ASPECTS OF UTILITY MAXIMIZATION WITH HABIT FORMATION: DYNAMIC PROGRAMMING AND STOCHASTIC PDE'S *

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Abstract. This paper studies the habit-forming preference problem of maximizing total expected utility from consumption net of the *standard of living*, a weighted-average of past consumption. We describe the effective state space of the corresponding optimal wealth and standard of living processes, identify the associated value function as a generalized utility function, and exploit the interplay between dynamic programming and Feynman-Kac results via the theory of random fields and stochastic partial differential equations (SPDE's). The resulting value random field of the optimization problem satisfies a non-linear, backward SPDE of parabolic type, widely referred to as the *stochastic Hamilton-Jacobi-Bellman equation.* The dual value random field is characterized further in terms of a backward parabolic SPDE which is *linear*. Progressively measurable versions of stochastic feedback formulae for the optimal portfolio and consumption choices are obtained as well.

Key words. habit formation, generalized utility function, random fields, stochastic backward partial differential equations, feedback formulae, stochastic Hamilton-Jacobi-Bellman equation

AMS subject classifications. Primary 93E20, 60H15, 91B28; secondary 91B16, 35R60

1. Introduction. An important question in financial mathematics is to explain the effect of past consumption patterns on current and future economic decisions. A useful tool in this effort has been the concept of *habit formation*: an individual who consumes portions of his wealth over time is expected to develop habits which will have a decisive impact on his subsequent consumption behavior. Employed in a wide variety of economic applications, habit formation was in turn considered by several authors in the classical utility optimization problem (e.g. Sundaresan (1989), Constantinides (1991), Detemple & Zapatero (1991, 1992), Heaton (1993), Chapman (1998), Schroder & Skiadas (2002)).

The present paper returns to the stochastic control problem described in Detemple & Zapatero (1992) and explores in detail particular aspects of portfolio/consumption optimization under habit formation in complete markets. We adopt non-separable von Neumann-Morgestern preferences over a given time-horizon $[0, T]$, and maximize total expected utility $E\int_0^T u(t, c(t) - z(t; c))dt$ from consumption $c(\cdot)$ in excess of *standard of living* $z(\cdot; c)$; i.e., a habit-index defined as an average of past consumption, given by $z(t; c) \triangleq z\, e^{-\int_0^t \alpha(v)dv} + \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c(s)ds$, with $z \geq 0$ and nonnegative stochastic coefficients $\alpha(\cdot)$, $\delta(\cdot)$. Moreover, by assuming infinite marginal utility at zero, i.e., $u'(t, 0^+) = \infty$, we force consumption never to fall below the contemporaneous level of standard of living, thus triggering the development of "addictive" consumption patterns. Hence, an economic agent is constantly "forced" to consume more than he used to in the past. At $t = 0$ the assumption $u'(t, 0^+) = \infty$ postulates the condition $x > wz$, specifying the *wedge* $\mathcal{D}$ of Assumption 4.1 as the *domain of acceptability* for

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the initial wealth x and initial standard of living z. The quantity w stands for the cost, per unit of standard of living, of the *subsistence consumption*: the consumption policy that matches the standard of living exactly, at all times.

Existence of an optimal portfolio/consumption pair is proved in Detemple & Zapatero (1992) by establishing a recursive linear stochastic equation for the properly normalized marginal utility. In order to set up the mathematical background needed for further analysis, we present a brief formulation of their solution. Our contribution starts by characterizing the *effective state space* of the corresponding optimal wealth $X_0(\cdot)$ and standard of living $z_0(\cdot)$ processes as the *random wedge* $\mathcal{D}_t$ (cf. (5.19)) determined by the evolution $\mathcal{W}(t)$ of w as a random process. This result reveals the stochastic evolution of the imposed condition $x > wz$ over time, in the sense $X_0(t) > \mathcal{W}(t)z_0(t)$ for all $t \in [0, T)$, and motivates the study of the *dynamic* aspects of our stochastic control problem. Thus, we define the value function V of the optimization problem as a mapping that depends on both x and z. Considering the latter as a pair of variables running on $\mathcal{D}$, we classify V in a broad family of utility functions; in fact, $V(\cdot, z)$ and the utility function $u(t, \cdot)$ exhibit similar analytic properties. This is carried out through the *convex dual* of the value function, defined in (5.25), in conjunction with differential techniques developed in Rockafellar (1970).

In order to describe quantitatively the dependence of the agent's optimal investment $\pi_0(\cdot)$ on his wealth $X_0(\cdot)$ and standard of living $z_0(\cdot)$, Detemple & Zapatero (1992) restrict the utility function to have either the logarithmic $u(t, x) = \log x$ or the power $u(t, x) = x^p/p$ form for a model with nonrandom coefficients. Driven by ideas of dynamic programming, we pursue such formulae for the optimal policies, where now u can be an arbitrary utility function and the model coefficients may be random in general. Classical dynamic programming techniques are, however, inadequate for the analysis of non-Markovian models. On the other hand, the dynamic evolution of domain $\mathcal{D}$, represented by the stochastically evolving wedges $\mathcal{D}_t$, hints that the *basic principles* of dynamic programming might be applicable in more general settings as well. Indeed, Peng (1992) considered a stochastic control problem with stochastic coefficients, and made use of Bellman's optimality principle to formulate an associated *stochastic* Hamilton-Jacobi-Bellman equation. The discussion in that paper was formal, due to insufficient regularity of the value function. The present paper culminates with *an explicit application and validation of Peng's ideas for the utility maximization problem.*

Since stock prices and the money-market price are not necessarily Markov processes, we are now required to work with conditional expectations; these take into account the market history up to the present, and thereby lead to the consideration of *random fields*. In this context, an important role is played by certain linear, *backward* parabolic *stochastic* partial differential equations which characterize the resulting random fields as their *unique adapted* solutions; in other words, *adapted* versions of *stochastic* Feynman-Kac formulas are established.

Under reasonable assumptions on the utility preferences, the adapted *value random field* of the stochastic control problem solves, in the classical sense, a non-linear, backward *stochastic* partial differential equation of parabolic type. To wit, the value random field possesses sufficient smoothness, such that all the spatial derivatives involved in the equation exist almost surely. This equation is the stochastic Hamilton-Jacobi-Bellman equation one would expect, according to the program of Peng (1992), and is derived from two *linear* Cauchy problems, which admit *unique* solutions subject to certain regularity conditions. Apart from the classical linear/quadratic case discussed in Peng (1992), and to the best of our knowledge, *this work is the first to illustrate explicitly, directly and completely the role of backward stochastic partial differential equations (SPDE's) in the study of stochastic control problems in any generality*; see Remarks 7.5 and 7.6 in this respect.

We also characterize the *dual* value random field as the unique adapted solution of a *linear,* parabolic backward SPDE. We conclude by deriving *stochastic* "feedback formulae" for the optimal portfolio-consumption decisions, in terms of the pair consisting of the current level of wealth and standard of living. In the special case of deterministic coefficients, these formulae establish this pair as a *sufficient statistic* for the optimal investment and consumption actions of an economic agent in this market.

Preview: In Sections 2−5 we introduce the market model, and go over the optimal portfolio-consumption solution of the stochastic control problem. Section 6 investigates the interrelation of dynamic programming with the theory of stochastic partial differential equations, which establishes the optimal policies in "feedback form". In Section 7 we develop the stochastic Hamilton-Jacobi-Bellman equation satisfied by the value random field, and conclusions follow in Section 8.

Literature Overview: Duality methods in stochastic control were introduced in Bismut (1973) and elaborated further in Xu (1990), Karatzas & Shreve (1998). Detemple & Zapatero (1991, 1992) employ martingale methods (Cox & Huang (1989), Karatzas (1989), Karatzas, Lehockzy & Shreve (1987), and Pliska (1986)) to derive a closed-form solution for the optimal consumption policy, denoted by $c_0(\cdot)$. They also provide insights about the structure of the optimal portfolio investment $\pi_0(\cdot)$, that finances the policy $c_0(\cdot)$, via an application of the Clark (1970) formula due to Ocone & Karatzas (1991). A case of non-addictive habits was explored in Detemple & Karatzas (2003).

The use of dynamic programming techniques on stochastic control problems was originated by Merton (1969, 1971), aiming closed-form solutions in the special case of constant coefficients for models without habit formation. The infinite-horizon case was generalized by Karatzas, Lehoczky, Sethi & Shreve (1986). Karatzas, Lehoczky & Shreve (1987) coupled martingale with convexity methods to allow random, adapted model coefficients for general preferences; nonetheless, they reinstated the Markovian framework with constant coefficients to obtain the optimal portfolio in closed-form. A study on the case of deterministic coefficients in markets without habits can be found in Karatzas & Shreve (1998).

"Pathwise" stochastic control problems were studied recently by Lions & Souganidis (1998a, 1998b), who proposed a new notion of *stochastic* viscosity solutions for the associated fully non-linear *stochastic* Hamilton-Jacobi-Bellman equations. In two subsequent papers, Buckdahn and Ma (2001 Parts I, II) employ a Doss-Sussmann-type transformation to extend this notion in a "point-wise" manner, and obtain accordingly existence and uniqueness results for similar stochastic partial differential equations. A problem of "pathwise" stochastic optimization, that emerges from mathematical finance and concerns the dependence on the paths of an exogenous noise, is considered by Buckdahn & Ma (2006).

Results concerning the existence, uniqueness and regularity of adapted solutions to stochastic partial differential equations of the type considered in the present paper, were obtained in Ma & Yong (1997, 1999). Kunita (1990) contains a systematic study of semimartingales with spatial parameters, including the derivation of the generalized Itô-Kunita-Wentzell formula that is put to significant use throughout our analysis.

2. The Model. We adopt a model for the financial market $\mathcal{M}_0$ which consists of one riskless asset (money market) with price $S_0(t)$ given by:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = 1, \tag{2.1}$$

and m risky securities (stocks) with prices per share $\{S_i(t)\}_{1 \le i \le m}$, satisfying the stochastic differential equations

$$dS_i(t) = S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad i = 1, ..., m. \tag{2.2}$$

Here $W(\cdot) = (W_1(\cdot), ..., W_d(\cdot))^*$ is a d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{F} = \{\mathcal{F}(t); 0 \le t \le T\}$ will denote the P-augmentation of the Brownian filtration $\mathcal{F}^W(t) \triangleq \sigma(W(s);\ s \in [0,t])$. We assume that $d \ge m$, i.e., the number of sources of uncertainty in the model is at least as large as the number of stocks available for investment. All processes encountered in this paper are defined on a finite time-horizon $[0,T]$ where T is the *terminal time* for our market.

