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EQUILIBRIUM IN A SIMPLIFIED DYNAMIC, STOCHASTIC ECONOMY WITH HETEROGENEOUS AGENTS

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Abstract

We study a dynamic, stochastic economy with several agents, who may differ in their endowments (of a single commodity) and in their utilities. An equilibrium financial market is constructed, under the condition that all agents have infinite marginal utility at zero. If, in addition, the Arrow–Pratt indices of relative risk aversion for all agents are less than or equal to one, then uniqueness of equilibrium is also proved. When agents consume and invest in this equilibrium market so as to maximize their expected utility of consumption, their aggregate endowment is consumed as it enters the economy and all financial instruments are held in zero net supply. Explicit examples are provided.

1. Introduction

A fairly complete theory has been developed recently for the optimal consumption/investment problem of a small investor with a general utility function [3,4,13]. Using tools from stochastic calculus, explicit expressions for the optimal consumption policy and terminal wealth can be provided when stock prices are modelled by Itô processes. The present paper draws on the methodology of [3,4,13] to construct *equilibrium* in a multi–agent economy, and to establish uniqueness.

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We suppose there is a finite number, N, of agents (small investors), each of whom receives an endowment stream denominated in units of a single, infinitely divisible commodity. The agents may have different endowment streams and utility functions. Each agent attempts to maximize his expected total utility from consumption of this commodity, over a finite horizon $[0,T]$. We shall construct a financial market, consisting of a bond and a finite number of stocks, which provides a vehicle for trading among the agents and thereby allows them to hedge the risk and smooth the nonuniformity associated with their respective endowments. The equilibrium problem is to construct this market in such a way that, when the stock and bond prices are accepted by the individual agents in the determination of their optimal policies, all the commodity is entirely consumed as it enters the economy and all the financial assets are held in zero net supply.

The present paper is quite similar to Duffie & Zame [9]. Both Duffie & Zame [9] and this work generalize the results of Cox, Ingersoll & Ross [5] in two important directions. First, heterogeneous agents are allowed, whereas in [5] all agents have the same endowments and the same utility functions. Secondly, endowment processes are adapted in a general way to an underlying d–dimensional Brownian motion, whereas in [5] this dependence on the underlying Brownian motion must be via a state process so that Markov methods could be employed. Duffie & Zame [9] and this paper both derive a formula for the endogenously determined equilibrium interest rate which agrees with that of [5] when specialized to their model. Both [9] and this paper derive formulas for the coefficients of the stock processes and the optimal consumption processes of the individual agents.

The Cox, Ingersoll & Ross interest rate formula is given in terms of an indirect utility function, J, derived from the single direct utility function, U, in their model. In our model, each agent has a utility function, U_n, and we construct a "representative agent" whose utility function will play the role of the Cox, Ingersoll & Ross function U. Roughly speaking, this representative agent acts as a proxy for the individual agents by receiving their aggregate endowment, solving his own optimization problem with utility function

$$U(t,c;\Lambda) \triangleq \max_{\substack{c_1\geq 0,\,\ldots,\,c_N\geq 0\\ c_1+\ldots+c_N=c}} \Sigma_{n=1}^{N}\lambda_n U_n(t,c_n), \tag{1.1}$$

and then apportioning his optimal commodity consumption process to the agents, instead of actually consuming it. The search for equilibrium is reduced to a search for an appropriate vector $\Lambda \in (0,\infty)^N$ in (1.1); cf. Sections 9 and 12. At this point, our work differs from Duffie & Zame [9], who introduce the representative agent but construct equilibrium in an infinite–dimensional functional space. One advantage of posing the equilibrium problem in a finite–dimensional space is that in this context, one can develop arguments resolving the question of uniqueness, an issue not addressed by Duffie & Zame [9] and largely ignored in the finance literature.

We use the Knaster–Kuratowski–Mazurkiewicz lemma [2, p. 26] to give a very simple proof of the existence of equilibrium. A different proof under slightly different assumptions on the endowment processes can be obtained directly from Araujo and Monteiro [1]. Under the assumption that the agents' measure of relative risk aversion is less than or equal to one, we show by a separate simple argument that the agents' equilibrium optimal consumption processes, as well as the equilibrium interest rate, are unique. Furthermore, the coefficients of the equilibrium stock price processes are unique up to the formation of mutual funds.

Some generalizations of this model are possible. First, one could easily include capital assets which are owned by the N agents, pay dividends, and can be traded among the agents. The additional condition of equilibrium, i.e., that all such assets are exactly owned by the agents, can be easily met. A formula for the arbitrage–free price of such assets is given in Section 13. Secondly, throughout this paper we consider only individual agent utility functions satisfying the condition $U'_k(t,0) = \infty$. Generalization to the case in which $U'_k(t,0) < \infty$ for at least one of the agents is possible, but care is required.

To accommodate this case within our framework, one needs a more general model of the financial markets than we define in Section 2. For equilibrium to hold in general, both the stock and bond price processes must have singularly continuous components. One can describe the bond price process, but due to the singularly continuous component, there will be no interest rate process; see [14] and the earlier work contained in the appendix of [9]. There is an alternative model presented in [15], following the formulation of Duffie [7] and Duffie and Huang [8], which avoids requiring the financial assets to have singularly continuous components. We refer to this as the *moneyed model*; in it, prices are denominated in some currency, rather than in units of the commodity. There is also a commodity spot price process which gives the value of the commodity in that currency. In [15], the agents' commodity endowments and the prices of the financial assets are given exogenously, and the commodity spot price is determined endogenously by the equilibrium conditions. The existence and essential uniqueness of equilibrium are proved in [15] without any condition on $U_k'(t,0)$, $1 \le k \le N$. None of the financial assets will have singularly continuous parts in their price processes, but when those prices are divided by the commodity spot price to value them in commodity units, singularly continuous components can arise.

