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EQUILIBRIUM MODELS WITH SINGULAR ASSET PRICES

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General equilibrium models in which economic agents have finite marginal utility from consumption at the origin lead to financial assets having continuous prices with singular components. In particular, there is no bona fide "interest rate" in such models, although asset prices can be determined by equilibrium considerations (and uniquely, up to the formation of mutual funds). The singularly continuous processes in question charge precisely the set of time points at which some agent "drops out" of the economy, or "comes back" into it, between intervals of zero consumption. Not surprisingly, these processes are governed by local time.

KEYWORDS: equilibrium analysis, financial economics, stochastic calculus

1. INTRODUCTION

A primary objective of consumption-based, capital asset pricing theory has been to model the relationship between rates of return and aggregate consumption. In continuous-time models, a number of researchers (e.g., Merton 1973, Breeden 1979, 1986, Cox, Ingersoll, and Ross 1985a,b, Lucas 1978, and Duffie and Zame 1989) have studied this relationship. Two key results which emerge from these papers are that, in equilibrium,

(1.1) The rate of return from a riskless asset should be the negative of the growth rate of the marginal utility for consumption of a representative agent.

(1.2) The excess (above the risk-free rate) rate of return from a risky asset should be proportional to the covariance between the price of that asset and the aggregate consump-

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tion, with the constant of proportionality independent of the asset and equal to the relative index of risk-aversion for a representative agent.

Under some regularity conditions, including the strict positivity of optimal consumption processes, equilibrium prices in continuous-time capital asset pricing models have been shown to enjoy these two properties, but the existence of suitable equilibrium prices in a multiagent economy has until recently been an open question. Price processes reside in an infinite-dimensional space, and one method of proving existence of equilibrium in such spaces is based on a fixed-point result of Mas-Colell (1986) (see, e.g., Duffie 1986). Mas-Colell's theorem assumes "uniform properness" of utility functions, which in the time-additive case requires finite marginal utility at the zero level of consumption. On the other hand, the derivations of statements (1.1) and (1.2) require positivity of consumption at all times, a situation which is known not to prevail when the marginal utility of zero consumption is finite. Araujo and Monteiro (1989a,b) have obtained equilibrium without assuming "uniform properness," but the ramifications of their results for continuous-time, capital asset pricing theory have yet to be explored.

This paper concerns the existence of equilibrium and the extent to which (1.1) and (1.2) hold. *The main result is that equilibrium does exist, but if some agents have finite marginal utility at zero while others do not, then the riskless asset can fail to have a rate of return in the traditional sense*; i.e., there may be no processes $r(t)$ such that the price $P_0(t)$ of the riskless asset satisfies

$$P_0(t) = P_0(0) \exp\left(\int_0^t r(s)\, ds\right). \tag{1.3}$$

However, (1.1) holds in a more general sense, made precise in Remark 8.2. Likewise, the price processes of the risky assets may not have rates of return in the traditional sense. Nevertheless, the difference between any risky asset and the riskless asset will have a traditional rate of return, and if we define the "excess rate of return" to be the rate for this difference, then (1.2) holds. All these difficulties arise because some agents may see their optimal consumption fall to zero. If assumptions are made to prevent this, then a process $r(t)$ satisfying (1.3) can be found, and the characterizations (1.1) and (1.2) hold.

Duffie and Zame (1989) were the first to prove the existence of an equilibrium satisfying (1.1) and (1.2) in a continuous-time, consumption-based, capital asset pricing model. They assumed infinite marginal utility at zero for every agent and avoided Mas-Colell's uniform properness condition by a functional analytic argument. Consequently, the anomaly addressed here did not arise. Duffie and Zame's model also included a *spot price process,* which denominated the consumption good in terms of a "numeraire." Such a process obscures the difficulty we address here because rates of return for assets denominated in terms of a numeraire can exist even when their rates denominated in terms of the consumption good fail to exist.

Karatzas, Lehoczky, and Shreve (1990) established the existence of equilibrium by reducing the problem to a finite-dimensional fixed-point problem. (Some of the results of Karatzas et al. have been sharpened by Dana and Pontier 1990.) The variables in the finite-dimensional problem of Karatzas et al. are the weights needed to form the appropriate representative agent, an idea known as the "Negishi method" and borrowed in the present context from Huang (1987). The method does not require any conditions on marginal utilities at zero, but existence of equilibrium is obtained only if the model includes a spot price process. (Such a model is referred to as the *moneyed model* in Karatzas et al.) In the model without a spot price (the *moneyless model*), equilibrium is obtained in Karatzas et al. only when all agents have infinite marginal utility at zero.

In this paper we consider a multiagent model without a spot price and with no condition on marginal utilities at zero. For the sake of simplicity, we set up the model in a pure-exchange economy; it is not difficult to combine this paper with Karatzas, Lehoczky, and Shreve (1990) to obtain analogous results for a production economy. Martingale methods are used to solve the optimization problems for the individual agents, so there is no need to introduce a state vector or otherwise attempt to create Markov processes. This was also the case in Duffie (1986), Duffie and Zame (1989), and Karatzas, Lehoczky, and Shreve (1990), but not in previous equilibrium papers.

To obtain equilibrium, we postulate at the outset a riskless asset (called a *bond*) whose price is continuous and of bounded variation, but which is not necessarily absolutely continuous. Thus, there is no "interest rate" which can be used to recover the price process for this bond. The prices of the risky assets (called *stocks*) are continuous, positive semimartingales. In particular, the bounded variation part of these processes may have singularly continuous components. We assume that the bounded variation parts of *discounted* stock price processes are absolutely continuous; discounting is accomplished through division by the bond price. We show in Section 11 that failure of this assumption would allow arbitrage. Following Karatzas, Lehoczky, and Shreve (1990), we reduce the equilibrium problem to a finite-dimensional fixed-point problem, the solution of which allows us to define a representative agent utility function. Related to the fact that some agents can see their optimal consumption fall to zero, this representative agent utility function may have a discontinuous first derivative. Itô rule computations for such a function introduce semimartingale local times, and these in turn lead to singularly continuous components in the asset prices. Section 8 provides formulas for the equilibrium asset prices and interprets them in light of (1.1) and (1.2). Section 9 shows by example that the singularly continuous components in the asset prices can be nontrivial, even in apparently innocuous situations, with no singular component in the aggregate endowment process.

Finally, we note that by allowing asset prices to be possibly discontinuous semimartingales, Back (1990) proposes a consumption-based, capital asset pricing model even more general than ours. In this context, and under the assumption of existence of equilibrium, he obtains a counterpart to (1.2). Our paper presents a rationale for moving at least some distance from the traditional model (with asset price processes whose bounded variation parts are absolutely continuous) in the direction of the one proposed by Back.