The interest rate $r(\cdot)$, as well as the instantaneous rate of return vector $b(\cdot) = (b_1(\cdot), ..., b_m(\cdot))^*$ and the volatility matrix $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \le i \le m,\ 1 \le j \le d}$, are taken to be $\mathbb{F}$-progressively measurable random processes and to satisfy

$$\int_0^T \|b(t)\|dt < \infty, \qquad \int_0^T |r(t)|dt \le \varrho \tag{2.3}$$

almost surely, for some given real constant $\varrho > 0$. It will be assumed that $\sigma(\cdot)$ is bounded and that the matrix $\sigma(t)$ has full rank for every t . Under the latter assumption the matrix $\sigma(\cdot)\sigma^*(\cdot)$ is invertible, so its inverse and the progressively measurable *relative risk process*

$$\vartheta(t) \triangleq \sigma^*(t)(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1}_m] \tag{2.4}$$

are well defined; here we denote by $\mathbf{1}_k$ the k-dimensional vector whose every component is one. We make the additional assumption that $\vartheta(\cdot)$ satisfies

$$E\int_0^T \|\vartheta(t)\|^2 dt < \infty. \tag{2.5}$$

We shall use quite often the exponential local martingale process

$$Z(t) \triangleq \exp\left\{-\int_0^t \vartheta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\vartheta(s)\|^2 ds\right\}; \tag{2.6}$$

the discount process

$$\beta(t) \triangleq \exp\left\{-\int_0^t r(s)ds\right\}; \tag{2.7}$$

their product, that is, the so-called *state-price density process*

$$H(t) \triangleq \beta(t)Z(t); \tag{2.8}$$

as well as the process

$$W_0(t) \triangleq W(t) + \int_0^t \vartheta(s)ds \qquad 0 \le t \le T. \tag{2.9}$$

We envision an economic agent who starts with a given initial endowment $x > 0$, and whose actions cannot affect the market prices. At any time $t \in [0,T]$ the agent can decide both the proportion $\pi_i(t)$ of his wealth $X(t)$ to be invested in the ith stock $(1 \le i \le m)$, and his consumption rate $c(t) \ge 0$. These decisions cannot anticipate the future, but must depend only on the current and past information $\mathcal{F}(t)$. The remaining amount $[1 - \sum_{i=1}^m \pi_i(t)]X(t)$ is invested in the money market. Here the investor is allowed both to sell stocks short, and to borrow money at the bond interest rate $r(\cdot)$; that is, the $\pi_i(\cdot)$ above are not restricted to take values only in $[0,1]$, and their sum may exceed 1.

The resulting *portfolio strategy* $\pi = (\pi_1, ..., \pi_m)^* : [0,T] \times \Omega \to \mathbb{R}^m$ and *consumption strategy* $c : [0,T] \times \Omega \to [0,\infty)$, are assumed to be $\mathbb{F}$-progressively measurable processes and to satisfy the integrability condition $\int_0^T \big(c(t) + \|\pi(t)\|^2\big)dt < \infty$, a.s.

According to the model dynamics of (2.1) and (2.2), the *wealth process* $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$, corresponding to the portofolio/consumption pair (π, c) and initial capital $x \in (0,\infty)$, is the solution of the linear stochastic differential equation

$$
\begin{aligned}
(2.10)\qquad dX(t) &= \sum_{i=1}^m \pi_i(t)X(t)\left\{ b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW_j(t)\right\} \\
(2.11)\qquad &\quad + \left\{1 - \sum_{i=1}^m \pi_i(t)\right\} X(t)r(t)dt - c(t)dt \\
&= [r(t)X(t) - c(t)]dt + X(t)\pi^*(t)\sigma(t)dW_0(t)\ ,
\end{aligned}
$$

subject to the initial condition $X(0) = x > 0$. Equivalently, we have

$$
\beta(t)X(t) + \int_0^t \beta(s)c(s)ds = x + \int_0^t \beta(s)X(s)\pi^*(s)\sigma(s)dW_0(s), \tag{2.12}
$$

and from Itô's lemma, applied to the product of $Z(\cdot)$ and $\beta(\cdot)X(\cdot)$, we obtain

$$
H(t)X(t) + \int_0^t H(s)c(s)ds = x + \int_0^t H(s)X(s)[\sigma^*(s)\pi(s) - \vartheta(s)]^*dW(s)\ . \tag{2.13}
$$

A portofolio/consumption process pair (π, c) is called *admissible for the initial capital* $x \in (0,\infty)$, if the agent's wealth remains nonnegative at all times, i.e., if

$$
X(t) \ge 0, \qquad \text{for all } t \in [0,T], \tag{2.14}
$$

almost surely. We shall denote the family of admissible pairs (π, c) by $\mathcal{A}(x)$.

For any $(\pi, c) \in \mathcal{A}(x)$, the left-hand side of (2.13) is a continuous and nonnegative local martingale, thus a supermartingale. Consequently,

$$
E\left(\int_0^T H(s)c(s)ds\right) \le x\,, \forall\ \ (\pi, c) \in \mathcal{A}(x)\,. \tag{2.15}
$$

Let $\mathcal{B}(x)$ denote the set of consumption policies $c : [0,T] \times \Omega \to [0,\infty)$ which are progressively measurable and satisfy (2.15). We have just verified that $c(\cdot) \in \mathcal{B}(x)$, for all pairs $(\pi, c) \in \mathcal{A}(x)$. In a complete market, where the number of stocks available for trading matches exactly the dimension of the "driving" Brownian motion, the converse holds as well, in the sense that any consumption strategy $c(\cdot)$ satisfying

(2.15) can be financed by some portfolio policy $\pi(\cdot)$. For this reason, (2.15) can be interpreted as a "budget constraint".

LEMMA 2.1. *Let the market model of (2.1), (2.2) be complete, namely $m = d$. Then, for every consumption process $c(\cdot) \in \mathcal{B}(x)$ there exists a portfolio process $\pi(\cdot)$ such that $(\pi, c) \in \mathcal{A}(x)$, and the associated wealth process $X(\cdot) \equiv X^{x,\pi,c}(\cdot)$ is given by*

$$H(t)X(t) = x + E_t(D(t)) - E(D(0)), \quad t \in [0,T], \qquad \textit{where} \quad D(t) \triangleq \int_t^T H(s)c(s)ds\,.$$

Here and in the sequel, $E_t[\,\cdot\,]$ denotes conditional expectation $E[\,\cdot\,|\mathcal{F}(t)]$ with respect to the probability measure P, given the σ-algebra $\mathcal{F}(t)$. For the proof of Lemma 2.1, see Karatzas & Shreve (1998), pp. 166-169.

3. Utility Functions. A *utility function* is a jointly continuous mapping $u : [0,T] \times (0,\infty) \to \mathbb{R}$ such that, for every $t \in [0,T]$, the function $u(t,\cdot)$ is strictly increasing, strictly concave, of class $C^1((0,\infty))$, and its derivative $u'(t,x) \triangleq \frac{\partial}{\partial x}u(t,x)$ satisfies

$$u'(t,0^+) = \infty, \qquad u'(t,\infty) = 0\,. \tag{3.1}$$

Due to these assumptions, the inverse $I(t,\cdot) : (0,\infty) \to (0,\infty)$ of the function $u'(t,\cdot)$ exists for every $t \in [0,T]$, and is continuous and strictly decreasing with

$$I(t,0^+) = \infty, \qquad I(t,\infty) = 0. \tag{3.2}$$

Furthermore, one can easily see the stronger assertion

$$\lim_{x\to\infty} \max_{t\in[0,T]} u'(t,x) = 0. \tag{3.3}$$

Let us now introduce, for each $t \in [0,T]$, the Legendre-Fenchel transform $\tilde{u}(t,\cdot) : (0,\infty) \to \mathbb{R}$ of the convex function $-u(t,-x)$, namely

$$\tilde{u}(t,y) \triangleq \max_{x>0}[u(t,x) - xy] = u(t,I(t,y)) - yI(t,y), \qquad 0 < y < \infty. \tag{3.4}$$

The function $\tilde{u}(t,\cdot)$ is strictly decreasing, strictly convex, and satisfies

$$\frac{\partial}{\partial y}\tilde{u}(t,y) = -I(t,y), \qquad 0 < y < \infty. \tag{3.5}$$

We note here that $\tilde{u} : [0,T] \times (0,\infty) \to \mathbb{R}$ is jointly continuous as well.

4. The Maximization Problem. For given utility function u and initial capital $x > 0$, we shall consider von Neumann-Morgenstern preferences with expected utility

$$J(z;\pi,c) \equiv J(z;c) \triangleq E\left[\int_0^T u(t,c(t) - z(t;c))dt\right], \tag{4.1}$$

corresponding to any given pair $(\pi,c) \in \mathcal{A}(x)$ and its associated index-process $z(\cdot) \equiv z(\cdot;c)$ defined in (4.3), (4.5) below. This process represents the "standard of living" of the decision-maker, an index that captures past consumption behavior and conditions the current consumption felicity by developing "habits". Of course, in order to ensure that the above expectation exists and is finite, we shall take into account only consumption strategies $c(\cdot)$ that satisfy

$$c(t) - z(t;c) > 0, \qquad \forall \;\; 0 \le t \le T, \tag{4.2}$$

almost surely. This additional budget specification insists that consumption must always exceed the standard of living, establishing incentives for a systematic built-up of habits over time and leading to "addiction patterns".

We shall stipulate that the standard of living follows the dynamics

$$\begin{aligned} dz(t) &= \big(\delta(t)c(t) - \alpha(t)z(t)\big)dt, \qquad t \in [0,T], \\ z(0) &= z, \end{aligned} \tag{4.3}$$

where $\alpha(\cdot)$ and $\delta(\cdot)$ are nonnegative, bounded and $\mathbb{F}$-adapted processes and $z \ge 0$ is a given real number. Thus, there exist constants $A > 0$ and $\Delta > 0$ such that

$$0 \le \alpha(t) \le A, \quad 0 \le \delta(t) \le \Delta, \qquad \forall \; t \in [0,T], \tag{4.4}$$

hold almost surely. Equivalently, (4.3) stipulates

$$z(t) \equiv z(t;c) = z\; e^{-\int_0^t \alpha(v)dv} + \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv} c(s)ds \tag{4.5}$$

and expresses $z(\cdot)$ as an exponentially-weighted average of past consumption.

In light of the constraint (4.2), we see that consumption $c(\cdot)$ must always exceed the "subsistence consumption" $\hat{c}(\cdot)$ for which $\hat{c}(\cdot) = z(\cdot;\hat{c})$, namely, that consumption pattern which barely meets the standard of living. From (4.3), this subsistence consumption satisfies

$$d\hat{c}(t) = \big(\delta(t) - \alpha(t)\big)\hat{c}(t)dt\,, \quad t \in [0,T]\,, \qquad \text{and} \qquad \hat{c}(0) = z\,,$$

and therefore with $\hat{z}(\cdot) \equiv z(\cdot;\hat{c})$ we have

$$c(t) > \hat{c}(t) = \hat{z}(t) = z\; e^{\int_0^t (\delta(v) - \alpha(v))dv}, \;\; \forall \; t \in [0,T].$$

Back into the budget constraint (2.15), this inequality gives $x > wz$, where

$$w \triangleq E\left[\int_0^T e^{\int_0^t (\delta(v)-\alpha(v))dv} H(t)dt\right] \tag{4.6}$$

represents the "marginal" cost of subsistence consumption, per unit of standard of living. Therefore, we need to impose the following restriction on the initial capital x and the initial standard of living level z.