The present work is a self–contained companion to the more detailed and comprehensive article [15]. It is designed to be more accessible than [15] in that it deals exclusively with the moneyless model when all agents have infinite marginal utility at zero. These conditions obviate a number of complex technicalities; in particular, they permit a different proof of uniqueness for equilibrium, which is simpler than that appearing in [15]. Since this paper was first drafted, Dana & Pontier [6] have provided an equilibrium existence which does not require our assumption (3.2) and which accomodates a weakening of our assumption of a bounded aggregate endowment process. The existence proof of Dana & Pontier [6] is considerably simpler than our original proof, but similar to the proof we give here.

2. The Agents and their Endowments

We consider an economy consisting of N agents. Each agent, n, receives a nonnegative exogenous endowment process of a single commodity $\epsilon_n = \{\epsilon_n(t); 0 \le t \le T\}$, where T is the fixed, positive planning horizon. These endowment processes are uncertain, and we model them as Itô processes taking values in $[0,\infty)$. More precisely, let $W = (W_1,\dots,W_d)^*$ be a d–dimensional Brownian motion on a complete probability space $(\Omega,\mathscr{F},\mathbb{P})$, and let $\{\mathscr{F}_t\}$ denote the augmentation by null sets of the filtration generated by W. Assume that for $n = 1,\dots,N$, there are bounded, $\{\mathscr{F}_t\}$–progressively measurable processes μ_n and ρ_n taking values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, such that

$$(2.1) \qquad \epsilon_n(t) = \epsilon_n(0) + \int_0^t \mu_n(s)ds + \int_0^t \rho_n^*(s)dW(s), \quad 0 \le t \le T,$$

where $\epsilon_n(0)$ is a deterministic, nonnegative constant.

We define the aggregate endowment $\epsilon(t) \triangleq \sum_{n=1}^N \epsilon_n(t)$, $0 \le t \le T$, and define also $\mu(t) \triangleq \sum_{n=1}^N \mu_n(t)$, $\rho(t) \triangleq \sum_{n=1}^N \rho_n(t)$, $0 \le t \le T$. Then

$$(2.2) \qquad \epsilon(t) = \epsilon(0) + \int_0^t \mu(s)ds + \int_0^t \rho^*(s)dW(s), \quad 0 \le t \le T.$$

We assume that for each n, ϵ_n is not identically zero, and that there exist positive constants k and K for which $k \le \epsilon(t) \le K$, $0 \le t \le T$, a.s.

3. The Agents' Utility Functions

We suppose that each agent, n, has a utility function $U_n : [0,T] \times (0,\infty) \to \mathbb{R}$ which is continuous and enjoys the following properties:

(i) for every $t \in [0,T]$, $U_n(t,\cdot)$ is strictly increasing and strictly concave;

(ii) the derivatives $\frac{\partial}{\partial t} U_n$, $\frac{\partial}{\partial c} U_n$, $\frac{\partial^2}{\partial t \partial c} U_n$, $\frac{\partial^2}{\partial c^2} U_n$ and $\frac{\partial^3}{\partial c^3} U_n$ exist and are continuous on $[0,T] \times (0,\infty)$;

(iii) for every $t \in [0,T]$, $U_n' \triangleq \frac{\partial}{\partial c} U_n$ satisfies

$$U_n'(t,\infty) \triangleq \lim_{c\to\infty} U_n'(t,c) = 0, \quad U_n'(t,0) \triangleq \lim_{c\downarrow 0} U_n'(t,c) = \infty. \tag{3.1}$$

We define $U_n(t,0) \triangleq \lim_{c\downarrow 0} U_n(t,c)$, which may be $-\infty$.

In order to prove the uniqueness of equilibrium, we shall impose in Section 12 the additional condition

(iv) for every $t \in [0,T]$, the function $c \mapsto c\, U_n'(t,c)$ is nondecreasing.

Condition (iv) is equivalent to assuming that the Arrow–Pratt measure of relative risk aversion, $-cU_n''(t,c)/U_n'(t,c)$, is less than or equal to one [17, p. 69].

Examples of functions which satisfy conditions (i) – (iv) are $e^{-\alpha t}\log c$ and $\frac{1}{\gamma} e^{-\alpha t} c^\gamma$, where $\alpha \in \mathbb{R}$ and $0 < \gamma < 1$. When $\gamma < 0$, the function $\frac{1}{\gamma} e^{-\alpha t} c^\gamma$ violates condition (iv), but if all agents have this utility function, the uniqueness of equilibrium can be established by explicit computations; see Example 11.1.

4. The Financial Market

The agents in our model receive utility from consumption of the single commodity with which they are endowed. Because an individual agent's endowment process is typically random and non–uniform, he would find it advantageous to participate in a market which allows him both to hedge risk and to smooth his consumption. We shall create such a market endogenously by equilibrium considerations.

We introduce the financial market in this section; its coefficients will be specified in section 10, in terms of the endowment processes and utility functions of the individual agents. The market has $d + 1$ assets. One of them is a pure discount bond, with price

$$P_0(t) = P_0(0)\exp\{\int_0^t r(s)ds\} \tag{4.1}$$

at time t. The remaining d assets are risky stocks, and the price per share $P_i(t)$ of the i^{th} stock is modelled by the linear stochastic differential equation

$$dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]; \quad i = 1,\ldots,d. \tag{4.2}$$

All these prices are denominated in units of the commodity with which the agents are endowed. The interest rate $r(\cdot)$ of the bond, the mean rate of return vector $b(\cdot) = (b_1(\cdot),\ldots,b_d(\cdot))^*$ of the stocks, and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1\le i,j\le d}$, will all be bounded, $\{\mathcal{F}_t\}$–progressively measurable processes. In addition, we shall impose the uniform nondegeneracy condition

$$\xi^* \sigma(t)\sigma^*(t)\xi \ge \delta\, \|\xi\|^2, \quad 0 \le t \le T, \text{ a.s.,} \tag{4.3}$$

for some $\delta > 0$. Under (4.3), the inverses of both $\sigma(\cdot)$ and $\sigma^*(\cdot)$ exist and are bounded. In particular, the relative risk process

$$\theta(t) \triangleq (\sigma(t))^{-1}[b(t) - r(t)\underset{\sim}{1}], \quad 0 \le t \le T, \tag{4.4}$$

is bounded and progressively measurable, where $\underset{\sim}{1}$ denotes the d–dimensional vector with every component equal to 1.