2. AGENTS AND ENDOWMENT PROCESSES

We consider an economy consisting of N agents. Each agent receives an *exogenous endowment process* $\varepsilon_n = \{\varepsilon_n(t); 0 \le t \le T\}$ which is positive and progressively measurable with respect to the filtration $\{\mathcal{F}_t\}$. We assume throughout that $\{\mathcal{F}_t\}$ is the augmentation by null sets of the natural filtration

$$\mathcal{F}_t^W = \sigma(W(s); 0 \le s \le t), \qquad t \in [0, T],$$

generated by a d-dimensional Brownian motion $W(t) = (W_1(t), \dots, W_d(t))^*$ on the complete probability space $(\Omega, \mathcal{F}, P)$ where $*$ denotes transposition. All economic activity takes place on the finite horizon $[0, T]$.

The aggregate endowment $\varepsilon(t) \triangleq \sum_{n=1}^{N} \varepsilon_n(t)$ will be assumed to be a continuous semi-martingale of the form

$$(2.1) \quad \varepsilon(t) = \varepsilon(0) + \int_0^t \varepsilon(s)a(s)\, d\xi(s) + \int_0^t \varepsilon(s)\nu(s)\, ds + \int_0^t \varepsilon(s)\rho^*(s)\, dW(s).$$

Here ξ is an $\{\mathcal{F}_t\}$-adapted process with paths which are continuous but singular with respect to Lebesgue measure and of bounded variation on $[0, T]$, and ρ, ν, a are bounded, $\{\mathcal{F}_t\}$-progressively measurable processes with values in $\mathbb{R}^d$, $\mathbb{R}$, and $\mathbb{R}$, respectively. We assume that there are positive, finite constants $k < K$ such that

$$(2.2) \qquad k \le \varepsilon(t) \le K, \qquad \forall t \in [0, T],$$

holds almost surely. To establish the uniqueness of equilibrium, we also need the condition

$$(2.3) \qquad \varepsilon_n(t) > 0 \quad \text{a.s.}, \qquad \forall t \in [0, T] \text{ and } n = 1, \dots, N.$$

REMARK 2.1 The term $\int_0^t \varepsilon(s)a(s)\, d\xi(s)$ is included in (2.1) only because it affords extra generality and requires no additional analysis. The nonexistence of an interest rate for the equilibrium bond price *can* occur even when this term is excluded; see the example of Section 9, where $a \equiv 0$ and the interest rate fails to exist.

3. UTILITY FUNCTIONS

Each agent is endowed with a *utility function* $U_n\colon (0, \infty) \to \mathbb{R}$ which is of class C^3, strictly increasing and strictly concave, and satisfies $U_n'(\infty) \triangleq \lim_{c\to\infty} U_n'(c) = 0$. From strict concavity, we have that $U'(0) \triangleq \lim_{c\downarrow 0} U_n')(c)$ exists in $(0, \infty]$.

For the uniqueness of equilibrium, we also need the condition

$$(3.1) \qquad c \mapsto cU_n'(c) \text{ is nondecreasing}, \qquad \forall n = 1, \dots, N.$$

This condition amounts to assuming that $-cU_n''(c)/U_n'(c)$, the Arrow-Pratt measure of relative risk aversion, is less than or equal to unity.

We denote by I_n the inverse of the function U_n'; this is a strictly decreasing mapping of $(0, U_n'(0))$ onto $(0, \infty)$, and we extend it on all of $(0, \infty)$ by setting $I_n(y) = 0$ for $y \ge U_n'(0)$.

In this model, agents derive utility by consuming parts of the aggregate commodity endowment. Because such endowments will typically be random and time-varying, the agents will find it useful to participate in a market which allows them both to hedge their risk and smooth out their consumption. A model for such a market is introduced in the next section; its coefficients will be determined in Section 8, by equilibrium considerations, in terms of the endowment processes and utility functions of the individual agents.

4. A FINANCIAL MARKET WITH SINGULAR BOND PRICES

A financial market with singular bond prices has $d + 1$ assets; one of them is a pure discount *bond*, with price $P_0(t)$ at time t which satisfies

$$dP_0(t) = P_0(t)[r(t)\,dt + dA(t)], \qquad P_0(0) = 1. \tag{4.1}$$

The remaining assets are risky *stocks*, with prices per share $P_i(t)$ given by

$$dP_i(t) = P_i(t)[b_i(t)\,dt + dA_i(t) + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)], \qquad 1 \le i \le d. \tag{4.2}$$

The processes $r(\cdot)$, $A(\cdot)$, $b_i(\cdot)$, $A_i(\cdot)$, and $\sigma_{ij}(\cdot)$ will be referred to collectively as the *coefficients* of the model. They are all $\{\mathcal{F}_t\}$-progressively measurable. The processes $r(\cdot)$, $b_i(\cdot)$, and $\sigma_{ij}(\cdot)$ are bounded uniformly in (t, ω), the matrix $\sigma(t) = \{\sigma_{ij}(t)\}_{1 \le i,j \le d}$ satisfies the strong nondegeneracy condition

$$z^*\sigma(t)\sigma^*(t)z \ge \delta\|z\|^2, \qquad \forall t \in [0, T], \quad \forall z \in \mathbb{R}^d \tag{4.3}$$

almost surely (for some given $\delta > 0$), and the processes A, A_i have P-almost every path continuous, of bounded variation on $[0, T]$ (uniformly in ω), and *singular* with respect to Lebesgue measure, with $A(0) = A_i(0) = 0$, $i = 1, \ldots, d$.

We see in Section 10 that we have to assume

$$A_i(t) = A(t), \quad \forall t \in [0, T], \quad i = 1, \ldots, d, \tag{4.4}$$

almost surely, to exclude arbitrage opportunities. This condition will be imposed from now on.

The "relative risk process"

$$\theta(t) \triangleq (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}], \qquad 0 \le t \le T, \tag{4.5}$$

is important in the sequel. It is progressively measurable with respect to $\{\mathcal{F}_t\}$ and, thanks to (4.3), bounded. (Here and elsewhere we denote $\mathbf{1} = (1, \ldots, 1)^*$.