Assumption 4.1. *In the notation of (4.6), the pair (x,z) belongs to the set*

$$\mathcal{D} \triangleq \Big\{(x',z') \in (0,\infty) \times [0,\infty);\;\; x' > wz'\Big\}.$$

DEFINITION 4.2. *The* Dynamic Optimization *problem is to maximize the expression of (4.1) over the class $\mathcal{A}'(x,z)$ of* admissible portfolio/consumption pairs $(\pi,c) \in \mathcal{A}(x)$ *that satisfy (4.2) and*

$$E\left[\int_0^T u^-(t, c(t) - z(t;c))dt\right] < \infty \tag{4.7}$$

(here and in the sequel, b^- denotes the negative part of the real number b). The value function of this problem will be denoted by

$$V(x,z) \triangleq \sup_{(\pi,c)\in\mathcal{A}'(x,z)} J(z;\pi,c), \qquad (x,z)\in\mathcal{D}. \tag{4.8}$$

DEFINITION 4.3. *The* Static Optimization *problem is to maximize the expression (4.1) over the set $\mathcal{B}'(x,z)$ of* consumption processes $c(\cdot)\in\mathcal{B}(x)$ *that satisfy (4.2) and (4.7). The value function of this problem will be denoted by*

$$U(x,z) \triangleq \sup_{c(\cdot)\in\mathcal{B}'(x,z)} J(z;c), \qquad (x,z)\in\mathcal{D}. \tag{4.9}$$

We obtain from (2.15) that $V(x,z)\le U(x,z)$, $\forall\,(x,z)\in\mathcal{D}$. In fact, equality prevails here: it suffices to solve only the static maximization problem, since for a *static* consumption optimizer process $c_0(\cdot)\in\mathcal{B}'(x,z)$ in (4.9) we can always construct, according to Lemma 2.1, a portfolio process $\pi_0(\cdot)$ such that $(\pi_0,c_0)\in\mathcal{A}'(x,z)$ satisfies

$$U(x,z)=J(z;c_0)=J(z;c_0,\pi_0)=V(x,z), \quad \forall\ (x,z)\in\mathcal{D}\,,$$

and constitutes a *dynamic* portfolio/consumption maximizing process pair for (4.8).

We also note that the set $\mathcal{B}'(x,z)$ of Definition 4.3 is convex, thanks to the linearity of $c\mapsto z(t;c)$ and the concavity of $x\mapsto u(t,x)$.

5. Solution of the Optimization Problem in Complete Markets. The static optimization problem of Definition 4.3 is treated as a typical maximization problem with constraints (2.15) and (4.2) in the case $m=d$ of a complete market, and admits a solution derived in Detemple & Zapatero (1992). In this section, we shall follow briefly their analysis, obtaining further results associated with the value function V and related features. More precisely, we shall identify the *effective state space* of the optimal wealth/standard of living vector process, generated by the optimal portfolio/consumption pair, as a *random wedge*, spanned by the temporal variable $t\in[0,T]$ and a family of suitable random half-planes (cf. Theorem 5.5). Theorem 5.8 below describes the relation of the value function V with a utility function as defined in Section 3, and begins the study of its dual value function $\widetilde{V}$. An alternative representation for the quantity w of (4.6) is provided as well.

In providing constructive arguments for the existence of an optimal consumption policy to the static problem, a prominent role will be played by the "adjusted" state-price density process

$$\Gamma(t) \triangleq H(t)+\delta(t)\cdot E_t\left(\int_t^T e^{\int_t^s(\delta(v)-\alpha(v))dv}H(s)ds\right), \qquad t\in[0,T], \tag{5.1}$$

which solves the *recursive linear stochastic equation*

$$\Gamma(t) = H(t)+\delta(t)\cdot E_t\left(\int_t^T e^{-\int_t^s \alpha(v)dv}\Gamma(s)ds\right), \qquad t\in[0,T]; \tag{5.2}$$

cf. Detemple & Zapatero (1992). The process $\Gamma(\cdot)$ is the state-price density process $H(\cdot)$ compensated by an additional term that reflects the effect of habits. Furthermore, we shall need to impose the following conditions:

Assumption 5.1. It will be assumed throughout that, for every $y \in (0,\infty)$,

$$E\left(\int_0^T H(t)I(t, y\Gamma(t))dt\right) < \infty \qquad \text{and} \qquad E\left(\int_0^T \big|u\big(t, I(t, y\Gamma(t))\big)\big|dt\right) < \infty.$$

In the sequel we shall provide conditions, on both the utility preferences and the model coefficients, which ensure the validity of the above assumption; cf. Remarks 5.7 and 6.3. Under Assumption 5.1, the function

$$\mathcal{X}(y) \triangleq E\left[\int_0^T \Gamma(t)I(t, y\Gamma(t))dt\right], \qquad 0 < y < \infty \tag{5.3}$$

inherits from $I(t,\cdot)$ its continuity and strict decrease, as well as $\mathcal{X}(0^+) = \infty$ and $\mathcal{X}(\infty) = 0$. We shall denote the (continuous, strictly decreasing, onto) inverse of this function by $\mathcal{Y}(\cdot)$. Obviously then, Assumption 4.1 ensures the existence of a number $y_0 \triangleq \mathcal{Y}(x - wz) \in (0,\infty)$ that satisfies

$$\mathcal{X}(y_0) = x - wz. \tag{5.4}$$

With this $y_0 > 0$, we consider now the process of net consumption given by

$$c_0(t) - z(t; c_0) \triangleq I(t, y_0\Gamma(t)) \quad \text{for} \;\; t \in [0,T]. \tag{5.5}$$

Inverting (5.5), we derive the relationship

$$\Gamma(t) = \frac{1}{y_0} u'(t, c_0(t) - z(t; c_0)), \qquad t \in [0,T], \tag{5.6}$$

which identifies the "adjusted" state-price density process $\Gamma(\cdot)$ as a "normalized marginal utility" process. Through substitution back to (4.3) the standard of living process $z_0(\cdot) \equiv z(\cdot\,; c_0)$ obtains the dynamics

$$dz_0(t) = \big[\delta(t)I(t, y_0\Gamma(t)) + (\delta(t) - \alpha(t))z_0(t)\big]dt, \qquad z_0(0) = z, \tag{5.7}$$

and by solving the first-order linear ordinary differential equation (5.7) we arrive at the expression

$$z_0(t) = e^{\int_0^t (\delta(v)-\alpha(v))dv}\left[z + \int_0^t \delta(s)F_0(s)ds\right], \qquad t \in [0,T] \tag{5.8}$$

with

$$F_0(t) \triangleq e^{\int_0^t (\alpha(v)-\delta(v))dv} I(t, y_0\Gamma(t)), \qquad t \in [0,T]. \tag{5.9}$$

From (5.5) and (5.8), the consumption process follows immediately:

$$c_0(t) = e^{\int_0^t (\delta(v)-\alpha(v))dv}\left[F_0(t) + z + \int_0^t \delta(s)F_0(s)ds\right], \qquad t \in [0,T]. \tag{5.10}$$

Theorem 5.2. *The consumption process $c_0(\cdot)$ of (5.10) solves the static optimization problem, satisfying the budget constraint (2.15) without slackness; that is, $c_0(\cdot) \in \mathcal{B}'(x,z)$ with*

$$E\left[\int_0^T H(t)c_0(t)dt\right] = x\,; \tag{5.11}$$

and that $J(z;c) \le J(z;c_0) < \infty$ *holds for any* $c(\cdot) \in \mathcal{B}'(x,z)$.

Proof. From Assumption 5.1, we have $J(z;c_0) < \infty$. On the other hand,

$$u(t, c_0(t) - z_0(t)) \ge u(t,1) + y_0\Gamma(t)\big[I(t, y_0\Gamma(t)) - 1\big] \ge -|u(t,1)| - y_0\Gamma(t)\ ;$$

from (3.4); through the observation

$$E[\Gamma(t)] \le E[H(t)] + \Delta e^{\Delta T} \cdot \left(\int_t^T E[H(s)]ds\right) \le e^{\varrho}\left(1 + \Delta T e^{\Delta T}\right) < \infty, \tag{5.12}$$

where we have used (2.3), (4.4) and the supermartingale property of $Z(\cdot)$, it is apparent that $c_0(\cdot)$ satisfies condition (4.7). Making use of (5.10) we have that

$$\begin{aligned}
E\left[\int_0^T H(t)c_0(t)dt\right] &= E\bigg[\int_0^T e^{\int_0^t(\delta(v)-\alpha(v))dv}H(t)\big(F_0(t)+z\big)dt \\
&\qquad + \int_0^T e^{\int_0^t(\delta(v)-\alpha(v))dv}H(t)\left(\int_0^t \delta(s)F_0(s)ds\right)dt\bigg] \\
&= E\bigg[\int_0^T e^{\int_0^t(\delta(v)-\alpha(v))dv}H(t)\big(F_0(t)+z\big)dt \\
&\qquad + \int_0^T \delta(t)F_0(t)\cdot E_t\left(\int_t^T e^{\int_0^s(\delta(v)-\alpha(v))dv}H(s)ds\right)dt\bigg] \\
&= E\left[\int_0^T \Gamma(t)I(t, y_0\Gamma(t))dt\right] + wz = x
\end{aligned}$$

[the next-to-last equation comes from the definitions (5.1), (5.9), and the last equation from (5.3), (5.4)], and (4.2) is also verified by definition (5.5) and the property of infinite marginal utility imposed in (3.1). It follows readily that $c_0(\cdot) \in \mathcal{B}'(x,z)$. A proof for the last assertion of the theorem was given by Detemple & Karatzas (2003), in the case of non-addictive habits. $\square$

Remark 5.3. From (5.4), (5.11), (5.3), (5.5), (5.2) and (4.5), we have for $z > 0$ the computations

$$\begin{aligned}
zw &= E\int_0^T H(t)c_0(t)dt - \mathcal{X}(y_0) = E\int_0^T \Big[H(t)c_0(t) - \Gamma(t)\big(c_0(t) - z_0(t)\big)\Big]dt \\
&= E\int_0^T \left[-\delta(t)E_t\left(\int_t^T e^{-\int_t^s \alpha(v)dv}\Gamma(s)ds\right)c_0(t) + z_0(t)\Gamma(t)\right]dt \\
&= E\left[-\int_0^T \Gamma(s)\left(\int_0^s \delta(t)e^{-\int_t^s \alpha(v)dv}c_0(t)dt\right)ds + \int_0^T z_0(t)\Gamma(t)dt\right] \\
&= E\int_0^T \left(z_0(t) - \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c_0(s)ds\right)\Gamma(t)dt = z\cdot E\int_0^T e^{-\int_0^t \alpha(v)dv}\Gamma(t)dt.
\end{aligned}$$

We obtain the expression

$$w = E\left[\int_0^T e^{-\int_0^t \alpha(v)dv}\Gamma(t)dt\right], \tag{5.13}$$

which re-casts the subsistence consumption cost per unit of standard of living w of (4.6), as a weighted average of the "adjusted" state-price density process $\Gamma(\cdot)$ of (5.1), discounted at the rate $\alpha(\cdot)$. This representation of w makes the terminology "adjusted" state-price density for $\Gamma(\cdot)$ quite intuitive: namely, a comparison of (5.13) with (4.6), which involves only the density process $H(\cdot)$, suggest the significance of $\Gamma(\cdot)$ as a modified state-price density process that takes habit-formation into account.