It follows then from the Girsanov theorem (e.g. [16, section 3.5]) that the exponential supermartingale

$$Z(t) \triangleq \exp\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\}, \quad \mathscr{F}_t; \quad 0 \le t \le T, \tag{4.5}$$

is actually a martingale, and that $\tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)ds$; is Brownian motion under the probability measure $\tilde{\mathbb{P}}(A) \triangleq E(Z(T)1_A)$; $A \in \mathscr{F}_T$. Under this measure, the discounted stock price processes $\beta(t)P_i(t)$, with

$$\beta(t) \triangleq (P_0(t))^{-1} = \frac{1}{P_0(0)} \exp\{-\int_0^t r(s)ds\} \tag{4.6}$$

are martingales, a fact of great importance in the modern theory of continuous trading (cf. [10,11,18] for its connections with the notions of "absence of arbitrage opportunities" and "completeness" in the market model). We shall see in Remark 7.1 that the process

$$\zeta(t) \triangleq \beta(t)Z(t) \;;\; 0 \le t \le T, \tag{4.7}$$

acts as a "deflator", in the sense that multiplication by $\zeta(t)$ converts wealth held at time t to the equivalent amount of wealth at time zero.

We impose on ζ the condition

$$0 < k \le \zeta(t) \le K, \quad 0 \le t \le T, \text{ a.s.}, \tag{4.8}$$

for some constants k and K.

5. The Individual Agents' Optimization Problems

Once a financial market is specified, as it will be in Section 10, each agent, n, acts as a price–taker. He has at his disposal the choice of an $\mathbb{R}^d$–valued <u>portfolio process</u> $\pi_n(t) = (\pi_{n1}(t),\dots,\pi_{nd}(t))^*$ and a nonnegative <u>consumption</u> rate <u>process</u> $c_n(t)$, $0 \le t \le T$. He must choose both these processes to be $\{\mathscr{F}_t\}$–progressively measurable and to satisfy $\int_0^T (c_n(t) + \|\pi_n(t)\|^2)dt < \infty$, almost surely. The interpretation here is that $\pi_{ni}(t)$ represents the amount of commodity invested at time t by the n^{th} investor in the i^{th} stock.

If we denote by $X_n(t)$ the wealth of the n^{th} investor at time t, then $X_n(t) - \sum_{i=1}^d \pi_{ni}(t)$ is the amount invested in the bond. Neither this quantity nor the individual $\pi_{ni}(t)$'s are constrained to be nonnegative, i.e., borrowing at the interest rate $r(t)$ and short–selling of stocks are permitted.

The wealth X_n corresponding to a given portfolio/consumption pair (π_n, c_n) satisfies the equation

$$(5.1)\qquad dX_n(t) = [\epsilon_n(t) - c_n(t)]dt + \sum_{i=1}^{d} \pi_{ni}(t)[b_i(t) + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)]$$

$$+ [X_n(t) - \sum_{i=1}^{d} \pi_{ni}(t)]r(t)dt$$

$$= r(t)X_n(t)dt + [\epsilon_n(t) - c_n(t)]dt + \pi_n^*(t)\sigma(t)d\tilde{W}(t)$$

whose solution is

$$(5.2)\beta(t)X_n(t) = \int_0^t \beta(s)[\epsilon_n(s) - c_n(s)]ds + \int_0^t \beta(s)\pi_n^*(s)\sigma(s)d\tilde{W}(s), \quad 0 \le t \le T.$$

5.1 Definition
A portfolio/consumption pair (π_n, c_n) is called admissible for agent n if the corresponding wealth process, X_n, is bounded from below and satisfies $X_n(T) \ge 0$, almost surely.

The n^{th} agent's optimization problem is to maximize the expected total utility from consumption $E\int_0^T U_n(t,c_n(t))dt$ over all admissible pairs (π_n, c_n) that satisfy

$$(5.3)\qquad E\int_0^T \max\{0, -U_n(t,c_n(t))\}dt < \infty.$$

Condition (5.3) is imposed to ensure that $E\int_0^T U_n(t,c_n(t))dt$ is defined. We shall let $(\hat{\pi}_n, \hat{c}_n)$ denote an optimal pair for this problem, and let $\hat{X}_n$ denote the associated wealth process. The existence of $(\hat{\pi}_n, \hat{c}_n)$ is established in Section 7.

6. The Definition of Equilibrium

We are now in a position to define the notion of equilibrium.

6.1 Definition
We say that the financial market (more specifically, the processes $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$) introduced in Section 4 results in equilibrium if, in the notation of Section 5, we have almost surely

$$(6.1)\qquad \sum_{n=1}^{N} \hat{c}_n(t) = \epsilon(t), \quad 0 \le t \le T,$$

$$(6.2)\qquad \sum_{n=1}^{N} \hat{\pi}_{ni}(t) = 0, \quad 0 \le t \le T \text{ and } 1 \le i \le d,$$

$$(6.3)\qquad \sum_{n=1}^{N} \hat{X}_n(t) = 0, \quad 0 \le t \le T.$$

The above conditions enforce the clearing of the spot market in the commmodity, and the clearing of the stock and bond markets, respectively.