5. PORTFOLIO AND CONSUMPTION POLICIES

Each agent may choose an $\mathbb{R}^d$-valued *portfolio process* $\pi_n(t) = (\pi_{n1}(t), \ldots, \pi_{nd}(t))^*$, and a nonnegative *consumption rate process* $c_n(t)$, $0 \le t \le T$; these processes are $\{\mathcal{F}_t\}$-progressively measurable, and satisfy $\int_0^T \{\|\pi_n(t)\|^2 + c_n(t)\}\,dt < \infty$, a.s. For every such pair (π_n, c_n), the corresponding wealth process X_n has initial value $X_n(0) = 0$ and obeys the equation[3]

$$dX_n(t) = \sum_{i=1}^{d} \pi_{ni}(t)\left[b_i(t)\,dt + dA_i(t) + \sum_{j=1}^{d} \sigma_{ij}(t)\,dW_j(t)\right]$$

[3]The interpretation here is that $\pi_{ni}(t)$ represents the amount invested by the nth agent in the ith stock, at time t, for $i = 1, \ldots, d$; the amount $X_n(t) - \sum_{i=1}^{d} \pi_i(t)$ is invested in the bond.

$$
\begin{aligned}
&+ \Big(X_n(t) - \sum_{i=1}^{d} \pi_{ni}(t)\Big)[r(t)\,dt + dA(t)] + (\varepsilon_n(t) - c_n(t))\,dt \\
(5.1)\qquad &= X_n(t)[r(t)\,dt + dA(t)] + (\varepsilon_n(t) - c_n(t))\,dt + \sum_{i=1}^{d} \pi_{ni}(t)\,dG_i(t) \\
&+ \pi_n^*(t)[(b(t) - r(t)\mathbf{1})\,dt + \sigma(t)\,dW(t)],
\end{aligned}
$$

where $b(t) \triangleq (b_1(t), \ldots, b_d(t))^*$, and $G_i(t) \triangleq A_i(t) - A(t)$ for $i = 1, \ldots, d$.

Recall the process θ of (4.5) and introduce the exponential martingale

$$
(5.2)\qquad Z(t) \triangleq \exp\left[-\int_0^t \theta^*(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2\,ds\right], \qquad 0 \le t \le T.
$$

According to the Girsanov theorem,

$$
(5.3)\qquad \tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)\,ds, \qquad 0 \le t \le T,
$$

is then a Brownian motion under the new probability measure $\tilde{P}(A) = E[Z(T)1_A]$, $A \in \mathcal{F}_T$ (cf. Karatzas and Shreve 1988, Section 3.5). With this notation, and taking (4.4) into account, the solution of (5.1) is given by

$$
\begin{aligned}
\beta(t)X_n(t) &= \int_0^t \beta(s)(\varepsilon_n(s) - c_n(s))\,ds \\
(5.4)\qquad &+ \int_0^t \beta(s)\pi_n^*(s)\sigma(s)\,d\tilde{W}(s), \qquad 0 \le t \le T,
\end{aligned}
$$

where

$$
(5.5)\qquad \beta(t) \triangleq \frac{1}{P_0(t)} = \exp\left\{-\int_0^t r(s)\,ds - A(t)\right\}.
$$

REMARK 5.1. The martingale Z of (5.2) satisfies the equation

$$
(5.6)\qquad Z(t) = 1 - \int_0^t Z(s)\theta^*(s)\,dW(s).
$$

Applying integration by parts to the product of βX_n and Z yields, in conjunction with (5.4) and (5.6),

$$
\begin{aligned}
\zeta(t)X_n(t) &= \int_0^t \zeta(s)[\varepsilon_n(s) - c_n(s)]\,ds \\
(5.7)\qquad &+ \int_0^t \zeta(s)[\sigma^*(s)\pi_n(s) - X_n(s)\theta(s)]^*\,dW(s), \qquad 0 \le t \le T.
\end{aligned}
$$

Here

$$\zeta(t) \triangleq \beta(t)Z(t), \tag{5.8}$$

and it is easily verified that (5.7) is actually equivalent to (5.4). □

We assume for the present that the process ζ of (5.8) satisfies the condition

$$0 < \delta \leq \zeta(t) \leq \Delta, \qquad \forall 0 \leq t \leq T, \tag{5.9}$$

almost surely, for some finite constants $\Delta > \delta > 0$. This assumption will be justified at the end of Section 7.

Definition 5.1. The portfolio/consumption process pair (π_n, c_n) for the nth agent is called *admissible* if the corresponding wealth process of (5.4) satisfies

$$\beta(t)X_n(t) + \tilde{E}\left(\int_t^T \beta(s)\varepsilon_n(s)\, ds \mid \mathcal{F}_t\right) \geq 0, \qquad \forall t \in [0, T],$$

or, equivalently (by virtue of the Bayes' rule, p. 193 in Karatzas and Shreve 1988),

$$\zeta(t)X_n(t) + E\left(\int_t^T \zeta(s)\varepsilon_n(s)\, ds \mid \mathcal{F}_t\right) \geq 0, \qquad \forall\, t \in [0, T], \tag{5.10}$$

almost surely. □

In particular, it follows from (5.7) to (5.10) and (2.2) that for an admissible pair (π_n, c_n), the process

$$\zeta(t)X_n(t) + \int_0^t \zeta(s)(c_n(s) - \varepsilon_n(s))\, ds, \qquad 0 \leq t \leq T,$$

is a local martingale bounded from below. It is, therefore, also a supermartingale with initial value equal to zero, and this implies

$$\begin{aligned} E\int_0^T \zeta(s)c_n(s)\, ds &\leq E\left[\zeta(T)X_n(T) + \int_0^T \zeta(s)c_n(s)\, ds\right] \\ &\leq E\int_0^T \zeta(s)\varepsilon_n(s)\, ds. \end{aligned} \tag{5.11}$$

PROPOSITION 5.1. *Let $\hat{c}_n$ be a consumption process which satisfies*

$$E\int_0^T \zeta(s)\hat{c}_n(s)\, ds = E\int_0^T \zeta(s)\varepsilon_n(s)\, ds. \tag{5.12}$$

Then there exists a portfolio process $\hat{\pi}_n$ such that the pair $(\hat{\pi}_n, \hat{c}_n)$ is admissible, and the corresponding wealth process $\hat{X}_n$ is given by

$$(5.13)\qquad \beta(t)\hat{X}_n(t) = \tilde{E}\left[\int_t^T \beta(s)(\hat{c}_n(s) - \varepsilon_n(s))\,ds \mid \mathcal{F}_t\right], \qquad 0 \le t \le T.$$

Proof. According to (5.12), the $\tilde{P}$-martingale

$$(5.14)\qquad M_n(t) \triangleq \tilde{E}\left[\int_0^T \beta(s)(\hat{c}_n(s) - \varepsilon(s))\,ds \mid \mathcal{F}_t\right], \qquad 0 \le t \le T,$$

has zero expectation; from the fundamental martingale representation theorem, it admits the stochastic integral representation

$$(5.15)\qquad M_n(t) = \int_0^t \beta(s)\pi_n^*(s)\sigma(s)\,d\tilde{W}(s)$$

for some portfolio process π_n (cf. Karatzas and Shreve 1988, Problem 3.4.16 and proof of Proposition 5.8.6). It follows then, from (5.4), (5.14), and (5.15), that the wealth process $\hat{X}_n$ corresponding to $(\hat{\pi}_n, \hat{c}_n)$ is given by (5.13) and satisfies the admissibility requirement of Definition 5.1. □