COROLLARY 5.4. *There exists a portfolio process $\pi_0(\cdot)$ such that the pair of policies $(\pi_0, c_0) \in \mathcal{A}'(x,z)$ attains the supremum of $J(z;\pi,c)$ over $\mathcal{A}'(x,z)$ in (4.8) and the corresponding wealth process $X_0(\cdot) \equiv X^{x,\pi_0,c_0}(\cdot)$ is given by*

$$X_0(t) = \frac{1}{H(t)} E_t\left[\int_t^T H(s)c_0(s)ds\right], \quad t \in [0,T]. \tag{5.14}$$

This optimal investment $\pi_0(\cdot)$ has the representation

$$\pi_0(t) = \left(\sigma(t)\sigma^*(t)\right)^{-1}\sigma(t)\left[\frac{\psi_0(t)}{X_0(t)H(t)} + \vartheta(t)\right], \tag{5.15}$$

in terms of the $\mathbb{R}^d$-valued, $\mathbb{F}$-progressively measurable, almost surely square-integrable process $\psi_0(\cdot)$ that represents as a stochastic integral $M_0(t) = x + \int_0^t \psi_0^*(s)dW(s)$ the martingale

$$M_0(t) \triangleq E_t\left[\int_0^T H(s)c_0(s)ds\right], \quad t \in [0,T]. \tag{5.16}$$

Furthermore, the value function V of the dynamic maximization problem (4.8) is

$$V(x,z) = G(\mathcal{Y}(x - wz)), \quad (x,z) \in \mathcal{D}; \tag{5.17}$$

here $\mathcal{Y}(\cdot)$ is the inverse of the function $\mathcal{X}(\cdot)$, defined in (5.3), and

$$G(y) \triangleq E\left[\int_0^T u\big(t, I(t, y\Gamma(t))\big)dt\right], \quad y \in (0,\infty). \tag{5.18}$$

Proof. The existence of the optimal portfolio $\pi_0(\cdot)$, along with the validation of (5.14), (5.15) and (5.16), is a consequence of Lemma 2.1 and (5.11). From the optimality of (π_0, c_0) we get

$$V(x,z) = E\left[\int_0^T u(t, c_0(t) - z_0(t))dt\right], \quad (x,z) \in \mathcal{D},$$

and (5.17) follows readily from (5.5), (5.4). ◻

Note that, under the optimal policies (π_0, c_0), the investor goes bankrupt at time $t = T$: $X_0(T) = 0$, almost surely. This is natural, since utility is desired here only from consumption, not from terminal wealth.

Assumption 4.1 determines the "domain of acceptability" $\mathcal{D}$ for the initial values of wealth and standard of living. The next reasonable issue to be explored is the temporal evolution of these quantities as random processes, under the optimal pair policy (π_0, c_0) and for all times $t \in [0,T]$.

THEOREM 5.5. *The effective state space of optimal wealth/standard of living process* $\big(X_0(\cdot), z_0(\cdot)\big)$ *is given by the family of* random wedges

$$\mathcal{D}_t \triangleq \Big\{(x', z') \in (0,\infty) \times [0,\infty);\ x' > \mathcal{W}(t) z'\Big\}, \quad 0 \le t < T, \tag{5.19}$$
$$\mathcal{D}_T \triangleq \Big\{(0, z');\ z' \in [0,\infty)\Big\},$$

where

$$\mathcal{W}(t) \triangleq \frac{1}{H(t)} E_t \left[\int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s) ds \right], \quad 0 \le t \le T \tag{5.20}$$

stands for the cost of subsistence consumption, per unit of standard of living, at time t*. In other words, we have, almost surely:*

$$\big(X_0(t), z_0(t)\big) \in \mathcal{D}_t, \quad \text{for all} \ \ t \in [0,T]. \tag{5.21}$$

Note $\mathcal{W}(0) = w$ and $\mathcal{D}_0 = \mathcal{D}$, the quantities of Assumption 4.1; thus, the random wedges $\mathcal{D}_t$ determine dynamically the range where the vector process of wealth/standard of living $\big(X_0(\cdot), z_0(\cdot)\big)$ takes values under the optimal regime.

Proof of Theorem 5.5. Consider the optimal pair (π_0, c_0) and the resulting standard of living $z_0(\cdot)$ processes, specified by (5.15), (5.10) and (5.8), successively. Recalling the definitions of (5.1) and (5.20), the corresponding wealth process $X_0(\cdot)$ of (5.14) may be reformulated as

$$\begin{aligned}
X_0(t) &= \frac{1}{H(t)}\ E_t \Bigg[\int_t^T H(s) \Big\{ I(s, y_0\Gamma(s)) + z e^{\int_0^s (\delta(v)-\alpha(v))dv} \\
&\qquad\qquad + \int_0^s \delta(\theta) e^{\int_\theta^s (\delta(v)-\alpha(v))dv} I(\theta, y_0\Gamma(\theta)) d\theta \Big\} ds \Bigg] \\
&= \frac{1}{H(t)}\ E_t \Bigg[\int_t^T H(s) \Big\{ z e^{\int_0^s (\delta(v)-\alpha(v))dv} \\
&\qquad\qquad + \int_0^t \delta(\theta) e^{\int_\theta^s (\delta(v)-\alpha(v))dv} I(\theta, y_0\Gamma(\theta)) d\theta \Big\} ds \\
&\qquad + \int_t^T H(s) I(s, y_0\Gamma(s)) ds \\
&\qquad + \int_t^T \delta(\theta) I(\theta, y_0\Gamma(\theta)) \left(\int_\theta^T e^{\int_\theta^s (\delta(v)-\alpha(v))dv} H(s) ds \right) d\theta \Bigg] \\
&= \frac{1}{H(t)}\ E_t \Bigg[z_0(t) \int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s) ds \\
&\qquad + \int_t^T \left\{ H(s) + \delta(s) E_s \left(\int_s^T H(\theta) e^{\int_s^\theta (\delta(v)-\alpha(v))dv} d\theta \right) \right\} I(s, y_0\Gamma(s)) ds \Bigg] \\
&= \mathcal{W}(t) z_0(t) + \frac{1}{H(t)} E_t \left[\int_t^T \Gamma(s) I(s, y_0\Gamma(s)) ds \right], \quad 0 \le t \le T.
\end{aligned}$$

Therefore,

$$X_0(t) - \mathcal{W}(t)z_0(t) = \frac{1}{H(t)} E_t \left[\int_t^T \Gamma(s) I(s, y_0 \Gamma(s)) ds \right] > 0, \quad \forall\, t \in [0,T),$$

almost surely, and (5.21) holds on $[0,T)$. The remaining assertions of the theorem follow directly from (5.14). ☐

Example 5.6. *(Logarithmic utility).* Consider $u(t,x) = \log x$, $\forall\ (t,x) \in [0,T] \times (0,\infty)$. Then $I(t,y) = 1/y$ for $(t,y) \in [0,T] \times (0,\infty)$, $\mathcal{X}(y) = T/y$ for $y \in (0,\infty)$, and $\mathcal{Y}(x) = T/x$ for $x \in (0,\infty)$. The optimal consumption, standard of living, and wealth processes are as follows:

$$c_0(t) = z e^{\int_0^t (\delta(v)-\alpha(v))dv} + \frac{x - wz}{T} \left[\frac{1}{\Gamma(t)} + \int_0^t \frac{\delta(s)}{\Gamma(s)} e^{-\int_s^t (\delta(v)-\alpha(v))dv} ds \right],$$

$$z_0(t) = z e^{\int_0^t (\delta(v)-\alpha(v))dv} + \frac{x - wz}{T} \int_0^t \frac{\delta(s)}{\Gamma(s)} e^{-\int_s^t (\delta(v)-\alpha(v))dv} ds\,,$$

$$X_0(t) = \frac{1}{H(t)} \left[z_0(t) E_t \left(\int_t^T e^{\int_t^s (\delta(v)-\alpha(v))dv} H(s) ds \right) + \frac{T-t}{T}(x - wz) \right]$$

for $0 \le t \le T$. Moreover,

$$G(y) = -T \log y - E\left[\int_0^T \log \Gamma(t) dt \right], \quad y \in (0,\infty),$$

and the value function is

$$V(x,z) = T \log\left(\frac{x - wz}{T} \right) - E\left[\int_0^T \log \Gamma(t) dt \right], \quad (x,z) \in \mathcal{D}.$$

Note here that the conditions of Assumption 5.1 are satisfied; the first holds trivially, and the second is implied by the observation

$$E\big(\log \Gamma(t)\big) \le \log\big(E(\Gamma(t))\big) \le \varrho + \log\big(1 + \Delta T e^{\Delta T}\big) < \infty, \quad 0 \le t \le T,$$

where we used Jensen's inequality, (2.3) and the supermartingale property of $Z(\cdot)$. Finally, one may ascertain an explicit stochastic integral representation for $M_0(\cdot)$, defined in (5.16), under the additional assumption of *deterministic* model coefficients; cf. Example 7.9. The optimal portfolio process $\pi_0(\cdot)$ follows then by (5.15).

Remark 5.7. Consider utility functions such that

$$\sup_{0 \le t \le T} I(t,y) \le \kappa y^{-\rho}, \quad \forall\ y \in (0,\infty), \tag{5.22}$$

holds for some $\kappa > 0$, $\rho > 0$. Then, the first condition of Assumption 5.1 holds under *at least one* of the subsequent conditions:

$$0 < \rho \le 1, \tag{5.23}$$

or

$$\vartheta(\cdot) \text{ is bounded uniformly on } [0,T]\times\Omega. \tag{5.24}$$

In particular, (5.22) and (5.23) yield

$$\mathcal{X}(y) \le \kappa y^{-\rho} E\left[\int_0^T (1\vee\Gamma(t))\right] < \infty, \quad y\in(0,\infty).$$

Otherwise, use (5.24), (2.3) and Novikov condition to set $(H(t))^{1-\rho} = m(t)L(t)$, in terms of the uniformly bounded process

$$m(t) \triangleq \exp\left\{(\rho-1)\int_0^t r(v)dv + \frac{1}{2}\rho(\rho-1)\int_0^t \|\vartheta(v)\|^2 dv\right\},$$

and the martingale

$$L(t) \triangleq \exp\left\{(\rho-1)\int_0^t \vartheta^*(v)dW(v) - \frac{1}{2}(\rho-1)^2\int_0^t \|\vartheta(v)\|^2 dv\right\}.$$

Then (5.22) implies that

$$\mathcal{X}(y) \le \kappa y^{-\rho}\left(1+\Delta T e^{\varrho+\Delta T}\right)^{(1-\rho)} E\left[\int_0^T m(t)L(t)dt\right] < \infty, \quad y\in(0,\infty).$$

The function $V(\cdot,z)$ satisfies all the conditions of a utility function as defined in Section 3, for any given $z\ge 0$; we formalize this aspect of the value function in the result that follows, leading to the notion of a *generalized utility function* and to the explicit computation of its *convex dual*

$$\widetilde{V}(y) \triangleq \sup_{(x,z)\in\mathcal{D}}\left\{V(x,z)-(x-wz)y\right\}, \quad y\in\mathbb{R}. \tag{5.25}$$