7. Solution of the n^{th} Agent's Problem

In order to characterize an equilibrium financial market, we let a financial market be given and study individual agent behavior in its presence. Let us therefore consider an admissible pair (π_n, c_n) and evaluate the corresponding wealth process X_n at the stopping time $\tau_m \overset{\Delta}{=} T \wedge \inf\{t\in[0,T];$ $\int_0^t \beta^2(s)\|\pi_n^*(s)\sigma(s)\|^2 ds \geq m\}$ for an arbitrary positive integer m. Taking expectation under $\tilde{\mathbb{P}}$ in (5.2) evaluated at $t = \tau_m$, we obtain $E\int_0^{\tau_m} \zeta(s)c_n(s)ds$ $= E\int_0^{\tau_m} \zeta(s)\epsilon_n(s)ds - E[\zeta(\tau_m)X_n(\tau_m)]$. Now we let $m \to \infty$. Admissibility and Fatou's lemma give $\underset{m\to\infty}{\underline{\text{Lim}}}\, E[\zeta(\tau_m)X_n(\tau_m)] \geq E[\zeta(T)X_n(T)] \geq 0$. This, coupled with the Monotone Convergence Theorem, yields in (7.1):

$$E\int_0^T \zeta(s)c_n(s)ds \leq E\int_0^T \zeta(s)\epsilon_n(s)ds. \tag{7.1}$$

7.1 Remark

Inequality (7.1) can be regarded as a budget constraint, and it justifies the terminology "deflator" for the process ζ of (4.7). It mandates that the expected total value of consumption, deflated back to the original time, does not exceed the expected total deflated value of endowment.

7.2 Proposition

Let a financial market be given. If (π_n, c_n) is an admissible pair for agent n, then (7.1) holds. Conversely, for any consumption process c_n satisfying (7.1), there exists a portfolio process π_n such that the pair (π_n, c_n) is admissible.

Proof:

It remains to justify the second claim; for any consumption process c_n satisfying (7.1), introduce the random variable $D_n \overset{\Delta}{=} \int_0^T \beta(s)[\epsilon_n(s) - c_n(s)]ds$ and observe that (7.2) amounts to $\tilde{E}D_n \geq 0$. Now the $\tilde{\mathbb{P}}$–martingale $M_n(t) \overset{\Delta}{=} \tilde{E}D_n - \tilde{E}(D_n | \mathcal{F}_t)$, can be written as a stochastic integral $M_n(t) = \int_0^t \beta(s)\pi_n^*(s)\sigma(s)d\tilde{W}(s)$ for a suitable portfolio process π_n, by virtue of the martingale representation theorem (cf. [16, Problem 3.4.16 and proof of Proposition 5.8.6]). Finally, the process

$$X_n(t) = \frac{1}{\beta(t)}\{\int_0^t \beta(s)[\epsilon_n(s) - c_n(s)]ds + M_n(t)\} \tag{7.2}$$

is obviously, from (5.2), the wealth associated with the pair (π_n, c_n) and satisfies

$$\zeta(t)X_n(t) = Z(t)\tilde{E}D_n - E\{\int_t^T \zeta(s)[\epsilon_n(s) - c_n(s)]ds \,|\, \mathcal{F}_t\}; \quad 0 \leq t \leq T, \text{ a.s.}$$

Both requirements of Definition 5.1 for admissibility follow easily from this representation, the boundedness of ϵ, and (4.8). □

We conclude from Proposition 7.2 that the n^{th} agent's optimization problem can be cast thus: to maximize the expected utility from consumption $E\int_0^T U_n(t, c_n(t))dt$ over consumption processes c_n which satisfy (7.1) and (5.3).

In order to solve this problem, we introduce $I_n(t,\cdot)$, the inverse of the strictly decreasing mapping $U_n'(t,\cdot)$ from $(0,\infty)$ onto itself. It is a straightforward verification that

$$(7.3)\quad U_n(t,I_n(t,y)) - yI_n(t,y) = \max_{c \geq 0}[U_n(t,c) - yc];\ \forall\,(t,y) \in [0,T] \times (0,\infty).$$

Because I_n is jointly continuous (in fact, jointly C^1 because of condition (ii) of Section 3) and ζ satisfies (4.8), the function $\mathscr{X}_n(y) \triangleq E\int_0^T \zeta(t)I_n(t,y\zeta(t))dt$ maps $(0,\infty)$ onto itself and is continuous and strictly decreasing. Define y_n to be the unique positive number for which

$$(7.4)\qquad \mathscr{X}_n(y_n) = E\int_0^T \zeta(t)\epsilon_n(t)dt,$$

and set

$$(7.5)\qquad \hat{c}_n(t) \triangleq I_n(t,y_n\zeta(t)),\quad 0 \leq t \leq T.$$

Then $\hat{c}_n$ satisfies (7.1) with equality, and is bounded away from zero because ζ is bounded, so (5.3) holds. Let c_n be another consumption process satisfying (5.3) and (7.1). From (7.3) we have

$$\begin{aligned}
&E\int_0^T U(t,\hat{c}_n(t))dt - E\int_0^T U(t,c_n(t))dt \\
&\geq E\int_0^T [U(t,I(t,y_n\zeta(t))) - y_n\zeta(t)I(t,y_n\zeta(t))]dt \\
&\quad - E\int_0^T [U(t,c_n(t)) - y_n\zeta(t)c_n(t)]dt \geq 0.
\end{aligned}$$

Therefore, $\hat{c}_n$ is optimal. Proposition 7.2 guarantees the existence of $\hat{\pi}_n$.

8. Characterization of Equilibrium

The issue now is how to choose the market coefficients $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ so that when, for each n, $\hat{c}_n$ is given by (7.5) and $\hat{\pi}_n$ is the corresponding portfolio process whose existence is guaranteed by Proposition 7.2, relations (6.1) – (6.3) are satisfied. It turns out that the only relevant aspect of $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ is the process ζ they lead to, as shown by the following proposition.

8.1 Proposition

Let $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$, as described in Section 4, be given, and suppose that the equilibrium conditions (6.1) – (6.3) are satisfied. Then

$$(8.1)\qquad \epsilon(t) = \sum_{n=1}^{N} I_n(t,y_n\zeta(t)),\quad 0 \leq t \leq T,$$

where y_n is defined by (7.4) and ζ is given by (4.7). Conversely, suppose there exist $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ whose corresponding process ζ satisfies (8.1); then the equilibrium conditions (6.1) – (6.3) are also satisfied.