6. THE n^{TH} AGENT'S OPTIMIZATION PROBLEM

Each agent's goal is to maximize the expected discounted utility from consumption

$$(6.1)\qquad E\int_0^T \exp\left\{-\int_0^t \mu(s)\,ds\right\} U_n(c_n(t))\,dt,$$

over all admissible pairs (π_n, c_n) which satisfy

$$(6.2)\qquad E\int_0^T \exp\left\{-\int_0^t \mu(s)\,ds\right\}\max[0, -U_n(c_n(t))]\,dt < \infty.$$

Here $\mu : [0, T] \to \mathbb{R}$ is a given bounded, measurable function. A pair $(\hat{\pi}_n, \hat{c}_n)$ that achieves the supremum of (6.1) over such pairs is called *optimal.*

We can describe the optimal $(\hat{\pi}_n, \hat{c}_n)$ in the manner of Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) as follows: there is a unique positive number y_n for which

$$(6.3)\qquad E\int_0^T \zeta(t)I_n\left(y_n\zeta(t)\exp\left\{\int_0^t \mu(s)\,ds\right\}\right)dt = E\int_0^T \zeta(t)\varepsilon_n(t)\,dt.$$

Then the consumption process

$$\hat{c}_n(t) \triangleq I_n\left(y_n\zeta(t)\exp\left\{\int_0^t \mu(s)\,ds\right\}\right), \qquad 0 \le t \le T, \tag{6.4}$$

satisfies (5.12), and from Proposition 5.1 there exists a portfolio process $\hat{\pi}_n$ such that $(\hat{\pi}_n, \hat{c}_n)$ is admissible, with associated wealth process given by (5.13).

For any other admissible pair (π_n, c_n), the elementary consequence of concavity,

$$U_n(I_n(y)) \ge U_n(c) + y[I_n(y) - c], \qquad \forall y \in [0, \infty), \quad c \in [0, \infty),$$

gives (when applied to $y = y_n\,\zeta(t)\exp\{\int_0^t \mu(s)\,ds\}$ and $c = c_n(t)$), after multiplying by $\exp\{-\int_0^t \mu(s)\,ds\}$ and integrating $dt \times dP$,

$$E\int_0^T \exp\left\{-\int_0^t \mu(s)\,ds\right\}U_n(\hat{c}_n(t))\,dt - E\int_0^T \exp\left\{-\int_0^t \mu(s)\,ds\right\}U_n(c_n(t))\,dt$$

$$\ge y_n\left[E\int_0^T \zeta(t)\hat{c}_n(t)\,dt - E\int_0^T \zeta(t)c_n(t)\,dt\right].$$

But this last term is nonnegative, thanks to (5.11) and (5.12), and the optimality of $(\hat{\pi}_n, \hat{c}_n)$ follows. (By taking $c_n(t)$ to be a suitable constant in the above argument, we see that $\hat{c}_n(\cdot)$ satisfies the requirement (6.2).)

7. EQUILIBRIUM AND THE "REPRESENTATIVE AGENT"

We say that the financial market of Section 4 results in *equilibrium* if, in the notation of Section 5, we have the following conditions:

(i) *Clearing of the commodity market:*

$$\sum_{n=1}^N \hat{c}_n(t) = \varepsilon(t), \qquad 0 \le t \le T, \tag{7.1}$$

(ii) *Clearing of the stock market:*

$$\sum_{n=1}^N \hat{\pi}_{ni}(t) = 0, \qquad 0 \le t \le T, \quad i = 1, \ldots, d, \tag{7.2}$$

(iii) *Clearing of the bond market:*

$$\sum_{n=1}^N \hat{X}_n(t) = 0, \qquad 0 \le t \le T. \tag{7.3}$$

In this context, $\hat{c}_n$, $\hat{\pi}_n$, and $\hat{X}_n$ denote the optimal processes for the nth agent.

PROPOSITION 7.1. *Conditions* (7.1)–(7.3) *lead to the a.s. identity*

$$\varepsilon(t) = \sum_{n=1}^{N} I_n \left(y_n \zeta(t) \exp\left\{ \int_0^t \mu(s)\, ds \right\} \right), \qquad 0 \le t \le T, \tag{7.4}$$

where y_n *is defined by* (6.3) *for* $n = 1, \ldots, N$.

Conversely, suppose there exists a financial market for which the process ζ *of* (5.8) *satisfies* (7.4) *and* (6.3) *for suitable positive numbers* $y_1, \ldots, y_N$. *Then this financial market results in equilibrium.*

Proof. For the first claim, simply observe that (7.4) follows from (7.1) and (6.4). For the converse, note that for the financial market in question the optimal consumption processes $\{\hat{c}_n\}_{n=1}^N$ are again given by (6.4), and the corresponding wealth processes $\{\hat{X}_n\}_{n=1}^N$ by (5.13). Condition (7.1) follows directly from (7.4) and (6.4) and leads, in conjunction with (5.13) and (5.14), to (7.3) and $\Sigma_{n=1}^N \hat{M}_n(t) \equiv 0$, respectively. Now this last condition, together with (5.15) and the nondegeneracy of σ^*, gives (7.2). □

To facilitate the search for an equilibrium financial market, let us introduce for every vector $\Lambda \in (0, \infty)^N$ the function

$$U(c; \Lambda) \triangleq \max_{\substack{c_1 \ge 0, \ldots, c_N \ge 0 \\ \Sigma_{n=1}^N c_n = c}} \sum_{n=1}^{N} \lambda_n U_n(c_n), \qquad 0 < c < \infty. \tag{7.5}$$

It can be seen as in Karatzas, Lehoczky and Shreve (1990, Section 10) that the maximum is achieved at

$$\bar{c}_n = I_n\left(\frac{H(c; \Lambda)}{\lambda_n} \right), \qquad n = 1, \ldots, N, \tag{7.6}$$

where $H(\cdot; \Lambda)$ is the inverse of the continuous, decreasing function

$$I(y; \Lambda) \triangleq \sum_{n=1}^{N} I_n\left(\frac{y}{\lambda_n} \right), \qquad 0 < y < \infty. \tag{7.7}$$

Thus,

$$U(c; \Lambda) = \sum_{n=1}^{N} \lambda_n U_n\left(I_n\left(\frac{H(c; \Lambda)}{\lambda_n} \right) \right),$$

and it follows from this representation that $U(\cdot; \Lambda)$ is continuous and continuously differentiable on $(0, \infty)$ with $U'(c; \Lambda) = H(c; \Lambda)$, and of class C^3 away from the set

$$\mathcal{A} = \{\alpha_1, \ldots, \alpha_N\}, \qquad \text{with } \alpha_n \triangleq I(\lambda_n U_n'(0); \Lambda). \tag{7.8}$$

We interpret the function $U(\cdot;\ \Lambda)$ of (7.5) as the *utility function of a representative agent,* who assigns weights $\lambda_1, \ldots, \lambda_N$ to the individual agents in the economy.

