u)\,du$$

$$+\int_t^s(D_t\theta(u))\,d\tilde{W}(u)\Bigg\}\,ds\Bigg|\mathscr{F}_t\Bigg)\Bigg]. \qquad (5.19)$$

In the case of deterministic $r(\cdot)$ and $\theta(\cdot)$, this portfolio takes the form (4.22).

6. DETERMINISTIC COEFFICIENTS

This section continues the discussion on models with deterministic coefficients, started in Example 4.4. The basic observation is that, in this case, *the driving Brownian motion W and the process $\tilde{W}$ of (2.2) generate the same filtration.* This means, in particular, that $\{\mathscr{F}_t\}$ is also the $\tilde{P}$-augmentation of $\{\mathscr{F}_t^{\tilde{W}}\}$. Thus, when representing $\tilde{P}$-martingales as stochastic integrals with respect to $\tilde{W}$ via the Clark formula, we can work on the probability space $(\Omega, \mathscr{F}, \tilde{P})$. *There will be no need to transform back to the original probability space* $(\Omega, \mathscr{F}, P)$, as was the case in the more general representation formulae (2.20) and (2.25). Of course, the formula we get could be derived from (2.20) with $D_t\theta\equiv 0$. However, the proof of formula (2.20) required assumptions (2.17) and (2.18), and these become unnecessary if we apply the Clark formula directly to $\tilde{W}$. In this way we avoid unnecessarily stringent conditions.

We place ourselves in the setting of the financial market model of Section 3 with deterministic coefficients, and address first the question of *maximizing the expected utility from terminal wealth* (4.1) over admissible pairs $(\pi, c)\in\mathscr{A}(x)$. The utility function U will be as in the opening paragraph of Section 4. Recalling the notation of (4.25)–(4.28), we write the optimal wealth process $\hat{X}$ of (4.3) in the form

$$\hat{X}(t)=\frac{\beta(T)}{\beta(t)}M(t), \quad \text{where} \qquad (6.1)$$

$$M(t)\doteq\tilde{E}[I(\gamma e^{Y(T)})|\mathscr{F}_t]=K(t, Y(t)), \qquad (6.2)$$

$$K(t,x)\doteq\int_{\mathscr{R}} I(\gamma e^{x+z})\frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}dz, \qquad (t,x)\in[0.\ T)\times\mathscr{R}. \qquad (4.28)$$

We also introduce the function

$$L(t,x)\doteq\int_{\mathscr{R}}\frac{z}{\alpha(t)}I(\gamma e^{x+z})\frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}dz; \qquad (t,x)\in[0, T)\times\mathscr{R}. \qquad (6.3)$$

Under the condition

$$\alpha(t) > 0, \qquad \forall\, 0 \leqq t < T, \tag{6.4}$$

the functions K and L of (4.31) and (6.3) are well-defined on $[0, T] \times \mathscr{R}$, provided that $(1+|z|)I(e^{x+z})$ is integrable with respect to $\exp(-(z^2/2\alpha(t)))\,dz$, for every $(t, x) \in [0, T) \times \mathscr{R}$. A sufficient condition will be imposed below (cf. (6.7)), which will guarantee this.

Remark 6.1 If I is smooth, a formal integration by parts in (6.3) suggests the computations

$$\begin{aligned} L(t,x) &= -\int_{\mathscr{R}} I(\gamma e^{x+z}) \frac{\partial}{\partial z}\left(\frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\right) dz \\ &= \int_{\mathscr{R}} \gamma e^{x+z} I'(\gamma e^{x+z}) \left(\frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\right) dz \\ &= K_x(t, x), \end{aligned} \tag{6.5}$$

if one ignores boundary terms and recalls the other heuristic computation (4.30). It follows then, from this formula and (4.31):

$$\hat{\pi}(t) = -(\sigma^*(t))^{-1}\theta(t)\frac{\beta(T)}{\beta(t)} L(t, Y(t)). \tag{6.6}$$

Despite the heuristic character of both (4.31) and (6.5), their consequence (6.6) can be made completely rigorous, as the following theorem demonstrates.