Proof:
For the first assertion, recall that for $n = 1,\dots,N$, the optimal consumption processes are given by (7.5). The spot market clearing condition (6.1) leads to (8.1).

For the converse assertion, note that for the ζ in question, the optimal consumption processes $\hat{c}_n$, $1 \le n \le N$, are again given by (7.5). Denote by $\hat{D}_n$, $\hat{M}_n$, $\hat{\pi}_n$ and $\hat{X}_n$ the corresponding processes constructed in Section 7, which now satisfy $\tilde{E}\hat{D}_n = 0$ and $\hat{X}_n(T) = 0$ a.s. From (8.1) we have $\sum_{n=1}^{N} \hat{D}_n = 0$, a.s. It follows then that $\sum_{n=1}^{N} \hat{M}_n(t) = \sum_{n=1}^{N} \hat{X}_n(t) = 0$, $0 \le t \le T$, a.s. Thus (6.1) and (6.3) are satisfied. Furthermore, the quadratic variation of $\sum_{n=1}^{N} \hat{M}_n$ on $[0,T]$, is equal to $\int_0^T \beta^2(s) \|\sigma^*(s) \sum_{n=1}^{N} \pi_n(s)\|^2 ds$, so this quantity is zero. Because σ^* is nonsingular, (6.2) must hold. □

9. The Representative Agent

For every $\Lambda = (\lambda_1,\dots,\lambda_N) \in (0,\infty)^N$, let us introduce the function

$$\text{(9.1)}\quad U(t,c;\Lambda) = \max_{\substack{c_1 \ge 0,\dots,c_N \ge 0 \\ c_1+\dots+c_N = c}} \sum_{n=1}^{N} \lambda_n U_n(t,c_n); \quad (t,c) \in [0,T] \times (0,\infty),$$

which inherits the basic properties of the individual utility functions U_n, as set out below. It is easily checked that the maximization in (9.1) is achieved by

$$\text{(9.2)}\qquad c_n = I_n(t, \frac{1}{\lambda_n} H(t,c;\Lambda)),$$

where $H(t,\cdot;\Lambda)$ is the inverse of the strictly decreasing function $I(t,\cdot;\Lambda)$ from $(0,\infty)$ onto itself, defined by

$$\text{(9.3)}\qquad I(t,h;\Lambda) \triangleq \sum_{n=1}^{N} I_n(t, \frac{h}{\lambda_n}).$$

In order to examine the differentiability of $U(\cdot,\cdot;\Lambda)$, we first note that for each n, I_n is jointly C^1 because of condition (ii) of Section 3.1 and the Implicit Function Theorem. Differentiating the equation $U_n'(t, I_n(t,y)) = y$ twice with respect to y, one sees that $\frac{\partial^2}{\partial y^2} I_n$ exists and is continuous. Consequently, for each $\Lambda \in (0,\infty)^N$, $\frac{\partial}{\partial t} I(\cdot,\cdot;\Lambda)$, $\frac{\partial}{\partial y} I(\cdot,\cdot;\Lambda)$ and $\frac{\partial^2}{\partial y^2} I(\cdot,\cdot;\Lambda)$ exist and are continuous. Because $I(t,H(t,c;\Lambda);\Lambda) = c$ we can similarly conclude that $\frac{\partial}{\partial t} H$, $\frac{\partial}{\partial c} H$ and $\frac{\partial^2}{\partial c^2} H$ exist and are continuous. Finally

$$U(t,c;\Lambda) = \sum_{n=1}^{N} \lambda_n U_n(t, I_n(t, \frac{1}{\lambda_n} H(t,c;\Lambda))),$$

and differentiation with respect to c yields

$$U'(t,c;\Lambda) \triangleq \frac{\partial}{\partial c} U(t,c;\Lambda) = H(t,c;\Lambda) \frac{d}{dc} I(t,H(t,c;\Lambda);\Lambda) = H(t,c,\Lambda).$$

Therefore, $U_t'(t,c;\Lambda) \triangleq \frac{\partial^2}{\partial t \partial c} U(t,c;\Lambda)$, $U''(t,c;\Lambda) \triangleq \frac{\partial^2}{\partial c^2} U(t,c;\Lambda)$ and $U'''(t,c;\Lambda) \triangleq \frac{\partial^3}{\partial c^3} U(t,c;\Lambda)$ exist and are continuous on $[0,T] \times (0,\infty)$.

We have shown that $I(t,\cdot;\Lambda)$ defined by (9.3) is the inverse of $U'(t,\cdot;\Lambda)$, and so $U(\cdot,\cdot;\Lambda)$ satisfies conditions (i) – (iii) of Section 3. We call $U(\cdot,\cdot;\Lambda)$ the utility function of a representative agent who assigns weights $\lambda_1,\dots,\lambda_N$ to the individual agents in the economy.

Making the identification $\Lambda = (\lambda_1,\dots,\lambda_N) = (\frac{1}{y_1},\dots,\frac{1}{y_N})$, equations (7.4) – (7.5), (8.1) may be rewritten as

$$\zeta(t) = U'(t,\epsilon(t);\Lambda), \quad 0 \le t \le T, \tag{9.4}$$

$$E\int_0^T U'(t,\epsilon(t);\Lambda) I_n(t,\frac{1}{\lambda_n}U'(t,\epsilon(t);\Lambda))dt = E\int_0^T U'(t,\epsilon(t);\Lambda)\epsilon_n(t)dt, \quad 1 \le n \le N, \tag{9.5}$$

and the search for equilibrium is equivalent to the search for a vector $\Lambda \in (0,\infty)^N$ which satisfies (9.5). Once such a vector is found, the corresponding equilibrium ζ is given by (9.4), and the optimal consumption processes of the individual agents by

$$\hat{c}_n(t;\Lambda) \triangleq I_n(t,\frac{1}{\lambda_n}U'(t,\epsilon(t);\Lambda)), \quad 0 \le t \le T, 1 \le n \le N. \tag{9.6}$$

Note that ζ given by (9.4) satisfies (4.8) because of the assumption $k \le \epsilon(t) \le K$ and the continuity of $U'(\cdot,\cdot;\Lambda)$.