The problem of equilibrium can then be cast as that of determining the "right" way to assign these weights. Indeed, with the identification $\Lambda = (\lambda_1, \ldots, \lambda_N) = (1/y_1, \ldots, 1/y_N)$, (7.4) and (6.3) can be written as

$$\zeta(t) = \exp\left\{-\int_0^t \mu(s)\, ds\right\} U'(\varepsilon(t);\ \Lambda), \qquad 0 \le t \le T, \tag{7.9}$$

$$\begin{aligned} &E\int_0^T \exp\left\{-\int_0^t \mu(s)\, ds\right\} U'(\varepsilon(t);\ \Lambda)\, I_n\left(\frac{1}{\lambda_n} U'(\varepsilon(t);\ \Lambda)\right) dt \\ &\quad = E\int_0^T \exp\left\{-\int_0^t \mu(s)\, ds\right\} U'(\varepsilon(t);\ \Lambda)\varepsilon_n(t)\, dt, \qquad n = 1, \ldots, N, \end{aligned} \tag{7.10}$$

and *constructing equilibrium is equivalent to finding a vector* $\Lambda \in (0, \infty)^N$ *which satisfies* (7.10).

Once such a vector has been found, the process ζ of the corresponding financial market is given by (7.9) and satisfies the requirement (5.9), thanks to the assumption (2.2) and the continuity of $U'(\cdot;\ \Lambda)$. The optimal consumption processes of the individual agents are given by (6.4) as

$$\hat{c}_n(t;\ \Lambda) \triangleq I_n\left(\frac{1}{\lambda_n} U'(\varepsilon(t);\ \Lambda)\right), \qquad 0 \le t \le T, \quad n = 1, \ldots, N \tag{7.11}$$

(see also (7.6)).

8. CHARACTERIZATION OF EQUILIBRIUM ASSET PRICES

We quote from Karatzas, Lehoczky and Shreve (1990, Theorem 11.1) the following fundamental result.

THEOREM 8.1. *There exists a vector* $\Lambda \in (0, \infty)^N$ *which satisfies* (7.10). *Furthermore, if the endowment processes satisfy* (2.3) *and all utility functions* $\{U_n\}_{n=1}^N$ *obey* (3.1), *this vector is unique up to a multiplicative constant.*

Consider now the process $\eta(t) \triangleq \zeta(t) \exp\{\int_0^t \mu(s)\, ds\}$; from (5.8), (5.5), and (5.6) it follows that η satisfies the stochastic integral equation

$$\begin{aligned} \eta(t) = 1 &+ \int_0^t \eta(s)\{\mu(s) - r(s)\}\, ds - \int_0^t \eta(s)\, dA(s) \\ &- \int_0^t \eta(s)\theta^*(s)\, dW(s). \end{aligned} \tag{8.1}$$

On the other hand, (7.9) gives $\eta(t) = U'(\varepsilon(t); \Lambda)$; apply the generalized Itô rule for convex functions of semimartingales (e.g., Karatzas and Shreve 1988, Chapter 3, Theorems 6.22 and 7.1, and Problem 6.24) to obtain

$$\begin{aligned} \eta(t) &= U'(\varepsilon(0); \Lambda) \\ &\quad + \int_0^t \left[U''(\varepsilon(s); \Lambda)\varepsilon(s)\nu(s) + \frac{1}{2} U'''(\varepsilon(s); \Lambda)\varepsilon^2(s)\|\rho(s)\|^2\right] ds \\ &\quad + \int_0^t U''(\varepsilon(s); \Lambda)\varepsilon(s)a(s)\, d\xi(s) + \int_0^t U''(\varepsilon(s); \Lambda)\varepsilon(s)\rho^*(s)\, dW(s) \\ &\quad + \sum_{n=1}^N [U''(\alpha_n+; \Lambda) - U''(\alpha_n-; \Lambda)]\, L_t(\alpha_n) \end{aligned} \tag{8.2}$$

in conjunction with (2.1), where $L_t(\alpha)$ is the local time at α for the semimartingale ε, accumulated up to time t.

We can identify now various terms in the two semimartingale decompositions (8.1) and (8.2) for the same process η to get

$$U'(\varepsilon(0); \Lambda) = 1, \tag{8.3}$$

$$r(t) = \mu(t) - \frac{U''(\varepsilon(t); \Lambda)\varepsilon(t)\nu(t) + (1/2)U'''(\varepsilon(t); \Lambda)\|\rho(t)\|^2\varepsilon^2(t)}{U'(\varepsilon(t); \Lambda)}, \tag{8.4}$$

$$\theta(t) = -\frac{U''(\varepsilon(t); \Lambda)}{U'(\varepsilon(t); \Lambda)}\varepsilon(t)\rho(t), \tag{8.5}$$

and

$$\begin{aligned} A(t) &= -\int_0^t \frac{U''(\varepsilon(s); \Lambda)}{U'(\varepsilon(s); \Lambda)}\varepsilon(s)a(s)\, d\xi(s) \\ &\quad - \sum_{n=1}^N \frac{U''(\alpha_n+; \Lambda) - U''(\alpha_n-; \Lambda)}{U'(\alpha_n; \Lambda)} L_t(\alpha_n). \end{aligned} \tag{8.6}$$

Equation (8.6) shows that the singularly continuous component of the asset price process arises from two distinct sources: singularly continuous components in the agents' endowment streams and a lack of smoothness in the representative agent's utility function caused by individual agents making boundary consumption decisions. Condition (8.3) determines uniquely the vector Λ among those which satisfy (7.10). With Λ thus determined, (8.4)–(8.6) provide the equilibrium values for the processes r, θ, and A appearing in the financial market of Section 4. The equilibrium market is thus determined uniquely, *up to the formation of mutual funds* (in the sense that the coefficients b and σ are not individually determined, but only modulo the process $\theta(t) = (\sigma(t))^{-1}[b(t) - r(t)\mathbf{1}]$ that they give rise to).

REMARK 8.1. If $U_n'(0) = \infty$ for every $n = 1, \ldots, N$ and $a \equiv 0$ in (2.1), then (8.6) gives $A \equiv 0$. Formulas (8.4) and (8.5) for the equilibrium financial market model agree then with (11.8) and (11.9) of Karatzas, Lehoczky, and Shreve (1990). In this case, the process of r of (8.4) and (4.1) is a genuine interest rate.