THEOREM 6.2 *Let (6.4) hold, and assume*

$$\int_{\mathscr{R}} I^p(e^z)\, e^{-z^2/2a}\, dz < \infty \tag{6.7}$$

for some $p > 1$, $a > \alpha(0)$. *Then the* $\tilde{P}$*-martingale* M *of (6.2) admits the representation*

$$M(t) = \tilde{E} I(\gamma e^{Y(T)}) - \int_0^t L(s, Y(s))\theta^*(s)\, d\tilde{W}(s), \tag{6.8}$$

and the optimal portfolio $\hat{\pi}$ *of (4.3) is given by (6.6).*

Remark 6.3 It is a consequence of the proof that all terms in (6.8) make sense. This is interesting in the case $I \notin C^1$, for it is not easy then to see *a priori* that

$$\int_0^T L^2(s, Y(s))\, ds < \infty, \quad \text{a.s.}$$

Remark 6.4 Observe that the condition (6.7) implies

$$I(b\,e^z), |z|I(b\,e^z) \in L^p(e^{-z^2/2\alpha}\,dz) \tag{6.9}$$

for every $\alpha \in (0, a)$, $b \in \mathscr{R}$. In particular then, the functions K, L of (4.31), (6.3) are well-defined on $[0, T] \times \mathscr{R}$, and

$$\tilde{E}I^p(\gamma\,e^{Y(T)}) = \int_{\mathscr{R}} I^p(\gamma\,e^z)\frac{e^{-z^2/2\alpha(0)}}{\sqrt{2\pi\alpha(0)}}\,dz < \infty. \tag{6.10}$$

The *Proof of Theorem 6.2* will proceed in two steps:

Step 1 Theorem 6.2 holds if, in addition to the assumptions stated there, one imposes also $I \in C^1$.

Step 2 One may relax the assumption $I \in C^1$.

Proof of Step 2 from Step 1: Smoothing argument.
Let $\rho: \mathscr{R} \to [0, \infty)$ be a C^∞-function with compact support and $\int_{\mathscr{R}} \rho(x)\,dx = 1$. Let $\rho_n(x) \doteq n\rho(nx)$, $x \in \mathscr{R}$ for $n \geqq 1$, and define

$$f(z) \doteq I(e^z), \quad f_n \doteq f * \rho_n$$

(here $*$ denotes convolution) and

$$I_n(y) \doteq f_n(\log y), \quad \text{so} \quad f_n(z) = I_n(e^z).$$

Then it is easy to see that

$$\lim_{n\to\infty} I_n(y) = I(y), \qquad \forall\, y > 0 \tag{6.11}$$

$$\lim_{n\to\infty} \tilde{E}\left|I_n(\gamma\,e^{Y(T)}) - I(\gamma\,e^{Y(T)})\right|^p = 0, \tag{6.12}$$

$$L_n(t, x) \doteq \int_{\mathscr{R}} \frac{z}{\alpha(t)} I_n(\gamma\,e^{x+z})\frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\,dz \underset{n\to\infty}{\longrightarrow} L(t, x) \tag{6.13}$$

for every $(t, x) \in [0, T] \times \mathscr{R}$.
Consider now the $\tilde{P}$-martingales

$$\bar{M}(t) \doteq \tilde{E}[I(\gamma\,e^{Y(T)})\,|\,\mathscr{F}_t] - \tilde{E}I(\gamma\,e^{Y(T)}) \tag{6.14}$$

$$\bar{M}_n(t) \doteq \tilde{E}[I_n(\gamma\,e^{Y(T)})\,|\,\mathscr{F}_t] - \tilde{E}I_n(\gamma\,e^{Y(T)}) \tag{6.15}$$

for $n \geqq 1$, which admit the representations

$$\bar{M}(t)=\int_0^t \psi^*(s)\,d\tilde{W}(s), \qquad 0\leqq t\leqq T \tag{6.16}$$

for some $\{\mathscr{F}_t\}$-progressively measurable, $\mathscr{R}^d$-valued process ψ with $\int_0^T |\psi(t)|^2\,dt<\infty$, a.s. (from the martingale representation theorem), as well as

$$M_n(t)=-\int_0^t L_n(s, Y(s))\theta^*(s)\,d\tilde{W}(s), \qquad 0\leqq t\leqq T \tag{6.17}$$