10. The Equilibrium Financial Market

In this section, we assume the existence of $\Lambda \in (0,\infty)^N$ satisfying (9.5), and we draw conclusions about the equilibrium financial market. The existence of such a Λ is established by explicit computation for certain special cases in Section 11 and in full generality by a fixed point argument in Section 12. It is apparent from (9.1) that for any $\Lambda \in (0,\infty)^N$ and $\eta > 0$,

$$U(t,c;\eta\Lambda) = \eta\, U(t,c;\Lambda), \forall\ (t,c) \in [0,T] \times (0,\infty), \tag{10.1}$$

so a multiplicative constant on Λ cancels out of (9.5) and (9.6). Therefore, the existence of any solution Λ to (9.5) guarantees the existence of a one–parameter family of solutions. In Section 11 and under the additional assumption (iv) in Section 12, the solution to (9.5) is shown to be unique up to a positive multiplicative constant. It follows then from (9.6) and (10.1) that the equilibrium optimal consumption processes for the individual agents are uniquely determined.

10.1 Proposition

Assume that there exists $\Lambda \in (0,\infty)^N$ satisfying (9.5), and that this Λ is unique up to a positive multiplicative constant. Then an interest rate process $r(\cdot)$, a mean rate of return vector process $b(\cdot)$, and a volatility matrix process $\sigma(\cdot)$ lead to equilibrium if and only if

$$r(t) = -\frac{1}{U'(t,\epsilon(t);\Lambda)}[U_t'(t,\epsilon(t);\Lambda) + \mu(t)\, U''(t,\epsilon(t);\Lambda) + \frac{1}{2}\|\rho(t)\|^2\, U'''(t,\epsilon(t);\Lambda)], \tag{10.2}$$

$$\theta(t) \triangleq (\sigma(t))^{-1}[b(t) - r(t)\,\underset{\sim}{1}] = -\frac{U''(t,\epsilon(t);\Lambda)\;\rho(t)}{U'(t,\epsilon(t);\Lambda)}, \quad 0 \le t \le T, \tag{10.3}$$

where Λ is determined by $P_0(0) \cdot U'(0,\epsilon(0);\Lambda) = 1$.

Proof:

From (4.5), (4.7), we have

$$\zeta(t) = \frac{1}{P_0(0)} - \int_0^t r(s)\zeta(s)ds - \int_0^t \zeta(s)\,\theta^*(s)dW(s), \quad 0 \le t \le T. \tag{10.4}$$

Equilibrium occurs if and only if (9.4) holds, and recalling (2.2), we see that (9.4) is equivalent to

$$(10.5)\qquad \zeta(t) = U'(0,\epsilon(0);\Lambda) + \int_0^t [\mu(s)U''(s,\epsilon(s);\Lambda)$$
$$+ \tfrac{1}{2}\|\rho(s)\|^2 U'''(s,\epsilon(s);\Lambda)]ds + \int_0^t U''(s,\epsilon(s);\Lambda)\,\rho^*(s)dW(s), \quad 0 \le t \le T$$

Identifying coefficients in (10.4) and (10.5), we obtain $U'(0,\epsilon(0);\Lambda) = \frac{1}{P_0(0)}$, (10.2) and (10.3).

11. Examples

We cite a few special cases in which the equilibrium can be computed explicitly.

11.1 Example. $U_n(t,c) = \frac{1}{\gamma} e^{-\alpha t} c^\gamma$, $\forall\ (t,c) \in [0,T] \times (0,\infty)$, $n \in \{1,\dots,N\}$, where $\alpha \in \mathbb{R}$ and $\gamma < 1$, $\gamma \neq 0$.
In this case, the vector $\Lambda = (\lambda_1,\dots,\lambda_N) \in (0,\infty)^N$ with

$$\lambda_n^{\frac{1}{1-\gamma}} = [E\int_0^T e^{-\alpha t}\epsilon_n(t)\epsilon^{\gamma-1}(t)dt]\,[E\int_0^T e^{-\alpha t}\epsilon^\gamma(t)dt]^{-1}$$

is the unique solution to (9.5) subject to the normalizing condition $\sum_{n=1}^N \lambda_n^{\frac{1}{1-\gamma}} = 1$. The optimal consumption processes are $\hat{c}_n(t) = \lambda_n^{\frac{1}{1-\gamma}}\epsilon(t)$, and the equilibrium financial market satisfies

$$r(t) = \alpha + \frac{(1-\gamma)}{\epsilon(t)}\mu(t) - \frac{(1-\gamma)(2-\gamma)}{2\epsilon^2(t)}\|\rho(t)\|^2, \quad \theta(t) = \frac{1-\gamma}{\epsilon(t)}\rho(t).$$

The normalization of Λ we have adopted corresponds to $P_0(0) = \epsilon^{1-\gamma}(0)$.