On the other hand, if $U_n'(0) < \infty$ for some $n = 1, \ldots, N$, or if the process a in (2.1) is not identically equal to zero, then the resulting singularly continuous process A of (8.6) can be nontrivial as well. The resulting bond price process P_0, in the financial market model of Section 4, *does not then have a bona fide interest rate*. In the following section we present an example of this situation, with $N = 2$, $U_1'(0) = \infty$, $U_2'(0) < \infty$, and $a \equiv 0$.

REMARK 8.2. In light of (8.1), it is reasonable to define the growth rate of the marginal utility for consumption of the representative agent to be the stochastic differential

$$dG(t) \triangleq \frac{1}{U'(\varepsilon(t); \Lambda)} \Bigg[U''(\varepsilon(t); \Lambda)\varepsilon(t)\nu(t)\, dt + \frac{1}{2}U'''(\varepsilon(t); \Lambda)\varepsilon^2(t)\|\rho(t)\|^2 + U''(\varepsilon(t); \Lambda)\varepsilon(t)a(t)\, d\xi(t) + \sum_{n=1}^{N} (U''(\alpha_n+; \Lambda) - U''(\alpha_n-; \Lambda))\, dL_t(\alpha_n) \Bigg].$$

This quantity is equal to $-(r(t) - \mu(t))\, dt - dA(t)$. In particular, if there is no discounting ($\mu \equiv 0$),

$$\frac{dP_0(t)}{P_0(t)} = -dG(t), \tag{8.7}$$

which is a precise formulation of (1.1). From (8.5), we have

$$b(t) - r(t)\mathbf{1} = \sigma(t)\theta(t) = -\frac{\varepsilon(t)U''(\varepsilon(t); \Lambda)}{U'(\varepsilon(t); \Lambda)}\sigma(t)\rho(t), \tag{8.8}$$

a precise formulation of (1.2) (recall (4.4)).

9. AN EXAMPLE

Let us consider again the example of Section 11 in Karatzas, Lehoczky, and Shreve (1990); with $d = 1$, $N = 2$, and $\mu \equiv 0$, we take $U_1(c) = \log c$ and $U_2(c) = \log(1 + c)$. Then for any $\Lambda = (\lambda_1, \lambda_2) \in (0, \infty)^2$, we have

$$I(y; \Lambda) = \begin{cases} \dfrac{\lambda_1 + \lambda_2}{y} - 1, & 0 < y < \lambda_2, \\[2ex] \dfrac{\lambda_1}{y}, & y \geq \lambda_2, \end{cases} \tag{9.1}$$
$$H(c; \Lambda) = \begin{cases} \dfrac{\lambda_1}{c}, & 0 < c < \alpha(\Lambda), \\[2ex] \dfrac{\lambda_1 + \lambda_2}{1 + c}, & c \geq \alpha(\Lambda), \end{cases}$$

with $\alpha(\Lambda) = \lambda_1/\lambda_2$, and the numbers $\bar{c}_n$ of (7.6) are given by

$$\bar{c}_1(c;\Lambda) = \begin{cases} c, & 0 < c < \alpha(\Lambda), \\ \dfrac{\lambda_1(1+c)}{\lambda_1+\lambda_2}, & c \geq \alpha(\lambda), \end{cases} \tag{9.2}$$
$$\bar{c}_2(c;\Lambda) = \begin{cases} 0, & 0 < c < \alpha(\lambda), \\ \dfrac{\lambda_2(1+c)}{\lambda_1+\lambda_2} - 1, & c \geq \alpha(\Lambda). \end{cases}$$

The representative agent utility function $U(c;\Lambda) = \lambda_1 U_1(\bar{c}_1) + \lambda_2 U_2(\bar{c}_2)$ then becomes

$$U(c;\Lambda) = \begin{cases} \lambda_1 \log c, & 0 < c < \alpha(\lambda), \\ (\lambda_1+\lambda_2)\log(1+c) + \lambda_1 \log \dfrac{\lambda_1}{\lambda_1+\lambda_2} + \lambda_2 \log \dfrac{\lambda_2}{\lambda_1+\lambda_2}, & c \geq \alpha(\Lambda), \end{cases} \tag{9.3}$$

and we observe

$$U''(\alpha(\Lambda)+;\Lambda) - U''(\alpha(\Lambda)-;\Lambda) = \frac{\lambda_2^3}{\lambda_1(\lambda_1+\lambda_2)}. \tag{9.4}$$

For the aggregate endowment, we consider the process

$$\varepsilon(t) \triangleq 1 + \exp\left[W(t\wedge\tau) - \frac{(t\wedge\tau)^2}{2}\right], \qquad 0 \leq t \leq T,$$

with $\tau = \inf\{t \in [0,T]; W_t = 1\} \wedge T$, which is a continuous, bounded martingale and satisfies

$$d\varepsilon(t) = \varepsilon(t)\, 1_{\{t<\tau\}}\, dW(t), \qquad \varepsilon(0) = 2 \tag{9.5}$$

(i.e., (2.1) with $a \equiv 0$, $\nu \equiv 0$, $\rho(t) = \varepsilon(t)\, 1_{\{t\leq\tau\}}$), as well as (2.2). For a given number $k \in (0, 1)$ which also satisfies

$$kE\int_0^T \frac{2\varepsilon(t)}{1+\varepsilon(t)}\, dt > T, \tag{9.6}$$

we take

$$\varepsilon_1(t) \triangleq k\varepsilon(t), \qquad \varepsilon_2(t) \triangleq (1-k)\varepsilon(t). \tag{9.7}$$

With these choices, (7.9) and (7.10) become

$$\zeta(t) = U'(\varepsilon(t);\lambda), \tag{9.8}$$

$$\lambda_1 T = kE \int_0^T \zeta(t)\varepsilon(t)\, dt, \tag{9.9}$$

$$E \int_0^T \max[\lambda_2 - \zeta(t), 0]\, dt = (1 - k)E \int_0^T \zeta(t)\varepsilon(t)\, dt. \tag{9.10}$$

According to Theorem 8.1, there exists a *unique* $\Lambda \in (0, \infty)^2$ which satisfies (9.8)–(9.10) and $U'(1; \Lambda) = 2$. We henceforth deal with this Λ and denote the corresponding $\alpha(\Lambda) = \lambda_1/\lambda_2$ simply by α.