(from Step 1). From the Burkholder–Davis–Gundy and Doob inequalities, we have then

$$\begin{aligned}\tilde{E}\left[\int_0^T |\psi(s)+L_n(s, Y(s))\theta(s)|^2\,ds\right]^{p/2} &\leqq C_1\cdot\tilde{E}\left(\sup_{0\leqq t\leqq T}|\bar{M}_n(t)-\bar{M}(t)|^p\right)\\ &\leqq C_2\cdot\tilde{E}|\bar{M}_n(T)-\bar{M}(T)|^p\\ &\leqq C_3\cdot\tilde{E}|I_n(\gamma e^{Y(T)})-I(\gamma e^{Y(T)})|^p,\end{aligned}$$

and by virtue of (6.12) this last expression tends to zero, as $n\to\infty$. It follows then from (6.13) that $\psi(s,\omega)=-L(s, Y(s,\omega))\cdot\theta(s,\omega)$ holds for $dt\otimes d\tilde{P}$-a.e. (s,ω) on $[0,T]\times\Omega$, and therefore we may take $\psi(t)=-L(t, Y(t))\cdot\theta(t)$ in (6.16).

Proof of Step 1 : Cut-off argument.
Consider a C^∞-function $\phi:\mathscr{R}\to[0,1]$ with

$$\phi(x)=\begin{Bmatrix}1, & |x|\leqq 1\\ 0, & |x|\geqq 2\end{Bmatrix}$$

and define the functions

$$\phi_n(x)\doteq\phi\left(\frac{x}{n}\right), \qquad x\in\mathscr{R}$$

$$f_n(z)\doteq\phi_n(z)I(\gamma e^z), \qquad z\in\mathscr{R}$$

and the random variables

$$G\doteq I(\gamma e^{Y(T)}), \qquad G_n\doteq f_n(Y(T))$$

for $n=1,2,\ldots$. We are interested in representing the $\tilde{P}$-martingale $M(t)=\tilde{E}(G|\mathscr{F}_t)$, $0\leqq t\leqq T$ as a stochastic integral with respect to the $\tilde{P}$-Brownian motion $\tilde{W}$, in the specific form (6.8).

Now every G_n belongs to the space $\mathbf{D}_{2,1}$, and so from Proposition 2.1:

$$M_n(t) \doteq \tilde{E}(G_n|\mathscr{F}_t) = \tilde{E}(G_n) + \int_0^t \tilde{E}[D_s G_n|\mathscr{F}_s]\, d\tilde{W}(s) \tag{6.18}$$

for $0 \leqq t \leqq T$, $n = 1, 2, \ldots$. (Strictly speaking, in (6.18) D denotes the operation of differentiation with respect to $\tilde{W}$. This does not contradict the representation formula (2.20) because, when θ is deterministic, differentiation with respect to $\tilde{W}$ and with respect to W give the same result.) Now recalling (4.27′) and the fact that $\theta(\cdot)$ is non-random, we obtain

$$D_t G_n = f_n'(Y(T)) \cdot D_t Y(T) = -f_n'(Y(T)) \cdot \theta(t)$$

and

$$\begin{aligned} -\tilde{E}[D_t G_n|\mathscr{F}_t] &= \tilde{E}[f_n'(Y(T))|\mathscr{F}_t] \cdot \theta(t) \\ &= \int_{\mathscr{R}} f_n'(z + Y(t)) \frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\, dz \cdot \theta(t) \\ &= \int_{\mathscr{R}} f_n(z + Y(t)) \frac{z}{\alpha(t)} \frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\, dz \cdot \theta(t) \\ &= \int_{\mathscr{R}} \phi_n(z + Y(t)) I(\gamma e^{z+Y(t)}) \frac{z}{\alpha(t)} \frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\, dz \cdot \theta(t) \end{aligned}$$

