Fall Semester 2007  
Professor Ioannis Karatzas  
G4151-G6105: ANALYSIS AND PROBABILITY I  
TENTATIVE COURSE SYLLABUS  

I. Measure Theory  
i. Construction of the integral, limits and integration  
ii. $L^p$ - spaces of functions  
iii. Construction of measures, Lebesgue-Stieltjes and product measures  
iv. Examples: ergodicity, Liouville measure, Hausdorff measure  

II. Elements of Probability  
i. The coin-tossing or random walk model  
ii. Independent events and independent random variables  
iii. The Khintchin weak law and the Kolmogorov strong law of large numbers  
iv. Notions of convergence of random variables  
v. The Central Limit, Cramer and Iterated Logarithm Theorems  

III. Elements of Fourier Analysis  
i. Fourier transforms of measures, Fourier-Lévy Inversion Formula  
ii. Convergence of distributions and characteristic functions  
iii. Proof of the Central Limit Theorem  
iv. Fourier transforms on Euclidean spaces  
v. Fourier series, the Poisson summation formula  
vi. Spectral decompositions of the Laplacian  
vii. The Heat equation and heat kernel, the Wave equation and D’Alembert’s formula  

IV. Brownian Motion  
i. Brownian motion as a Gaussian process  
ii. Brownian motion as scaling limit of random walks  
iii. Brownian motion as random Fourier series  
iv. Brownian motion and the heat equation  
v. Elementary properties of Brownian paths  

Recommended Texts:  
G.B. FOLLAND: "Real Analysis: Modern Techniques And Applications"  
Recommended Text for a review of undergraduate Probability:

Also recommended:  P. Billingsley: Probability and Measure, 3rd Edition (Wiley)
C.D. Aliprantis & O. Burkinshaw: Problems in Real Analysis - A Workbook with Solutions,

Copies of the Lecture Notes for this class will be made available.

Prerequisites: A solid, working knowledge of Advanced Calculus, Linear Algebra, Principles of
Mathematical Analysis (at the level of Rudin's and Browder's books), and of Probability at the
undergraduate level.

COURSE SYLLABUS

.Lecture #1: Tuesday, 4 September.
Definition and properties of measure. Examples. Sigma-algebras of sets. Measurable
functions and their properties. Integration of simple functions, properties.

.Lecture #2: Thursday, 6 September.
Integration of measurable functions. Monotone Convergence Theorem, Fatou's lemma,
Dominated Convergence Theorem.

.Lecture #3: Tuesday, 11 September.
Approximation of measurable functions by simple functions. Linearity properties of the
integral. Outer measure, Karatheodory's theorem.

Assignment # 1: Elementary properties of measures.
Assignment # 3: Measurable and integrable functions.

.Lecture #4: Thursday, 13 September.
The Hahn extension theorem. Complete measures, notion of completion of a measure space.

.Lecture #5: Tuesday, 18 September.
Proof of the Hahn extension theorem. Construction of Lebesgue and Lebesgue-Stieltjes

Assignment # 2: Outer measure, completions, properties of Lebesgue-Stieltjes measures.

.Lecture #6: Thursday, 20 September.
$L^p$ - spaces and their norms. Holder, Minkowski and Chebyshev inequalities.
Convex functions, the Jensen inequality.

Assignment # 4: Elementary properties of integration.
Lecture #7: Tuesday, 25 September.
Product measure and Tonelli-Fubini theorems. Applications:
Young’s inequality. Definition and properties of convolution.

Lecture #8: Thursday, 27 September.
Modes of convergence for measurable functions:

Assignment # 5: Convergence in measure and in other modes.

Lecture #9: Tuesday, 2 October.
Monote Class Theorem, proof of the Product Measure and Fubini-Tonelli theorems.

Lecture #10: Thursday, 4 October.
Probability spaces, events, random variables. Example: Bernoulli, binomial, Poisson
and normal (Gaussian) variables. Poisson and normal approximations to the binomial.

Lecture #11: Tuesday, 9 October.
Independence of random variables; additivity of the variance. Elementary versions
of the Weak and Strong (Rajchman, Markov) Laws of Large Numbers.

Assignment # 6: Probability theory, modes of convergence, laws of large numbers.

Lecture #12: Thursday, 11 October.
Khinchin and Kolmogorov Laws of Large Numbers (with proofs). Statement and
significance of the Central Limit Theorem; statement of the Berry-Esseen Theorem.

Lecture #13: Tuesday, 16 October.
Fourier transforms of measures, and their elementary properties. Examples.
The determination of a distribution from its Fourier transform, Fourier-Levy
inversion theorem. The simple Fourier inversion theorems (for functions).

Lecture #14: Thursday, 18 October.
Convergence of probability measures, connections to other modes of convergence.
The Skorohod representation.

Assignment # 8: Modes of Convergence for Random Variables.

Lecture #15: Tuesday, 23 October. MID-TERM EXAMINATION.

Lecture #16: Thursday, 25 October.
The Helly-Bray lemma, tightness, the basic convergence result. Convergence results for characteristic functions, proof of the Central Limit Theorem.

_Lecture #17: Tuesday, 30 October._

**Assignment # 9:** On Fourier analysis.

_Lecture #18: Thursday, 1 November._
Applications of Fourier analysis to the Wave Equation and the Heat Equation. The D’Alembert and Laplace formulae.

_Lecture #17: Tuesday, 6 November._ UNIVERSITY HOLIDAY

_Lecture #18: Thursday, 8 November._
The Heat equation: fundamental solutions, existence, uniqueness, the method of images, initial and boundary-value problems.

**Assignment # 10:** Applications of Fourier analysis to differential equations.

_Lecture #19: Tuesday, 13 November._
Overview of Hilbert space theory: basic properties, the parallelogram law, the Pythagorean theorem. Orthonormal systems, the Bessel inequality and the Parseval identity.

_Lecture #20: Thursday, 15 November._

**Assignment # 7:** Uniform Integrability, properties, the Dunford-Pettis and Komlos theorems.

_Lecture #21: Tuesday, 20 November._
The Lebesgue Decomposition and Radon-Nikodym Theorems. The Daniell-Kolmogorov Theorem.

_Lecture #22: Thursday, 22 November._ THANKSGIVING HOLIDAY

_Lecture #23: Tuesday, 27 November._
Brownian Motion: definition, visualization as a limit of random walks, elementary properties, Law of Large Numbers, Quadratic Variation. Interpretation of the solution of the Heat Equation in terms of Brownian Motion.
. Lecture #24: Thursday, 29 November.

. Lecture #25: Tuesday, 4 December.

Assignment # 11: On Hilbert spaces.

. Lecture #26: Thursday, 6 December.

. Lecture #27: Tuesday, 11 December.
Notion and elementary properties of relative entropy. The Neyman-Pearson lemma.

. Lecture #28: Thursday, 13 December.
Review session; problem—solving.