THEOREM 5.8. *The function $V:\mathcal{D}\to\mathbb{R}$ is a generalized utility function, in the sense of being strictly concave and of class $C^{1,1}(\mathcal{D})$; it is strictly increasing in its first argument, strictly decreasing in the second, and satisfies $V_x((wz)^+,z)=\infty$, $V_x(\infty,z)=0$ for any $z\ge 0$. Additionally, for all pairs $(x,z)\in\mathcal{D}$, we have that*

$$\lim_{(x,z)\to(\chi,\zeta)} V(x,z) = \int_0^T u(t,0^+)dt, \quad \forall\ (\chi,\zeta)\in\partial\mathcal{D}, \tag{5.26}$$

where $\partial\mathcal{D}=\left\{(x',z')\in[0,\infty)^2;\ x'=wz'\right\}$ is the boundary of $\mathcal{D}$. Furthermore, with $\mathcal{X}(\cdot)$ and $G(\cdot)$ given by (5.3) and (5.18), respectively, we have

$$V_x(x,z)=\mathcal{Y}(x-wz), \qquad V_z(x,z)=-w\mathcal{Y}(x-wz), \quad \forall\ (x,z)\in\mathcal{D}, \tag{5.27}$$

$$\widetilde{V}(y) = G(y)-y\mathcal{X}(y) = E\int_0^T \tilde{u}(t,y\Gamma(t))dt, \quad \forall\ y>0, \tag{5.28}$$

$$\widetilde{V}'(y) = -\mathcal{X}(y), \quad \forall\ y>0\,. \tag{5.29}$$

Proof. We show first the strict concavity of V. Let $(x_1, z_1), (x_2, z_2) \in \mathcal{D}$ and $\lambda_1, \lambda_2 \in (0, 1)$ such that $\lambda_1 + \lambda_2 = 1$. For each (x_i, z_i) consider the optimal portfolio/consumption policy $(\pi_i, c_i) \in \mathcal{A}'(x_i, z_i)$ which generates the corresponding wealth process $X^{x_i,\pi_i,c_i}(\cdot)$, and the standard of living process $z_i(\cdot)$, $i = 1, 2$. Define now the portfolio/consumption plan $(\pi, c) \triangleq (\lambda_1\pi_1 + \lambda_2\pi_2, \lambda_1 c_1 + \lambda_2 c_2)$, denoting by $X^{x,\pi,c}(\cdot)$, $z(\cdot)$ the corresponding wealth and standard of living with $x \triangleq \lambda_1 x_1 + \lambda_2 x_2$ and $z \triangleq \lambda_1 z_1 + \lambda_2 z_2$. It is then easy to see that $(\pi, c) \in \mathcal{A}'(x, z)$ and

$$X^{x,\pi,c}(\cdot) = \lambda_1 X^{x_1,\pi_1,c_1}(\cdot) + \lambda_2 X^{x_2,\pi_2,c_2}(\cdot), \quad z(\cdot) = \lambda_1 z_1(\cdot) + \lambda_2 z_2(\cdot)$$

hold almost surely. Therefore, the strict concavity of $u(t, \cdot)$ implies

$$\begin{aligned}
&\lambda_1 V(x_1, z_1) + \lambda_2 V(x_2, z_2) \\
&= \lambda_1 E\left[\int_0^T u(t, c_1(t) - z_1(t))dt\right] + \lambda_2 E\left[\int_0^T u(t, c_2(t) - z_2(t))dt\right] \\
&< E\left[\int_0^T u(t, c(t) - z(t))dt\right] \le V(x, z) = V(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 z_1 + \lambda_2 z_2).
\end{aligned}$$

As a real-valued concave function on $\mathcal{D}$, V is continuous on its domain.

To establish (5.26), we consider pairs $(x, z) \in \mathcal{D}$, and observe from (5.17) that $\lim_{(x,z)\to(\chi,\zeta)} V(x, z) = \lim_{y\to\infty} G(y)$ holds for any $(\chi, \zeta) \in \partial\mathcal{D}$. But (3.2) indicates that $\lim_{y\to\infty} I(t, y\Gamma(t)) = 0$ for $0 \le t \le T$, and Assumption 5.1 ensures that $G(y)$ of (5.18) is finite for any $y \in (0, \infty)$; thus, (5.26) becomes a direct consequence of the monotone convergence theorem.

We next undertake (5.28). Its second equality is checked algebraically via (4.5), (5.3) and (5.18). Turning now to the first, for every $(x, z) \in \mathcal{D}$, $y > 0$ and $(\pi, c) \in \mathcal{A}'(x, z)$, the relation of (3.4) gives

$$u(t, c(t) - z(t)) \ \le\ \tilde{u}(t, y\Gamma(t)) + y\Gamma(t)(c(t) - z(t)). \tag{5.30}$$

Taking expectations, we employ (4.5), (5.2), (5.13) and the budget constraint (2.15) to obtain

$$\begin{aligned}
&E\int_0^T u(t, c(t) - z(t))dt \le E\int_0^T \Big[\tilde{u}(t, y\Gamma(t)) + y\Gamma(t)(c(t) - z(t))\Big]dt \\
&= E\int_0^T \tilde{u}(t, y\Gamma(t))dt \\
&\qquad + y\cdot E\int_0^T \Gamma(t)\left(c(t) - ze^{-\int_0^t \alpha(v)dv} - \int_0^t \delta(s)e^{-\int_s^t \alpha(v)dv}c(s)ds\right)dt \\
&= E\int_0^T \tilde{u}(t, y\Gamma(t))dt - ywz \\
&\qquad + y\cdot E\left[\int_0^T \Gamma(t)c(t)dt - \int_0^T \delta(s)\left(\int_s^T e^{-\int_s^t \alpha(v)dv}\Gamma(t)dt\right)c(s)ds\right] \qquad (5.31) \\
&= E\int_0^T \tilde{u}(t, y\Gamma(t))dt - ywz \\
&\qquad + y\cdot E\int_0^T \left\{\Gamma(t) - \delta(t)E_t\left(\int_t^T e^{-\int_t^s \alpha(v)dv}\Gamma(s)ds\right)\right\}c(t)dt
\end{aligned}$$

$$= E\int_0^T \tilde{u}(t, y\Gamma(t))dt - ywz + y \cdot E\int_0^T H(t)c(t)dt$$
$$\leq E\int_0^T \tilde{u}(t, y\Gamma(t))dt + y(x - wz) = G(y) - y\mathcal{X}(y) + y(x - wz).$$

The inequalities in (5.31) will hold as equalities, if and only if

$$c(t) - z(t) = I(t, y\Gamma(t)) \quad \text{and} \quad E\int_0^T H(t)c(t)dt = x. \tag{5.32}$$

Setting $Q(y) \triangleq G(y) - y\mathcal{X}(y)$ and maximizing over $(\pi, c) \in \mathcal{A}'(x, z)$, it follows from (5.31) that $V(x, z) \leq Q(y)+(x-wz)y$ for every $(x, z) \in \mathcal{D}$, and thereby $\widetilde{V}(y) \leq Q(y)$ for every $y > 0$. Conversely, (5.31) becomes an equality, if the first equation of (5.32) is satisfied and if $\mathcal{X}(y) = x - wz$, so $Q(y) = V(\mathcal{X}(y) + wz, z) - \mathcal{X}(y)y \leq \widetilde{V}(y)$. Hence (5.28) is established, and clearly the supremum in (5.25) is attained if $x - wz = \mathcal{X}(y)$.

We argue now (5.29) by bringing to our attention the identity

$$\begin{aligned} yI(t, y) - hI(t, h) - \int_h^y I(t, \lambda)d\lambda &= yI(t, y) - hI(t, h) + \tilde{u}(t, y) - \tilde{u}(t, h) \\ &= u(t, I(t, y)) - u(t, I(t, h)), \end{aligned} \tag{5.33}$$

which holds for any utility function u and $0 \leq t \leq T$, $0 < h < y < \infty$; recall (3.4) and (3.5). This enables us to compute

$$\begin{aligned} &y\mathcal{X}(y) - h\mathcal{X}(h) - \int_h^y \mathcal{X}(\xi)d\xi \\ &= E\int_0^T \left[yH(t)I(t, yH(t)) - hH(t)I(t, hH(t)) - \int_{hH(t)}^{yH(t)} I(t, \lambda)d\lambda \right] dt \\ &= E\int_0^T \Big[u\big(t, I(t, yH(t))\big) - u\big(t, I(t, hH(t))\big)\Big] dt = G(y) - G(h), \end{aligned} \tag{5.34}$$

which in conjunction with (5.28) leads to

$$\widetilde{V}(y) - \widetilde{V}(h) = -\int_h^y \mathcal{X}(\xi)d\xi, \quad 0 < h < y < \infty, \tag{5.35}$$

and (5.29) follows.

Finally, let us rewrite (5.25) in the more suggestive form

$$\widetilde{V}(y) = \sup_{(x,z)\in\mathcal{D}} \big\{ V(x, z) - (x, z) \cdot (y, -wy) \big\}, \quad y \in \mathbb{R},$$

where $v_1 \cdot v_2$ stands for the dot product between any two vectors v_1 and v_2. We recall that for $(x^*, z^*) \in \mathcal{D}$ and $y > 0$, we have $(y, -wy) \in \partial V(x^*, z^*)$ if and only if the maximum in the above expression is attained by (x^*, z^*) (e.g., Rockafellar (1970), Theorem 23.5). However, we have already shown that this maximum is attained by the pair (x^*, z^*) only if $x^* - wz^* = \mathcal{X}(y)$, implying

$$\partial V(x^*, z^*) = \big\{ \big(\mathcal{Y}(x^* - wz^*), -w\mathcal{Y}(x^* - wz^*)\big) \big\}.$$

Therefore, (5.27) is proved (e.g. Theorem 23.4 loc. cit.), and implies that $V_x(\cdot, z)$ is continuous, positive (thus $V(\cdot, z)$ strictly increasing), strictly decreasing on (wz, ∞), with $\lim_{x \downarrow wz} V_x(x,z) = \lim_{x \downarrow wz} \mathcal{Y}(x - wz) = \infty$ and $\lim_{x \uparrow \infty} V_x(x,z) = \lim_{x \uparrow \infty} \mathcal{Y}(x - wz) = 0$; while, $V_z(x,\cdot)$ is continuous, negative, and so $V(x,\cdot)$ decreases strictly. Consequently, V is a generalized utility function since it satisfies all its aforementioned properties. □

Remark 5.9. We note that given any $z \in [0,\infty)$, (5.17) can be written as $G(y) = V(\mathcal{X}(y) + wz, z)$ for every $y \in (0,\infty)$. Thus, if $\mathcal{X}(\cdot)$ is differentiable, then by (5.27), the function $G(\cdot)$ is also differentiable with

$$G'(y) = V_x(\mathcal{X}(y) + wz, z)\mathcal{X}'(y) = y\mathcal{X}'(y), \quad y \in (0,\infty). \tag{5.36}$$

6. The Role of Stochastic Partial Differential Equations. In Section 5 we established the existence and uniqueness, up to almost-everywhere equivalence, of a solution to our habit-modulated utility maximization problem in the case of a complete security market. The analysis resulted in a concrete representation for the optimal consumption process $c_0(\cdot)$, given by (5.10), but not for the optimal portfolio strategy $\pi_0(\cdot)$; we provided for it no useful expression aside from (5.15). In this section we shall confront this issue by deploying a technique based on the ideas of *dynamic programming.* Our motivation goes back to Theorem 5.5, which reveals the *dynamic* nature of the optimal wealth/standard of living pair $\big(X_0(\cdot), z_0(\cdot)\big)$ in terms of a stochastically evolving range.

Our analysis will be supported by the recently developed theory of backward *stochastic* partial differential equations and their interrelation with appropriate *adapted versions of stochastic* Feynman-Kac formulas. This interplay will be based on the generalized Itô-Kunita-Wentzell formula, and will show that the value function of problem (4.8) satisfies a *nonlinear*, backward *stochastic* Hamilton-Jacobi-Bellman partial differential equation of parabolic type.

We shall provide the optimal portfolio $\pi_0(t)$ and consumption policy $c_0(t)$ in closed, *stochastic* "feedback forms" on the current wealth $X_0(t)$ and the standard of living $z_0(t)$. In other words, we shall get hold of suitable *random fields* $C : [0,T) \times (0,\infty) \times [0,\infty) \times \Omega \to (0,\infty)$ and $\Pi : [0,T) \times (0,\infty) \times [0,\infty) \times \Omega \to \mathbb{R}^d$, for which

$$c_0(t) = C(t, X_0(t), z_0(t)) \quad \text{and} \quad \pi_0(t) = \Pi(t, X_0(t), z_0(t)), \qquad 0 \le t < T\,. \tag{6.1}$$

The conditions listed below will allow us to present the main concepts of our dynamic approach, with a minimum of technical fuss.