after integrating by parts. By the Dominated Convergence Theorem, this last expression converges to

$$\int_{\mathscr{R}} I(\gamma e^{z+Y(t)}) \frac{z}{\alpha(t)} \frac{e^{-z^2/2\alpha(t)}}{\sqrt{2\pi\alpha(t)}}\, dz \cdot \theta(t) = L(t, Y(t)) \cdot \theta(t)$$

as $n \to \infty$, for *every* $(t, \omega) \in [0, T] \times \Omega$. On the other hand, recalling (6.10) we obtain

$$\tilde{E}|G_n - G|^p = \tilde{E}|f_n(Y(T)) - I(\gamma e^{Y(T)})|^p \underset{n\to\infty}{\longrightarrow} 0,$$

and arguing as before we arrive from (6.18) to the desired representation (6.8). ∎

Finally, we discuss the question of *maximizing the expected utility from consumption* (5.1) over all admissible pairs $(\pi, c) \in \mathscr{A}(x)$ in the setting of a financial market with deterministic coefficients. The utility function $U(t, c)$ is as in the opening paragraph of Section 5, whose notation we recall. By analogy with (4.25), (4.26), (6.3) we introduce the additional notation

$$\alpha_s^t \doteq \int_t^s |\theta(u)|^2\, du, \quad \gamma(s) \doteq \mathscr{Y}(x)\beta(s)\, e^{\alpha_s^0/2}, \quad Y(s) \doteq -\int_0^s \theta^*(u)\, d\tilde{W}(u)$$

$$\hat{L}(s, x) \doteq \int_s^t \int_{\mathscr{R}} \frac{z}{\alpha_u^s} I(u, \gamma(u)\, e^{x+z}) \frac{e^{-z^2/2\alpha_u^s}}{\sqrt{2\pi\alpha_u^s}}\, dz\, du.$$

THEOREM 6.5 *Suppose that $I(t, y)$ is continuous on $[0, T] \times (0, \infty)$, and that*

$$0 < c_1 \leqq \|\theta(t, \omega)\|^2 \leqq c_2 < \infty; \qquad \forall (t, \omega) \in [0, T] \times \Omega$$

$$\int_0^T \int_{\mathscr{R}} I^p(t, e^z) \frac{e^{-z^2/2at}}{\sqrt{2\pi at}}\, dz\, dt < \infty$$

hold, for some constants $p > 1$, $a > c_2 > c_1 > 0$. Then the $\tilde{P}$-martingale

$$\hat{M}(t) \doteq \beta(t)\hat{X}(t) + \int_0^t \beta(s)\hat{c}(s)\, ds = \tilde{E}\left(\int_0^T \beta(s)\hat{c}(s)\, ds \middle| \mathscr{F}_t\right)$$

and the optimal portfolio $\hat{\pi}(t)$ of (5.3) admit the representations

$$\hat{M}(t) = \tilde{E}\left(\int_0^T \beta(s) I(s, \gamma(s)\, e^{Y(s)})\, ds\right) - \int_0^t \hat{L}(s, Y(s))\theta^*(s)\, d\tilde{W}(s)$$

and

$$\hat{\pi}(t) = -(\sigma^*(t))^{-1} \frac{\theta(t)}{\beta(t)} \hat{L}(t, Y(t)),$$

respectively. ∎

The proof is similar to that of Theorem 6.2; we omit the details.

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APPENDIX

In this appendix we collect and prove some technical results which are used throughout the paper. The first result is simply the chain rule with proper attention to function spaces. While the result is simple, it has not been previously stated in the setting of $\mathbf{D}_{1,1}$ except by Enchev [7], section 10, in the context of a weaker definition of the gradient D.

LEMMA A.1 *Let $F=(F_1,\ldots,F_k)\in(\mathbf{D}_{1,1})^k$. Let $\phi\in C^1(\mathcal{R}^k)$ be a real-valued function and assume that*

$$E\left\{|\phi(F)|+\left\|\sum\frac{\partial\phi}{\partial x_i}(F)DF_i\right\|\right\}<\infty.$$

Then $\phi(F) \in \mathbf{D}_{1,1}$ *and* $D\phi(F) = \sum (\partial\phi/\partial x_i)(F) DF_i$.