11.2 Example. $U_n(t,c) = e^{-\alpha t}\log c$ $\forall\ (t,c) \in [0,T] \times (0,\infty)$, $n \in \{1,\dots,N\}$, where $\alpha \in \mathbb{R}$.
In this case, we obtain the formulas of Example 11.1 but with $\gamma = 0$. In particular,

$$\lambda_n = \begin{cases} \dfrac{\alpha}{1-e^{-\alpha T}} E\displaystyle\int_0^T e^{-\alpha t}\frac{\epsilon_n(t)}{\epsilon(t)}dt, & \alpha \neq 0, \\[2ex] \dfrac{1}{T} E\displaystyle\int_0^T \frac{\epsilon_n(t)}{\epsilon(t)}dt, & \alpha = 0, \end{cases}$$

provides the unique solution to (9.5) subject to the normalizing condition $\sum_{n=1}^N \lambda_n = 1$. The optimal consumption processes are $\hat{c}_n(t) = \lambda_n\epsilon(t)$, and the equilibrium financial market satisfies

$$r(t) = \alpha + \frac{1}{\epsilon(t)}\mu(t) - \frac{1}{\epsilon^2(t)}\|\rho(t)\|^2, \quad \theta(t) = \frac{1}{\epsilon(t)}\rho(t). \quad \square$$

If agents have different utility functions, it is not in general possible to compute the solution of the equilibrium problem in closed form. A special case in which such computations can be carried out arises when $N = 2$, $U_1(c) = \log c$ and $U_2(c) = \sqrt{c}$. Another special case is the following.

11.3 Example. Constant aggregate endowment $\epsilon(t) \equiv \epsilon > 0$ and time-independent utility functions.
In this case, the optimal consumption rates are constant:

$\hat{c}_n(t) \equiv \hat{c}_n \triangleq \frac{1}{T} E \int_0^T \epsilon_n(t)dt$, and every solution of (9.5) is a multiple of $\Lambda = (\frac{1}{U_1'(\hat{c}_1)}, \dots, \frac{1}{U_N'(\hat{c}_N)})$. Constant aggregate endowment implies that $\mu \equiv 0$, $\rho \equiv 0$, so the equilibrium market must satisfy $r \equiv 0$ and $b \equiv 0$. The displayed Λ is normalized to correspond to $P_0(t) \equiv P_0(0) = 1$. Note, however, that in this model the individual agent endowments can be random and time–varying, in which case agents must trade with one another to finance their constant rates of consumption.

12. Existence and Uniqueness of Equilibrium

In this section we establish the major results of the paper: existence of an equilibrium financial market and its uniqueness in the sense of Proposition 10.1. The proof of existence is based on the Knaster–Kuratowski–Mazurkiewicz (KKM) Theorem [2, pg. 26] and requires only assumptions (i)–(iii) of Section 3, while our uniqueness proof requires the additional condition (iv). Example 11.1 shows, however, that condition (iv) is not necessary for uniqueness.

We begin with some notation adapted from [2]. Let $x^1, \dots, x^{(n)}$ denote the elementary vectors of R^N, and let $\mathscr{N} = \{1, \dots, N\}$. Suppose $A \subset \mathscr{N}$, then $\mathscr{S}_A$ denotes the convex hull of the elementary vectors $\{x^{(i)}; i \in A\}$, i.e., $\mathscr{S}_A = \{\sum_{i \in A} \lambda_i x^{(i)}; \lambda_i \geq 0 \ \forall i$ and $\sum_{i \in A} \lambda_i = 1\}$, and we define $\mathscr{S}_A^+ = \{\sum_{i \in A} \lambda_i x^{(i)}; \lambda_i > 0, \forall i$ and $\sum_{i \in A} \lambda_i = 1\}$. To set the stage for the next theorem, we define for $\Lambda \in \mathscr{S}_N$

$$R_n(\Lambda) \triangleq \begin{cases} E \int_0^T U'(t, \epsilon(t); \Lambda)[I_n(t, \frac{1}{\lambda_n} U'(t, \epsilon(t); \Lambda)) - \epsilon_n(t)]dt, & \text{if } \lambda_n > 0, \\ -E \int_0^T U'(t, \epsilon(t); \Lambda)\epsilon_n(t)dt, & \text{if } \lambda_n = 0, \end{cases}$$

and let $F_n = \{\Lambda \in \mathscr{S}_{\mathscr{N}}; R_n(\Lambda) \geq 0\}$.

12.1 Theorem

Under conditions (i) – (iii) of Section 3, there exists a vector $\Lambda \in \mathscr{S}_{\mathscr{N}}^+$ satisfying (9.5).

Proof:

With $\Lambda = (\lambda_1, \dots, \lambda_N)$, we have from the dominated convergence theorem that $\lim_{\lambda_n \downarrow 0} R_n(\Lambda) = -E \int_0^T U'(t, \epsilon(t); \Lambda)\epsilon_n(t)dt < 0$. This, coupled with the smoothness conditions on U_n, proves that $R_n(\Lambda)$ is continuous on $\mathscr{S}_{\mathscr{N}}$ and F_n is closed. From (9.3) we have $\sum_{n=1}^N R_n(\Lambda) = 0$ for every $\Lambda \in \mathscr{S}_{\mathscr{N}}$. Suppose there were a Λ^* in $\mathscr{S}_{\mathscr{N}}$ which was not in $\bigcup_{n \in \mathscr{N}} F_n$. This would imply

$\sum_{n=1}^N R_n(\Lambda^*) < 0$, a contradiction. Consequently, $\mathscr{S}_{\mathscr{N}} \subset \bigcup_{n \in \mathscr{N}} F_n$. More generally, if we let $A \subset \mathscr{N}$ and consider $\Lambda^* \in \mathscr{S}_A$, a similar argument shows that $\Lambda^* \in \bigcup_{n \in A} F_n$. Indeed, if $\Lambda^* \notin \bigcup_{n \in A} F_n$, then $R_n(\Lambda^*) < 0$ for all $n \in A$, again contradicting $\sum_{n=1}^N R_n(\Lambda^*) = 0$. By the KKM Theorem [2, page 26], $\bigcap_{n \in \mathscr{N}} F_n$ is nonempty. Choose $\hat{\Lambda} \in \bigcap_{n \in \mathscr{N}} F_n$. Then $R_n(\hat{\Lambda}) = 0, 1 \leq n \leq N$, for otherwise we would have $\sum_{n=1}^{N} R_n(\hat{\Lambda}) > 0$, a contradiction. Thus (9.5) is satisfied by $\hat{\Lambda}$. Finally, $\hat{\lambda}_n > 0$ or else $R_n(\hat{\Lambda})$ would be strictly negative. □

As observed following (10.1), once a vector in $\mathscr{S}_{\mathscr{N}}^{+}$ satisfying (9.5) is obtained, any positive multiple of this vector also satisfies (9.5). We next turn our attention to the question of uniqueness. Condition (iv) of Section 3 is equivalent to the assumption

$$\varphi_n(t,y) \triangleq y\, I_n(t,y) \quad \text{is nonincreasing in } y. \tag{12.1}$$

This leads to the following uniqueness result.