Assumption 6.1. *The model coefficients $r(\cdot)$, $b(\cdot)$, $\vartheta(\cdot)$, $\sigma(\cdot)$, $\alpha(\cdot)$ and $\delta(\cdot)$ are continuous, $\delta(\cdot)$ is differentiable, and $\|\vartheta(\cdot)\|$ is bounded away from zero and infinity:*

$$\exists\ k_1, k_2 > 0 \ \ \textit{such that} \quad 0 < k_1 \le \|\vartheta(t)\| \le k_2 < \infty, \quad \forall\ t \in [0,T]. \tag{6.2}$$

It will also be assumed that $r(\cdot) - \delta(\cdot) + \alpha(\cdot)$ is non-random.

This last assumption on $r(\cdot) - \delta(\cdot) + \alpha(\cdot)$ is rather severe, and can actually be omitted. It will be crucial, however, in our effort to keep the required analysis and notation at manageable levels, without obscuring by technicalities the essential ideas.

Since the market price of risk $\vartheta(\cdot)$ is bounded, the local martingale $Z(\cdot)$ of (2.6) is a martingale. Thus, by Girsanov's theorem, the process $W_0(\cdot)$ of (2.9) is standard, d-dimensional Brownian motion under the new probability measure

$$P^0(A) \triangleq E[Z(T)\mathbf{1}_A], \quad A \in \mathcal{F}(T). \tag{6.3}$$

We shall refer to P^0 as the *equivalent martingale measure* of the financial market $\mathcal{M}_0$, and denote expectation under this measure by E^0.

Assumption 6.2. *We shall assume that the utility function u satisfies* *(i) polynomial growth of I :*

$$\exists \ \gamma > 0 \quad \text{such that} \quad I(t,y) \leq \gamma + y^{-\gamma}, \quad \forall \ (t,y) \in [0,T] \times (0,\infty);$$

(ii) polynomial growth of $u \circ I$:

$$\exists \ \gamma > 0 \quad \text{such that} \quad u(t, I(t,y)) \geq -\gamma - y^{\gamma}, \ \ \forall \ (t,y) \in [0,T] \times (0,\infty);$$

(iii) for each $t \in [0,T]$, $y \mapsto u(t,y)$ and $y \mapsto I(t,y)$ are of class $C^4((0,\infty))$;
(iv) $I'(t,y) = \frac{\partial}{\partial y} I(t,y)$ is strictly negative for every $(t,y) \in [0,T] \times (0,\infty)$;
(v) for every $t \in [0,T]$, $y \mapsto g(t,y) \triangleq y I'(t,y)$ is increasing and concave.

Remark 6.3. Assumption 6.2(i), (ii), together with (3.3) and the strict decrease of $I(t,\cdot)$, yields that

$$\exists \ \gamma > 0 \ \text{ such that } |u(t, I(t,y))| \ \leq \gamma + y^{\gamma} + y^{-\gamma}, \quad \forall \ (t,y) \in [0,T] \times (0,\infty).$$

Notice that Assumptions 6.1 and 6.2(i), (ii) guarantee the validity of Assumption 5.1 in the preceding section; compare also with Remark 5.7. Moreover, the composite function $u(t, I(t,\cdot))$ inherits the order of smoothness posited in Assumption 6.2(iii) for its components, for every $t \in [0,T]$.

Preparing the ground of our approach, we state the following implication of the generalized Itô-Kunita-Wentzell formula (e.g. Kunita (1990), Section 3.3, pp 92-93). This will enable us to carry out computations in a stochastically modulated dynamic framework.

Proposition 6.4. *Suppose that the random field $\mathbf{F} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ is of class $C^{0,2}([0,T] \times \mathbb{R}^n)$ and satisfies*

$$\mathbf{F}(t,\mathrm{x}) = \mathbf{F}(0,\mathrm{x}) + \int_0^t \mathbf{f}(s,\mathrm{x}) ds + \int_0^t \mathbf{g}^*(s,\mathrm{x}) dW(s), \quad \forall \ (t,\mathrm{x}) \in [0,T] \times \mathbb{R}^n,$$

almost surely. Here $\mathbf{g} = (\mathbf{g}^{(1)}, \ldots, \mathbf{g}^{(d)})$, $\mathbf{g}^{(j)} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$, $j = 1, \ldots, d$ are $C^{0,2}([0,T] \times \mathbb{R}^n)$, $\mathbb{F}$-adapted random fields, and $\mathbf{f} : [0,T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ is a $C^{0,1}([0,T] \times \mathbb{R}^n)$ random field. Furthermore, let $\mathbf{X} = (\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(n)})$ be a vector of continuous semimartingales with decompositions

$$\mathbf{X}^{(i)}(t) = \mathbf{X}^{(i)}(0) + \int_0^t \mathbf{b}^{(i)}(s) ds + \int_0^t (\mathbf{h}^{(i)}(s))^* dW(s); \quad i = 1, \ldots, n,$$

where $\mathbf{h}^{(i)} = (\mathbf{h}^{(i,1)}, \ldots, \mathbf{h}^{(i,d)})$ is an $\mathbb{F}$-progressively measurable, almost surely square integrable vector process, and $\mathbf{b}^{(i)}(\cdot)$ is an almost surely integrable process. Then $\mathbf{F}(\cdot, \mathbf{X}(\cdot))$ is also a continuous semimartingale, with decomposition

$$\begin{aligned}
\mathbf{F}\big(t, \mathbf{X}(t)\big) &= \mathbf{F}\big(0, \mathbf{X}(0)\big) + \int_0^t \mathbf{f}\big(s, \mathbf{X}(s)\big) ds + \int_0^t \mathbf{g}^*\big(s, \mathbf{X}(s)\big) dW(s) \\
&\quad + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial \mathbf{x}_i} \mathbf{F}\big(s, \mathbf{X}(s)\big) \mathbf{b}^{(i)}(s) ds + \sum_{i=1}^n \int_0^t \frac{\partial}{\partial \mathbf{x}_i} \mathbf{F}\big(s, \mathbf{X}(s)\big) \Big(\mathbf{h}^{(i)}(s)\Big)^* dW(s) \\
&\quad + \sum_{j=1}^d \sum_{i=1}^n \int_0^t \frac{\partial}{\partial \mathbf{x}_i} \mathbf{g}^{(j)}\big(s, \mathbf{X}(s)\big) \mathbf{h}^{(i,j)}(s) ds \qquad (6.4)
\end{aligned}$$

$$+\frac{1}{2}\sum_{\ell=1}^{d}\sum_{i=1}^{n}\sum_{k=1}^{n}\int_0^t \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_k}\mathbf{F}\big(s,\mathbf{X}(s)\big)\mathbf{h}^{(i,\ell)}(s)\mathbf{h}^{(k,\ell)}(s)ds\,, \qquad 0\le t\le T\,.$$

The following notation will also be in use throughout the section.

Notation 6.5. For any integer $k \geq 0$, let $C^k(\mathbb{R}^n, \mathbb{R}^d)$ denote the set of functions from $\mathbb{R}^n$ to $\mathbb{R}^d$ that are continuously differentiable up to order k. In addition, for any $1 \leq p \leq \infty$, any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$, and any sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$, let

- $\mathbb{L}^p_{\mathcal{G}}(\Omega, \mathbb{X})$ denote the set of all $\mathbb{X}$-valued, $\mathcal{G}$-measurable random variables $\mathbf{X}$ such that $E\|\mathbf{X}\|^p_{\mathbb{X}} < \infty$;
- $\mathbb{L}^p_{\mathbb{F}}(0,T;\mathbb{X})$ denote the set of all $\mathbb{F}$-progressively measurable, $\mathbb{X}$-valued processes $\mathbf{X} : [0,T]\times\Omega \to \mathbb{X}$ such that $\int_0^T \|\mathbf{X}(t)\|^p_{\mathbb{X}}dt < \infty$, a.s.;
- $\mathbb{L}^p_{\mathbb{F}}(0,T;\mathbb{L}^p(\Omega;\mathbb{X}))$ denote the set of all $\mathbb{F}$-progressively measurable, $\mathbb{X}$-valued processes $\mathbf{X} : [0,T]\times\Omega \to \mathbb{X}$ such that $\int_0^T E\|\mathbf{X}(t)\|^p_{\mathbb{X}}dt < \infty$;
- $C_{\mathbb{F}}([0,T];\mathbb{X})$ denote the set of all continuous, $\mathbb{F}$-adapted processes $\mathbf{X}(\cdot,\omega) : [0,T] \to \mathbb{X}$ for P-a.e. $\omega \in \Omega$.

Define similarly the set $C_{\mathbb{F}}([0,T];\mathbb{L}^p(\Omega;\mathbb{X}))$, and let $\mathbb{R}^+$ stand for the positive real numbers.

For each $(t,y) \in [0,T]\times\mathbb{R}^+$ and $t \leq s \leq T$, we consider the stochastic processes

$$Z^t(s) \triangleq e^{-\int_t^s \vartheta^*(v)dW(v) - \frac{1}{2}\int_t^s \|\vartheta(v)\|^2 dv}, \qquad H^t(s) \triangleq e^{-\int_t^s r(v)dv} Z^t(s). \tag{6.5}$$

These extend the processes of (2.6) and (2.8), respectively, to initial times other than zero. In accordance with (5.1), we shall also consider the extended "adjusted" state-price density process

$$\begin{aligned}\Gamma^t(s) &\triangleq H^t(s) + \delta(s)\cdot E_s\left(\int_s^T e^{\int_s^\theta(\delta(v)-\alpha(v))dv}H^t(\theta)d\theta\right)\\ &= H^t(s)\left[1+\delta(s)\cdot E_s\left(\int_s^T e^{\int_s^\theta(\delta(v)-\alpha(v))dv}H^s(\theta)d\theta\right)\right]\\ &= H^t(s)\left[1+\delta(s)\int_s^T e^{\int_s^\theta(-r(v)+\delta(v)-\alpha(v))dv}d\theta\right] = H^t(s)\mu(s), \quad t\le s\le T.\end{aligned}\tag{6.6}$$

We have invoked here Assumption 6.1, the martingale property of $Z(\cdot)$, and have set

$$\mu(t) \triangleq 1 + \delta(t)w(t), \text{ where } w(t) \triangleq \int_t^T e^{\int_t^s(-r(v)+\delta(v)-\alpha(v))dv}ds, \quad t\in[0,T]\,, \tag{6.7}$$

$$w'(t) \triangleq \frac{d}{dt}w(t) = \big[r(t)+\alpha(t)-\delta(t)\big]w(t) - 1 = \big[r(t)+\alpha(t)\big]w(t) - \mu(t) \tag{6.8}$$

for $0 \leq t \leq T$. Note that $w(\cdot)$ is the deterministic reduction of $\mathcal{W}(\cdot)$ in (5.20); namely, $\mathcal{W}(\cdot) \equiv w(\cdot)$ within the context of the current section.