Suppose $\varepsilon(t) \le \alpha$, $\forall t \in [0, T]$ almost surely. Then from (9.3) and (9.8) we have $\zeta(t) = \lambda_1/\varepsilon(t)$, and (9.9) gives $k = 1$, a contradiction. On the other hand, suppose $\varepsilon(t) \ge \alpha$, $\forall t \in [0, T]$ almost surely; since $\varepsilon(\cdot)$ reaches values arbitrarily close to 1 with positive probability, we must have $\alpha \le 1$. Moreover, $\zeta(t) = (\lambda_1 + \lambda_2)/(1 + \varepsilon(t))$, and (9.8) and (9.9) give

$$\frac{\lambda_1 T}{\lambda_1 + \lambda_2} = kE \int_0^T \frac{\varepsilon(t)}{1 + \varepsilon(t)}\, dt,$$

which, in conjunction with (9.6), yields the contradiction $\lambda_1/\lambda_2 = \alpha > 1$. It develops from this analysis that *the process* $\varepsilon(\cdot)$ *crosses the level* α *during the interval* $[0, T]$, *with positive probability.*

From (8.4) to (8.6) we conclude that the equilibrium coefficients of the financial market are given by

$$r(t) = -\left[1_{\{\varepsilon(t)<\alpha\}} + \left(\frac{\varepsilon(t)}{1 + \varepsilon(t)}\right)^2 1_{\{\varepsilon(t)\ge\alpha\}}\right] 1_{\{t\le T\}}, \tag{9.11}$$

$$\theta(t) = \left[1_{\{\varepsilon(t)<\alpha\}} + \frac{\varepsilon(t)}{1 + \varepsilon(t)} 1_{\{\varepsilon(t)\ge\alpha\}}\right] 1_{\{t\le T\}}, \tag{9.12}$$

$$A(t) = -\frac{L_t(\alpha)}{\alpha(1 + \alpha)} \tag{9.13}$$

in this case. From the preceding analysis, it develops that the process (9.13) is nontrivial.

According to (6.4) and (9.2), the optimal consumption processes are given by

$$\hat{c}_1(t) = \varepsilon(t)\, 1_{\{\varepsilon(t)<\alpha\}} + \frac{\alpha(1 + \varepsilon(t))}{1 + \alpha} 1_{\{\varepsilon(t)\ge\alpha\}} \tag{9.14}$$

$$\hat{c}_2(t) = \frac{\varepsilon(t) - \alpha}{1 + \alpha} 1_{\{\varepsilon(t)\ge\alpha\}}. \tag{9.15}$$

REMARK 9.1. Note that $\{t \ge 0;\ \varepsilon(t) = \alpha\}$, the set of time points charged by the process A of (9.13), coincides with the set of time points at which switches from one régime to another occur in (9.14) and (9.15).

This actually holds in some generality; with $a \equiv 0$, the process A of (8.6) charges the set $\cup_{n=1}^{N} \{t \geq 0; \varepsilon(t) = \alpha_n\}$ and is flat away from it. Now for any fixed $n \in \{1, \ldots, N\}$ with $U_n'(0) < \infty$, the set $\{t \geq 0; \varepsilon(t) = \alpha_n\}$ is precisely the set of time points at which

$$\hat{c}_n(t) = I_n \left(\frac{1}{\lambda_n} U'(\varepsilon(t); \Lambda) \right),$$

the optimal consumption process for the nth agent, "switches from positive to zero value, or vice versa" (or, equivalently, the set of time points at which the nth agent "exits from" or "enters into" the economy). It is precisely at these instances of exit or entry that the singularly continuous process A makes itself felt. (Of course, when $\varepsilon(\cdot)$ has a nonzero diffusion coefficient ρ, these switches are not clean; every point of the set $\{t \geq 0; \varepsilon(t) = \alpha_n\}$ is a cluster point, and it is not possible in general to say, at any one of these points, whether the agent is exiting or entering the economy.)

10. APPENDIX

In this section we show that (4.4) or, equivalently,

(10.1) $G_i \triangleq A_i - A$ has almost all paths absolutely continuous with respect to Lebesgue measure, $\forall i = 1, \ldots, d,$

is *necessary for excluding arbitrage opportunities in the financial market of Section* 4. The sufficiency of (10.1) in this regard follows from (5.11).

Let us start by writing the solution of (5.1):

$$\begin{aligned} \beta(t)X_n(t) = & \int_0^t \beta(\theta)(\varepsilon_n(\theta) - c_n(\theta))\, d\theta \\ & + \sum_{i=1}^{d} \int_0^t \beta(\theta)\pi_{ni}(\theta)\, dG_i(\theta) + \int_0^t \beta(\theta)\pi_n^*(\theta)(b(\theta) - r(\theta)\mathbf{1})\, d\theta \\ & + \int_0^t \beta(\theta)\pi_n^*(\theta)\sigma(\theta)\, dW(\theta), \qquad t \geq 0. \end{aligned} \tag{10.2}$$

For any given function F : $[0, \infty) \to \mathbb{R}$ of bounded variation, let us denote by $\check{F}(t)$ its total variation on the interval $[0, t]$. We define

$$C(t) \triangleq t + \sum_{i=1}^{d} \check{F}_i(t), \quad F_i(t) \triangleq G_i(t) + \int_0^t (b_i(\theta) - r(\theta))\, d\theta, \qquad t \geq 0, \tag{10.3}$$

$$T(s) \triangleq \inf\{t \geq 0; C(t) > s\}, \qquad s \geq 0. \tag{10.4}$$

LEMMA 10.1. (i) *For every fixed $s \geq 0$, $T(s)$ is a stopping time of $\{\mathcal{F}_t\}$; the resulting filtration*

$$\{\mathcal{G}_s\} = \{\mathcal{F}_{T(S)}\}, \qquad s \geq 0, \tag{10.5}$$

satisfies the usual conditions (Karatzas and Shreve (1988), p. 10).

(ii) *Almost every path of the process* $\{T(s);\ s \geq 0\}$ is *absolutely continuous with respect to Lebesgue measure and is strictly increasing.*

(iii) *For every* $i = 1, \ldots, d$, *almost every path of the process* $\{\tilde{F}_i(s) \triangleq F_i(T(s));\ s \geq 0\}$ *is absolutely continuous with respect to Lebesgue measure.*

Proof. For (i), cf. Karatzas and Shreve (1988, Exercise 3.4.4 and Problem 3.4.5). For (ii) and (iii) we have from (10.3) almost surely that $C(t_2) - C(t_1) \geq \max(t_2 - t_1, \check{F}_i(t_2) - \check{F}_i(t_1))$, $\forall 0 \leq t_1 \leq t_2$. Therefore, for given $0 \leq s_1 < s_2$,

$$s_2 - s_1 = C(T(s_2)) - C(T(s_1)) \geq \max(T(s_2) - T(s_1), \check{F}_i(T(s_2)) - \check{F}_i(T(s_1))).$$

The conclusions on absolute continuity now follow easily. □

Consequently, we can write

$$T(s) = \int_0^s T'(\nu)\,d\nu, \qquad \tilde{F}_i(s) = \int_0^s \tilde{F}_i'(\nu)\,d\nu, \tag{10.6}$$

where T', $\tilde{F}_i'$ are $\{\mathcal{G}_s\}$-progressively measurable, locally integrable processes.