Proof It is easy to see that the lemma is true if $\phi \in C_b^1(\mathscr{R}^k)$, that is, if ϕ and its first derivatives are bounded. Now consider the general case. Let $\psi \in C_0^\infty(\mathscr{R})$ satisfy $\psi(z) = z$, if $|z| \leqq 1$, $|\psi(z)| \leqq |z|$ for all $z \in \mathscr{R}$. For any integer n, let $\phi_n(x) = n\psi(\phi(x)/n)$, $x \in \mathscr{R}^k$. For each n, $\phi_n \in C_b^1(\mathscr{R}^k)$, and thus $\phi_n(F) \in \mathbf{D}_{1,1}$ and $D\phi_n(F) = \psi'(\phi(F)/n) \sum (\partial\phi/\partial x_i)(F) DF_i$. Note that $|\phi_n(F)| \leqq |\phi(F)|$ for all n and $\lim_{n\to\infty} \phi_n(F) = \phi(F)$ almost surely. Likewise,

$$|D\phi_n(F)| \leqq \sup |\psi'(x)| \left| \sum \frac{\partial\phi}{\partial x_i}(F) DF_i \right|, \quad \text{and}$$

$$\lim_{n\to\infty} D\phi_n(F) = \sum \frac{\partial\phi}{\partial x_i}(F) DF_i, \quad \text{almost surely.}$$

Therefore,

$$\lim_{n\to\infty} E\left\{ |\phi_n(F) - \phi(F)| + \left\| D\phi_n(F) - \sum \frac{\partial\phi}{\partial x_i}(F) DF_i \right\| \right\} = 0$$

by dominated convergence. The result follows because D is a closed operator on $\mathbf{D}_{1,1}$. ∎

The next result extends a technique of Liptser and Shiryayev [19, pp. 92–95], which is useful for approximating stochastic integrals; see also Karatzas and Shreve [17, Problem 3.25]. The proof is a minor modification of the arguments in these references, and so we omit it.

LEMMA A.2 *Let* K *be a separable Hilbert space and let* $f:[0, T] \times \Omega \to K$ *be a measurable function such that*

$$E\left\{ \left(\int_0^T \|f(s)\|_K^2 \, ds \right)^{1/2} \right\} < \infty. \tag{A.1}$$

Let $\psi_n(t) = \sum_{-\infty}^{\infty} 1_{(j/2^n, (j+1)/2^n]}(t)$ *and extend* $f(t)$ *to all* $t \in \mathscr{R}$ *by setting* $f(t, \omega) = 0$ *for* $t \notin [0, T]$. *Then there exists a subsequence* $\{n_i\}$ *such that*

$$\lim_{n_i\to\infty} E\left[\left(\int_0^T \|f(s + \psi_{n_i}(t - s)) - f(t)\|_K^2 \, dt \right)^{1/2} \right] = 0,$$

for almost every $s \in [0, T]$.

Remark A.3 For every (n, s), $f^{(n)}(t) = f(s + \psi_n(t - s))$ is a simple function, and $f^{(n)}$ is adapted to a filtration if f is. Therefore, Lemma A.2 establishes the existence of adapted, simple approximations to adapted vector-valued processes. Recall now the definition of $\mathbf{L}^a_{1,1}$ from Section 2, and the norm

$$|||u|||^a_{1,1} = E\left\{\left(\int_0^T |u(s)|^2\, ds\right)^{1/2} + \left(\int_0^T \sum_1^d \|Du_i(s)\|^2\, ds\right)^{1/2}\right\}.$$

COROLLARY A.4 *Simple processes are dense in* $\mathbf{L}^a_{1,1}$ *with respect to* $|||\cdot|||^a_{1,1}$.