12.2 Theorem
Assume conditions (i)–(iv) of Section 3. Then the solution $\Lambda \in (0,\infty)^n$ of (9.5) is unique up to multiplication by a positive constant.

Proof:
We introduce the usual partial order in $(0,\infty)^N$: $\Lambda \le M$ if and only if $\lambda_n \le \mu_n$, $\forall\, n \in \{1,\dots,N\}$. We write $\Lambda < M$ if $\Lambda \le M$ and $\Lambda \ne M$. In particular, notice in (9.3) the implications

$$\Lambda \underset{(<)}{\le} M \Longrightarrow I(t,h;\Lambda) \underset{(<)}{\le} I(t,h;M) \ \ \forall (t,h) \in [0,T] \times (0,\infty). \tag{12.2}$$

For $\Lambda \le M$ we have from (12.2) that $U'(t,\epsilon(t);\Lambda) \le U'(t,\epsilon(t);M)$. Let Λ and $\tilde{\Lambda}$ be two solutions of (9.5) and define $\eta \triangleq \max_{1\le n\le N} \frac{\lambda_n}{\tilde{\lambda}_n}$ and $M = (\mu_1,\dots,\mu_n)$

$= \eta\tilde{\Lambda}$, so M is a solution of (9.5) and $\Lambda \le M$. If $\Lambda = M$, then $\tilde{\Lambda}$ is indeed a positive multiple of Λ. Therefore, it suffices to rule out the case $\Lambda < M$.

Suppose that $\Lambda < M$. From (12.2) we obtain $U'(t,\epsilon(t);\Lambda) < U'(t,\epsilon(t);M)$, $\forall\,(t,\omega) \in [0,T] \times \Omega$. Choose an integer $n \in \{1,\dots,N\}$ satisfying $\lambda_n = \eta\tilde{\lambda}_n$ (and hence also $\lambda_n = \mu_n$). We have

$$E\int_0^T \frac{1}{\lambda_n} U'(t,\epsilon(t);\Lambda)\epsilon_n(t)dt < E\int_0^T \frac{1}{\mu_n} U'(t,\epsilon(t);M)\epsilon_n(t)dt$$

$$E\int_0^T \varphi_n\Big(t, \frac{1}{\lambda_n} U'(t,\epsilon(t);\Lambda)\Big)dt \ge E\int_0^T \varphi_n\Big(t, \frac{1}{\mu_n} U'(t,\epsilon(t);M)\Big)dt,$$

where φ_n is given by (12.1). Taking the difference of these two relations, we obtain $\frac{1}{\lambda_n} R_n(\Lambda) > \frac{1}{\mu_n} R_n(M)$. But Λ and M both solve (9.5), so $R_n(\Lambda) = R_n(M) = 0$, and a contradiction is obtained.

13. Variations of the Model

In addition to the financial assets of Section 4, one can allow the agents to trade in capital assets, and one can associate to each one of these assets a dividend process $\delta_m(\cdot)$, $1 \le m \le M$, denominated in units of the commodity. In contrast to financial assets, which are essentially contracts between the agents, capital assets have to maintain a positive net supply. One can show that the prices $S_m(\cdot)$ of these new assets have to be given as

$$\zeta(t)S_m(t) = E\Big[\int_t^T \zeta(s)\delta_m(s)ds \,\big|\, \mathscr{F}_t\Big]; \quad 0 \le t \le T, \tag{13.1}$$

in order to prevent "arbitrage opportunities". Once the deflator ζ has been determined by equilibrium considerations, relation (13.1) allows the endogenous computation of the capital asset prices $S_m(\cdot)$, $1 \leq m \leq M$. The details appear in [15].

Consider now an economy with deterministic endowments and no financial market except for a bond with deterministic interest rate. Agents can consume but cannot borrow or invest, are bound simply by the budget constraints

$$\int_0^T \beta(s)c_n(s)ds \leq \int_0^T \beta(s)\epsilon_n(s)ds; \quad 1 \leq n \leq N,$$

(the deterministic analogue of (7.1)), and try to maximize their total utilities $\int_0^T U_n(t,c_n(t))dt$ from consumption. Equilibrium amounts to the requirements (6.1), (6.3) alone. In this simple model the results of sections 7–12 are valid, provided that one sets $\zeta(t) \equiv \beta(t)$, omits reference to θ, and drops the expectation signs in the formulas.

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Feynman-Kac Formula for a Degenerate Planar Diffusion and an Application in Stochastic Control

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Abstract

A formula of the Feynman-Kac type is established for the degenerate, two-dimensional diffusion process introduced by Beneš, Karatzas & Rishel [2]. With its aid, a stochastic control problem with partial observations is solved explicitly. Our derivation combines probabilistic techniques with use of the so-called *principle of smooth fit*.

1 Introduction

The degenerate, two-dimensional diffusion process $(Y^{y,\xi}, Z^{y,\xi})$ given by the stochastic equation

$$
\begin{aligned}
dY_t &= dW_t \ , \quad Y_0 = y \\
dZ_t &= -sgn(Y_t Z_t)dW_t \ , \quad Z_0 = \xi
\end{aligned}
\tag{1.1}
$$

with W a standard, one-dimensional Brownian motion, was introduced and studied in [2]. It was shown there that (1.1) admits a

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