Furthermore, we define the diffusion process

$$Y^{(t,y)}(s) \triangleq y\Gamma^t(s), \qquad t\le s\le T, \tag{6.9}$$

which, from (6.5) and (6.6), satisfies the linear stochastic differential equation

$$dY^{(t,y)}(s) = Y^{(t,y)}(s)\left[\left(\frac{\mu'(s)}{\mu(s)} - r(s)\right)ds - \vartheta^*(s)dW(s)\right], \tag{6.10}$$

or equivalently

$$dY^{(t,y)}(s) = Y^{(t,y)}(s)\left[\left(\frac{\mu'(s)}{\mu(s)} - r(s) + \|\vartheta(s)\|^2\right)ds - \vartheta^*(s)dW_0(s)\right], \tag{6.11}$$

and $Y^{(t,y)}(t) = y\mu(t)$, $Y^{(t,y)}(s) = yY^{(t,1)}(s) = yH(s)\mu(s)/H(t)$. Invoking the "Bayes rule" for conditional expectations, a computation akin to the one presented in the proof of Theorem 5.5 shows that the optimal wealth/standard of living vector process $\big(X_0(\cdot), z_0(\cdot)\big)$ of (5.8), (5.14), satisfies

$$\begin{aligned} X_0(t) - w(t)z_0(t) &= \frac{1}{\xi}\, E_t\left[\int_t^T Y^{(t,\xi)}(s)I(s, Y^{(0,\xi)}(s))ds\right] \\ &= E_t^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s, Y^{(0,\xi)}(s))ds\right] = \mathcal{X}\left(t, \frac{Y^{(0,\xi)}(t)}{\mu(t)}\right) \end{aligned} \tag{6.12}$$

for $0 \le t \le T$ and $\xi = \mathcal{Y}(x - wz)$. We have used here the definition (6.9), and introduced the random field $\mathcal{X} : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ defined as

$$\mathcal{X}(t,y) \triangleq E_t^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s, yY^{(t,1)}(s))ds\right]. \tag{6.13}$$

A comparison of (5.3), (6.12) and (6.13) divulges the dynamic and stochastic evolution of the function $\mathcal{X}(\cdot)$ as a random field in the sense that $\mathcal{X}(\cdot) = \mathcal{X}(0,\cdot)$.

We proceed with the derivation of the random fields C and Π in (6.1) by formulating first a semimartingale decomposition for the random field $\mathcal{X}$ of (6.13). A significant role in this program will be played by an appropriate backward stochastic partial differential equation, whose unique adapted solution will lead, via the generalized Itô-Kunita-Wentzell rule, to a stochastic Feynman-Kac formula and consequently to the desired decomposition for $\mathcal{X}$.

Let us start by looking at the Cauchy problem for the parabolic *Backward Stochastic PDE* (BSPDE for brevity):

$$\begin{aligned} -d\mathcal{U}(t,\eta) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2\mathcal{U}_{\eta\eta}(t,\eta) + \left(\frac{\mu'(t)}{\mu(t)} - r(t) + \frac{1}{2}\|\vartheta(t)\|^2\right)\mathcal{U}_\eta(t,\eta) \\ &- r(t)\mathcal{U}(t,\eta) - \vartheta^*(t)\Psi_\eta(t,\eta) + \mu(t)I(t,e^\eta)\Bigg]dt - \Psi^*(t,\eta)dW_0(t) \end{aligned} \tag{6.14}$$

for $\eta \in \mathbb{R},\ 0 \le t < T$, as well as the terminal condition

$$\mathcal{U}(T,\eta) = 0, \quad \eta \in \mathbb{R} \tag{6.15}$$

for the pair of $\mathbb{F}$-adapted random fields $\mathcal{U}$ and Ψ. According to Assumptions 6.1, 6.2, and the study of parabolic backward stochastic partial differential equations by Ma and Yong (1997), the problem (6.14), (6.15) admits a unique solution pair $(\mathcal{U},\Psi) \in C_{\mathbb{F}}\big([0,T];\mathbb{L}^2(\Omega;C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T;\mathbb{L}^2(\Omega;C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$. Apply the generalized Itô-Kunita-Wentzell formula (cf. Proposition 6.4) for a fixed pair $(t,y) \in [0,T) \times \mathbb{R}^+$, in conjunction with the dynamics of (6.11) and the equation of (6.14), to get

$$\begin{aligned} d\Big[e^{-\int_t^s r(v)dv}\mathcal{U}(s,\log Y^{(t,y)}(s))\Big] &= -e^{-\int_t^s r(v)dv}\mu(s)I(s,Y^{(t,y)}(s))ds \\ &- e^{-\int_t^s r(v)dv}\Big[\vartheta(s)\mathcal{U}_\eta(s,\log Y^{(t,y)}(s)) - \Psi(s,\log Y^{(t,y)}(s))\Big]^* dW_0(s), \end{aligned} \tag{6.16}$$

almost surely. Adopting the proof of Corollary 6.2 in the above citation (p. 76), integrate over $[t,T]$, take conditional expectations with respect to the martingale measure P^0, and make use of (6.13), (6.15) to end up with

$$\mathcal{X}(t,y) = \mathcal{U}(t, \log(y\mu(t))) \tag{6.17}$$

for every $(t,y) \in [0,T] \times \mathbb{R}^+$. We define, accordingly, the random field $\Psi^{\mathcal{X}} : [0,T] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}^d$ by

$$\Psi^{\mathcal{X}}(t,y) \triangleq \Psi(t, \log(y\mu(t)))\,. \tag{6.18}$$

Lemma 6.6. *Considering Assumptions 6.1 and 6.2, the pair of random fields* $(\mathcal{X}, \Psi^{\mathcal{X}})$*, where* $\mathcal{X}$ *is provided by (6.13) and* $\Psi^{\mathcal{X}}$ *by (6.18), belongs to the class* $C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ *and is the unique solution of the Cauchy problem*

$$\begin{aligned} -d\mathcal{X}(t,y) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2 y^2 \mathcal{X}_{yy}(t,y) + \big(\|\vartheta(t)\|^2 - r(t)\big)\, y\mathcal{X}_y(t,y) - r(t)\mathcal{X}(t,y) \\ &- \vartheta^*(t) y \Psi^{\mathcal{X}}_y(t,y) + \mu(t) I(t, y\mu(t))\Bigg] dt - \big(\Psi^{\mathcal{X}}(t,y)\big)^* \, dW_0(t) \end{aligned} \tag{6.19}$$

on $[0,T) \times \mathbb{R}^+$*, as well as the terminal condition*

$$\mathcal{X}(T,y) = 0 \quad \textit{on} \;\; \mathbb{R}^+, \tag{6.20}$$

almost surely. Furthermore, for each $t \in [0,T)$*, we have that* $\mathcal{X}(t,0^+) = \infty$*,* $\mathcal{X}(t,\infty) = 0$ *and* $\mathcal{X}(t,\cdot)$ *is strictly decreasing, establishing the existence of a strictly decreasing inverse random field* $\mathcal{Y}(t,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \xrightarrow{onto} \mathbb{R}^+$*, such as*

$$\mathcal{X}(t, \mathcal{Y}(t,x)) = x, \qquad \textit{for all } x \in \mathbb{R}^+, \tag{6.21}$$

almost surely. The random field $\mathcal{Y}$ *is of class* $C_{\mathbb{F}}\big([0,T); C^3(\mathbb{R}^+)\big)$.

Proof. From (6.14), (6.15), (6.17) and (6.18), it is verified directly that the pair of random fields $(\mathcal{X}, \Psi^{\mathcal{X}})$ possesses the desired regularity and constitutes the unique solution of the Cauchy problem (6.14), (6.15), almost surely.

Next, we shall verify that $\mathcal{X}_y(t,y)$ is strictly negative, almost surely. To this end, let $(t,y) \in [0,T) \times \mathbb{R}^+,\ h > 0$, and invoke the (strict) decrease of $I(t,\cdot)$, coupled with (2.3), to verify that

$$\frac{1}{h}\big[\mathcal{X}(t,y) - \mathcal{X}(t,y+h)\big] \geq E^0_t\left[\int_t^T \frac{e^{-\varrho}}{h}\Big\{I(s, yY^{(t,1)}(s)) - I\big(s, (y+h)Y^{(t,1)}(s)\big)\Big\} ds\right].$$

By the mean-value theorem, there is a real number $y_h \in [y, y+h]$ such that

$$I(s, yY^{(t,1)}(s)) - I\big(s, (y+h)Y^{(t,1)}(s)\big) = -hY^{(t,1)}(s) I'\big(s, y_h Y^{(t,1)}(s)\big),$$

and conditions (2.3), (4.4), (6.2) imply the inequality $Y^{(t,1)}(s) \leq \phi(s) Z^t_0(s)$, in terms of the deterministic function $\phi(t) \triangleq (1 + \Delta w(t)) e^{\varrho + \kappa_2^2 (T-t)}$ and the P^0-martingale

$$Z^t_0(s) \triangleq \exp\left\{-\int_t^s \vartheta^*(v) dW_0(v) - \frac{1}{2}\int_t^s \|\vartheta(v)\|^2 dv\right\}, \quad t \leq s \leq T. \tag{6.22}$$

Due to Assumption 6.2(v), the right-hand side of the former inequality attains the lower bounds

$$-E_t^0\left[\int_t^T \frac{e^{-\varrho}}{y_h}\, g\big(s, y_h\phi(s)Z_0^t(s)\big)ds\right] \geq -\int_t^T \frac{e^{-\varrho}}{y_h}\, g\Big(s, y_h\phi(s)E_t^0\big(Z_0^t(s)\big)\Big)ds$$
$$= -e^{-\varrho}\int_t^T I'(s, y_h\phi(s))\phi(s)ds,$$

where we have also used Jensen's inequality. Passing to the limit as $h \downarrow 0$, we obtain from Fatou's lemma

$$\mathcal{X}_y(t,y) \leq e^{-\varrho}\int_t^T I'(s, y\phi(s))\phi(s)ds < 0.$$

According to the implicit function theorem, the inverse random field $\mathcal{Y} : [0,T)\times\mathbb{R}^+\times\Omega \xrightarrow{onto} \mathbb{R}^+$ of $\mathcal{X}$ exists almost surely, in the context of (6.21); in fact, the two random fields enjoy the same order of regularity on their respective domains. Concluding, the claimed values of $\mathcal{X}(t,0^+)$ and $\mathcal{X}(t,\infty)$ are easily confirmed, respectively, by the monotone and dominated convergence theorem. □

Remark 6.7. At this point, we should note that Lemma 6.6 assigns to the pair of random fields $(\mathcal{X}, \Psi^{\mathcal{X}})$ an additional order of smoothness than is required in order to solve the stochastic partial differential equation (6.19), (6.20). Nevertheless, this extra smoothness allows us to apply the Itô-Kunita-Wentzell formula, as we did already in (6.16). Furthermore, the above lemma yields the representation

$$\mathcal{X}(t,y) = \int_t^T \Bigg[\frac{1}{2}\|\vartheta(s)\|^2 y^2\mathcal{X}_{yy}(s,y) + \big(\|\vartheta(s)\|^2 - r(s)\big)\, y\mathcal{X}_y(s,y) - r(s)\mathcal{X}(s,y)$$
$$- \vartheta^*(s)y\Psi_y^{\mathcal{X}}(s,y) + \mu(s)I(s,y\mu(s))\Bigg]ds - \int_t^T \big(\Psi^{\mathcal{X}}(s,y)\big)^* dW_0(s)$$

for the pair $(\mathcal{X}, \Psi^{\mathcal{X}})$, namely, the semimartingale decomposition of the stochastic processes $\mathcal{X}(\cdot,y)$ defined in (6.13) for each $y \in \mathbb{R}^+$.

The random field $\mathcal{Y}$ represents the random dynamic extension of the function $\mathcal{Y}(\cdot)$, established in Section 5. In particular, $\mathcal{Y}(\cdot) = \mathcal{Y}(0,\cdot)$.

Remark 6.8. Combining (2.7), (6.16), (6.17) and (6.18), we obtain the dynamics

$$d\left[\beta(s)\mathcal{X}\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right] = -\beta(s)\Bigg\{\mu(s)I(s, Y^{(0,y)}(s))ds$$
$$+ \left[\vartheta(s)\frac{Y^{(0,y)}(s)}{\mu(s)}\mathcal{X}_y\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right) - \Psi^{\mathcal{X}}\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right]^* dW_0(s)\Bigg\}.$$

Therefore, via integration, we arrive at the relationship

$$\beta(t)\mathcal{X}\left(t, \frac{Y^{(0,y)}(t)}{\mu(t)}\right) + \int_0^t \beta(s)\mu(s)I(s, Y^{(0,y)}(s))ds \tag{6.23}$$
$$= \mathcal{X}(0,y) - \int_0^t \beta(s)\left[\vartheta(s)\frac{Y^{(0,y)}(s)}{\mu(s)}\mathcal{X}_y\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right) - \Psi^{\mathcal{X}}\left(s, \frac{Y^{(0,y)}(s)}{\mu(s)}\right)\right]^* dW_0(s)$$

for every $(t, y) \in [0, T] \times \mathbb{R}^+$, almost surely.