Proof Apply Lemma A.2 to the process

$$f(t,\omega) = (u(t,\omega), Du_1(t,\omega), \ldots, Du_d(t,\omega)) \in \mathscr{R}^d \times (L^2([0,T]))^d \times \cdots \times (L^2([0,T]))^d.$$

It follows that we may choose a subsequence $\{n_i\}$, such that if we define

$$u^{\tilde{s},n_i}(s) = u(\tilde{s} + \psi_{n_i}(s - \tilde{s})),$$

then

$$|||u^{\tilde{s},n_i} - u|||^a_{1,1} \to 0, \quad \text{as} \quad n_i \to \infty, \quad \text{for } ds\text{-a.e. } \tilde{s}. \tag{A.2}$$

However, $u(s) \in \mathbf{D}_{1,1}$ only for Lebesgue-almost every s, and so we must in addition choose $\tilde{s}$ so that $u^{\tilde{s},n_i}(s) \in \mathbf{D}_{1,1}$ for all s. This can always be done. Let $N = \{s \mid u(s)$ fails to be in $\mathbf{D}_{1,1}\}$. If

$$\tilde{s} \notin B \doteq \bigcup_{j=-\infty}^{\infty} \bigcup_{n<0} \left(\frac{j}{2^n} + N\right),$$

then $u(\tilde{s} + j/2^n)$ belongs to $\mathbf{D}_{1,1}$ for all j, and hence so does

$$u^{\tilde{s},n}(s) = \sum u\left(\tilde{s} + \frac{j}{2^n}\right) 1_{\{j/2^n > s - \tilde{s} \leq (j+1)/2^n\}}$$

for all s. Because the Lebesgue measure of B is zero, we can choose $\tilde{s} \notin B$ so that (A.2) holds.

COROLLARY A.5 *Let* $f: [0,T] \times \Omega \to (L^2([0,T]))^d$ *be an* $\{\mathscr{F}_t\}$*-progressively measurable process satisfying*

$$|||f|||_1 \doteq E\left\{\left(\int_0^T \|f(s)\|^2\, ds\right)^{1/2}\right\} < \infty. \tag{A.3}$$

Let $\bar{Y}: [0,T] \times \Omega \to \mathscr{R}$ *be a measurable function such that for P-almost every* ω, $\bar{Y}(\cdot,\omega) = \int_0^T f(s)^*\, dW(s)$. *Then*

$$\tilde{Y}(t,\omega)=\int_0^T \tilde{f}(t,s,\omega)\,dW(s),\ \text{a.s., for almost every } t\in[0,T], \tag{A.4}$$

where $\tilde{f}:[0,T]\times[0,T]\times\Omega\to\mathscr{R}^d$ *is progressively measurable, and* $\tilde{f}(\cdot,s,\omega)=f(s,\omega)$ *for* $ds\otimes dP$ *almost every* (s,ω).

Remark Note that for almost every $t,(s,\omega)\mapsto\tilde{f}(t,s,\omega)$ is progressively measurable, and from (A.3) $P(\int_0^T|\tilde{f}(t,s,\omega)|^2\,ds<\infty)=1$. Hence the right-hand side of (A.4) is well-defined for almost every $t\in[0,T]$.

Proof Corollary A.5 is certainly true for simple processes. To prove the general case, let $f^{(n_i)}$ be the approximating sequence of processes obtained by Lemma A.3. Then pass to the limit using almost surely converging subsequences and the Burkholder–Davis–Gundy inequality for Hilbert space valued integrands (see Remark 2.4(i)).

PROPOSITION A.6 *Let* $u\in\mathbf{L}^a_{1,1}$. *Then* $\int_0^T u(s)\,dW(s)\in\mathbf{D}_{1,1}$, *and*

$$D_t\int_0^T u(s)\,dW(s)=\int_0^T D_t u(s)\,dW(s)+u(t). \tag{A.5}$$

Proof Eq. (A.5) is written as an identity between processes. However, from Corollary A.5 and the remarks preceding Proposition 2.3, Eq. (A.5) is equivalent to the identity between $(L^2([0,T]))^d$-valued random vectors:

$$D\int_0^T u(s)\,dW(s)=\int_0^T Du(s)\,dW(s)+u(\cdot). \tag{A.6}$$

We shall prove (A.6). Let $u^{n_i}(s)=u(\tilde{s}+\psi_{n_i}(s-\tilde{s}))$ be chosen as in the proof of Corollary A.4, such that $u^{n_i}(s)\in\mathbf{D}_{1,1}$ for every s, for all n_i, and

$$|||u^{n_i}-u|||^a_{1,1}\to 0 \quad\text{as}\quad n_i\to\infty. \tag{A.7}$$

Now observe that

$$\int_0^T (u^n(s))^*\,dW(s)=\sum_j u^*\left(\tilde{s}+\frac{j}{2^n}\right)\left[W\left(\tilde{s}+\frac{j+1}{2^n}\right)-W\left(\tilde{s}+\frac{j}{2^n}\right)\right]. \tag{A.8}$$

Because $u(\tilde{s}+j/2^n)\in\mathbf{D}_{1,1}$ for each j, and the random vectors $u(\tilde{s}+j/2^n)$, $W(\tilde{s}+(j+1)/2^n)-W(\tilde{s}+j/2^n)$ are independent, Lemma A.1 implies that each term on the right-hand side of (A.8) is in $\mathbf{D}_{1,1}$ and

$$D\left(u^*\left(\tilde{s}+\frac{j}{2^n}\right)[W(\tilde{s}+(j+1)/2^n)-W(\tilde{s}+j/2^n)]\right)$$

$$=\left[Du\left(\tilde{s}+\frac{j}{2^n}\right)\right]\left(W(\tilde{s}+(j+1)/2^n)-W(\tilde{s}+j/2^n]\right)$$

$$+u(\tilde{s}+j/2^n)1_{(\tilde{s}+j/2^n,\tilde{s}+(j+1)/2^n]}(\cdot).$$

It follows that

$$D\left(\int_0^T (u^n(s))^* \, dW(s)\right)=\int_0^T Du^n(s)\, dW(s)+u^n(\cdot).$$

Now, by the Burkholder–Davis–Gundy inequalities (2.15) and (2.16), we have

$$E\left|\int_0^T (u^{n_i}(s))^* \, dW(s)-\int_0^T u^*(s)\, dW(s)\right|\leqq cE\left(\int_0^T |u^{n_i}(s)-u(s)|^2 \, ds\right)^{1/2}$$

$$\leqq c\,|||u^{n_i}-u|||^a_{1,1}\to 0, \quad \text{as} \quad n_i\to\infty,$$

and

$$E\left\|\int_0^T Du(s)\, dW(s)+u(\cdot)-\int_0^T Du^{n_i}(s)\, dW(s)-u^{n_i}(\cdot)\right\|$$

$$\leqq cE\left\{\left(\int_0^T |u^{n_i}(s)-u(s)|^2 \, ds\right)^{1/2}\right\}+cE\left\{\left(\int_0^T \|Du(s)-Du^{n_i}(s)\|^2 \, ds\right)^{1/2}\right\}$$

$$=c\,|||u^{n_i}-u|||^a_{1,1}\to 0, \quad \text{as} \quad n_i\to\infty.$$

It follows that $\int_0^T u^*(s)\, dW(s)\in \mathbf{D}_{1,1}$, and that $D\int_0^T u^*(s)\, dW(s)$ is given by Eq. (A.6).

COROLLARY A.7 *$\int_0^T D_t u(s)\, dW(s)=\int_t^T D_t u(s)\, dW(s)$ for almost every t. Indeed, $D_t u(s,\omega)=0$ for $t>s$ for $ds\otimes dP$-a.e. (s,ω).*

Proof Let u^{n_i} be defined as in the proof of Proposition A.6. Then by the adaptedness of u, $D_t u^{n_i}(s,\omega)=0$, for $T\geqq t>s$. for $ds\otimes dP$-almost every (s,ω). Thus $\int_0^T D_t u^{n_i}(s)\, dW(s)=\int_t^T D_t u^{n_i}(s)\, dW(s)$ for every n_i. Now take limits as $n_i\to\infty$.