On the other hand, the processes $\tilde{\beta}(s) \triangleq \beta(T(s))$, $\tilde{X}(s) \triangleq X_n(T(s))$, $\tilde{\varepsilon}(s) \triangleq \varepsilon_n(T(s))$, $\tilde{c}(s) \triangleq c_n(T(s))$, $\tilde{\pi}(s) \triangleq \pi_n(T(s))$, $\tilde{b}(s) \triangleq b(T(s))$, $\tilde{r}(s) \triangleq r(T(s))$, $\tilde{\sigma}(s) \triangleq \sigma(T(s))$, and $\tilde{M}(s) \triangleq W(T(s))$ are all $\{\mathcal{G}_s\}$-progressively measurable. In terms of them, we have the following time-changed version of (10.2):

$$\begin{aligned}\tilde{\beta}(s)\tilde{X}(s) &= \int_0^s \tilde{\beta}(\nu)(\tilde{\varepsilon}(\nu) - \tilde{c}(\nu))T'(\nu)\,d\nu + \sum_{i=1}^d \int_0^s \tilde{\beta}(\nu)\tilde{\pi}_i(\nu)\tilde{F}_i'(\nu)\,d\nu \\ &\quad + \int_0^s \tilde{\beta}(\nu)\tilde{\pi}^*(\nu)\tilde{\sigma}(\nu)\,d\tilde{M}(\nu), \qquad s \geq 0.\end{aligned} \tag{10.7}$$

The process $\{\tilde{M}(s), \mathcal{G}_s;\ s \geq 0\}$ is a martingale on $(\Omega, \mathcal{F}, P)$ with quadratic variation $T(s)$; therefore, there exists a Brownian motion $\tilde{B}$ on this space (possibly extended, to accommodate an independent, one-dimensional Brownian motion process) such that

$$\tilde{M}(s) = \int_0^s \sqrt{T'(\nu)}\,d\tilde{B}(\nu), \qquad s \geq 0 \tag{10.8}$$

(Karatzas and Shreve (1988), Theorem 3.4.2).

Let us take now $\tilde{c}(s) \triangleq 0$, $\tilde{\pi}_i(s) \triangleq k\,\mathrm{sgn}(\tilde{F}_i'(s))\cdot 1_{\{T'(s)=0\}}$, $s \geq 0$, for some finite constant $k > 0$. The process $\tilde{\pi}_i$ is bounded and $\{\mathcal{G}_s\}$-progressively measurable, and thus the process $\pi_{ni}(t) \triangleq \tilde{\pi}_i(C(t))$ is bounded and $\{\mathcal{F}_t\}$-progressively measurable. If X_n is the wealth process corresponding to consumption $c_n \equiv 0$ and portfolio $\pi_n = (\pi_{n1}, \ldots, \pi_{nd})^*$ as above, the time-changed version $\tilde{X}(s) = X_n(T(s))$ is given, thanks to (10.7) and (10.8), by

$$\tilde{\beta}(s)\tilde{X}(s) = \int_0^s \tilde{\beta}(\nu)\tilde{\varepsilon}(\nu)T'(\nu)\,d\nu + k\int_0^s \tilde{\beta}(\nu) \sum_{i=1}^d |\tilde{F}_i'(\nu)|\, 1_{\{T'(\nu)=0\}}\,d\nu. \tag{10.9}$$

Now suppose we have, for some $i = 1, \ldots, d$, meas$\{s \geq 0; \tilde{F}_i'(s, \omega) \neq 0$ and $T'(s, \omega) = 0\} > 0$ for every ω in some event of positive probability (here and below, "meas" stands for "Lebesgue measure"). Then by selecting $k > 0$ sufficiently large, we can make $X(\cdot)$ a.s. nonnegative and arbitrarily large with positive probability. To exclude this "arbitrage possibility," we must have

$$\text{(10.10)} \qquad \begin{aligned} &\text{meas}\{s \geq 0; \tilde{F}_i'(s, \omega) \neq 0 \text{ and } T'(s, \omega) = 0\} = 0, \\ &\forall \omega \in \Omega^*, \quad i = 1, \ldots, d, \end{aligned}$$

for some event Ω^*, with $P(\Omega^*) = 1$.

LEMMA 10.2. *Equation* (10.10) *implies* (10.1).

Proof. Fix $\omega \in \Omega^*$ and $\varepsilon > 0$; then there is a $\delta > 0$ such that $\sum_{j=1}^m [C^{ac}(t_j', \omega) - C^{ac}(t_j, \omega)] < \varepsilon$ for every finite collection of nonoverlapping intervals $\{(t_j, t_j')\}_{j=1}^m$ in $[0, T]$ with $\sum_{j=1}^m (t_j' - t_j) < \delta$. (The superscript "ac" denotes absolutely continuous part.) Then for every $i = 1, \ldots, d$, we have

$$\text{(10.11)} \qquad \begin{aligned} \sum_{j=1}^m |F_i(t_j', \omega) - F_i(t_j, \omega)| &= \sum_{j=1}^m |\tilde{F}_i(C(t_j', \omega),\omega) - \tilde{F}_i(C(t_j, \omega),\omega)| \\ &= \sum_{j=1}^m \left| \int_{C(t_j,\omega)}^{C(t_j',\omega)} \tilde{F}_i'(\nu, \omega)\, d\nu \right| \\ &= \sum_{j=1}^m \left| \int_{C(t_j,\omega)}^{C(t_j',\omega)} 1_{\{T'(\nu,\omega)>0\}} \tilde{F}_i'(\nu, \omega)\, d\nu \right|, \end{aligned}$$

where the last equality follows from (10.10). Now the last quantity in (10.11) can be made arbitrarily small, because it amounts to integrating the integrable function $\tilde{F}_i'(\cdot, \omega)$ over a set with Lebesgue measure

$$\begin{aligned} \sum_{j=1}^m \int_{C(t_j,\omega)}^{C(t_j',\omega)} 1_{\{T'(\nu,\omega)>0\}}\, d\nu &= \sum_{j=1}^m \int_{C(t_j,\omega)}^{C(t_j',\omega)} C'(T(\nu, \omega))T'(\nu, \omega)\, d\nu \\ &= \sum_{j=1}^m \int_{t_j}^{t_j'} C'(\theta, \omega)\, d\theta \\ &= \sum_{j=1}^m [C^{ac}(t_j', \omega) - C^{ac}(t_j, \omega)] < \varepsilon. \end{aligned}$$

Thus the function $F_i(\cdot, \omega)$ is absolutely continuous with respect to Lebesgue measure, and by (10.3) the same is true for the function $G_i(\cdot, \omega)$ for every $i = 1, \ldots, d$. □

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