We are in position now to obtain *stochastic feedback formulae* for the optimal investment and consumption processes. In view of (6.12), for each $t \in [0, T)$, the effective range for the running optimal wealth $X_0(t)$ and for the associated standard of living $z_0(t)$ will be

$$\mathcal{D}_t \triangleq \big\{(x', z') \in \mathbb{R}^+ \times [0, \infty);\ x' > w(t)z'\big\}. \tag{6.24}$$

THEOREM 6.9. *Under the Assumptions 6.1 and 6.2, the optimal consumption* $c_0(\cdot)$ *and the optimal trading strategy* $\pi_0(\cdot)$ *of the dynamic optimization problem (4.8) admit the stochastic adapted feedback forms of (6.1), determined by the random fields*

$$C(t, x, z) \triangleq z + I\big(t, \mu(t)\mathcal{Y}(t, x - w(t)z)\big), \tag{6.25}$$

$$\Pi(t, x, z) \triangleq -\frac{1}{x}(\sigma^*(t))^{-1}\left[\vartheta(t)\frac{\mathcal{Y}\big(t, x - w(t)z\big)}{\mathcal{Y}_x\big(t, x - w(t)z\big)} - \Psi^{\mathcal{X}}\Big(t, \mathcal{Y}\big(t, x - w(t)z\big)\Big)\right], \tag{6.26}$$

for $t \in [0, T)$ *and any pair* $(x, z) \in \mathcal{D}_t$.

Proof. For any initial wealth x and standard of living z such that $(x, z) \in \mathcal{D}_0$ of (6.24), we may rewrite (6.12) as

$$Y^{(0,\mathcal{J})}(t)\Big|_{\mathcal{J}=\mathcal{Y}(0,x-wz)} = \mu(t)\mathcal{J}(t)$$

with $\mathcal{J}(t) \triangleq \mathcal{Y}\big(t, X_0(t) - w(t)z_0(t)\big)$. From (5.5) and (5.8), it develops that the optimal consumption process of (5.10) is expressed by

$$c_0(t) = z_0(t) + I\big(t, \mu(t)\mathcal{J}(t)\big)$$

for $0 \le t < T$, and (6.25) is proved. Considering (6.23) for $y = \mathcal{Y}(0, x - wz)$, in connection with (6.12), we obtain

$$\begin{aligned}\beta(t)\Big[X_0(t) - w(t)z_0(t)\Big] &+ \int_0^t \beta(s)\mu(s)\Big[c_0(s) - z_0(s)\Big]ds \\ &= x - wz - \int_0^t \beta(s)\Big[\vartheta(s)\mathcal{J}(s)\mathcal{X}_y\left(s, \mathcal{J}(s)\right) - \Psi^{\mathcal{X}}(s, \mathcal{J}(s))\Big]^* dW_0(s).\end{aligned}$$

Differentiating (6.21), we arrive at $\mathcal{X}_y\big(t, \mathcal{Y}(t, x - w(t)z)\big) = 1/\mathcal{Y}_x(t, x - w(t)z)$ for every $(x, z) \in \mathcal{D}_t$; setting $\mathcal{J}_x(t) \triangleq \mathcal{Y}_x\big(t, X_0(t) - w(t)z_0(t)\big)$ and using (6.7), the above equation becomes

$$\begin{aligned}\beta(t)X_0(t) &+ \int_0^t \beta(s)c_0(s)ds \\ &= x - \int_0^t \beta(s)\left[\vartheta(s)\frac{\mathcal{J}(s)}{\mathcal{J}_x(s)} - \Psi^{\mathcal{X}}(s, \mathcal{J}(s))\right]^* dW_0(s) + \beta(t)w(t)z_0(t) \\ &\quad - wz - \int_0^t \beta(s)\delta(s)w(s)\Big[c_0(s) - z_0(s)\Big]ds + \int_0^t \beta(s)z_0(s)ds.\end{aligned} \tag{6.27}$$

On the other hand, use (4.3) and (6.8) to compute

$$\begin{aligned}\beta(t)w(t)z_0(t) - wz &= \int_0^t d\Big(\beta(s)w(s)z_0(s)\Big) \\ &= \int_0^t \beta(s)\delta(s)w(s)\Big[c_0(s) - z_0(s)\Big]ds - \int_0^t \beta(s)z_0(s)ds,\end{aligned} \tag{6.28}$$

and conclude that (6.27) reads

$$\beta(t)X_0(t) + \int_0^t \beta(s)c_0(s)ds = x - \int_0^t \beta(s)\left[\vartheta(s)\frac{\mathcal{J}(s)}{\mathcal{J}_x(s)} - \Psi^{\mathcal{X}}(s,\mathcal{J}(s))\right]^* dW_0(s),$$

almost surely. A comparison of the later with the integral expression (2.12) implies that (6.26) follows from $X_0(t)\pi_0^*(t)\sigma(t) = -\left[\vartheta(t)\frac{\mathcal{J}(t)}{\mathcal{J}_x(t)} - \Psi^{\mathcal{X}}(t,\mathcal{J}(t))\right]^*$. □

Remark 6.10. Under the additional assumption of *deterministic* coefficients, $r(\cdot):[0,T]\to\mathbb{R}$, $\vartheta(\cdot):[0,T]\to\mathbb{R}^d$, $\sigma(\cdot):[0,T]\to L(\mathbb{R}^d;\mathbb{R}^d)$, the set of $(d\times d)$ matrices, $\alpha(\cdot):[0,T]\to[0,\infty)$ and $\delta(\cdot):[0,T]\to[0,\infty)$, the process $Y^{(t,y)}(\cdot)$ of (6.9) obtains the Markov property. Hence, the random fields of (6.25) and (6.26), which represent the optimal policies in feedback form, reduce to the deterministic functions

$$C(t,x,z) = z + I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big), \tag{6.29}$$

$$\Pi(t,x,z) = -(\sigma^*(t))^{-1}\vartheta(t)\cdot\frac{\mathcal{Y}\big(t,x-w(t)z\big)}{x\mathcal{Y}_x\big(t,x-w(t)z\big)}\ ; \tag{6.30}$$

here $\mathcal{Y}(t,\cdot)$ is the inverse of the function

$$\mathcal{X}(t,y) = E^0\left[\int_t^T e^{-\int_t^s r(v)dv}\mu(s)I(s,yY^{(t,1)}(s))ds\right], \qquad 0<y<\infty\,.$$

[cf. Lemma 6.6 and (6.13)]. It is then evident that the decision-maker needs only to keep track of his current level of wealth $X_0(t)$ and standard of living $z_0(t)$, not of the entire history of the market up to time t; in other words, these processes serve as *sufficient statistics* for the optimization problem (4.8)

7. The Stochastic Hamilton-Jacobi-Bellman Equation. We shall investigate now the analytical behavior of the value function for the optimization problem (4.8) as a solution of a nonlinear partial differential equation, widely referred to as the stochastic Hamilton-Jacobi-Bellman equation. In this vein, we find it useful to generalize the time-horizon of our asset market $\mathcal{M}_0$ by taking initial date $t\in[0,T]$ rather than zero. Hence, for a fixed starting time $t\in[0,T]$ and any given capital wealth/initial standard of living pair $(x,z)\in\mathcal{D}_t$ (cf. (6.24)), the wealth process $X^{t,x,\pi,c}(\cdot)$, corresponding to a portfolio strategy $\pi(\cdot)$ and a consumption process $c(\cdot)$, satisfies the stochastic integral equation

$$X(s) = x + \int_t^s [r(v)X(v) - c(v)]dv + \int_t^s X(v)\pi^*(v)\sigma(v)dW_0(v), \tag{7.1}$$

for $t\le s\le T$, and the respective standard of living process $z(\cdot)$ is developed by

$$z(s) = ze^{-\int_t^s \alpha(\theta)d\theta} + \int_t^s \delta(v)e^{-\int_v^s \alpha(\theta)d\theta}c(v)dv, \quad t\le s\le T. \tag{7.2}$$

In this context, we shall call *admissible at the initial condition* (t,x), and denote their class by $\mathcal{A}(t,x)$, all portfolio/consumption pairs (π,c) such that $X^{t,x,\pi,c}(s)\ge 0$, $\forall\ s\in[t,T]$, almost surely. Each of these pairs satisfies the budget constrain

$$E_t\left[\int_t^T H^t(s)c(s)ds\right] \le x. \tag{7.3}$$

Conversely, a variant of Lemma 2.1, subject to an initial date t that is not necessarily zero, shows that for every given consumption plan $c(\cdot)$ satisfying (7.3) we can fashion a portfolio strategy $\pi(\cdot)$ such that $(\pi, c) \in \mathcal{A}(t, x)$. Furthermore, we extend the optimization problem of Definition 4.2 by the random field

$$V(t,x,z) \triangleq \operatorname*{ess\,sup}_{(\pi,c)\in\mathcal{A}'(t,x,z)} E_t\left[\int_t^T u\left(s, c(s)-z(s)\right) ds\right], \tag{7.4}$$

where

$$\mathcal{A}'(t,x,z) \triangleq \left\{(\pi,c)\in\mathcal{A}(t,x);\ E_t\left[\int_t^T u^-\left(s, c(s)-z(s)\right) ds\right] < \infty,\ \text{a.s.}\right\},$$

and $V(0,\cdot,\cdot) = V(\cdot,\cdot)$. Summoning Assumptions 6.1 and 6.2, we obtain

$$V(t,x,z) = G\big(t, \mathcal{Y}(t, x - w(t)z)\big), \quad (x,z)\in\mathcal{D}_t, \quad t\in[0,T), \tag{7.5}$$

almost surely, where we have also introduced the real-valued random field

$$G(t,y) \triangleq E_t\left[\int_t^T u\big(s, I(s, yY^{(t,1)}(s))\big) ds\right], \qquad (t,y)\in[0,T]\times\mathbb{R}^+. \tag{7.6}$$

One observes that the random fields (7.4) and (7.6) constitute the dynamic, probabilistic analogues of those in (4.8) and (5.18) respectively, since $V(\cdot,\cdot) = V(0,\cdot,\cdot)$ and $G(\cdot) = G(0,\cdot)$; this complies with the temporal and stochastic evolution of the function $\mathcal{X}(\cdot)$ described in the previous section. Clearly

$$V(T,x,z) = 0, \quad \forall\ (x,z)\in\mathcal{D}; \tag{7.7}$$

in fact, $V(t,x,z) < \infty$ for every $t\in[0,T)$, $(x,z)\in\mathcal{D}_t$, and with $\partial\mathcal{D}_t = \big\{(x',z')\in[0,\infty)^2;\ x' = w(t)z'\big\}$ the boundary of $\mathcal{D}_t$ (cf. (5.26)) we have

$$\lim_{(x,z)\to(\chi,\zeta)} V(t,x,z) = \int_t^T u(s,0^+)ds, \quad \forall\ (\chi,\zeta)\in\partial\mathcal{D}_t. \tag{7.8}$$

We shall derive next a semimartingale decomposition for the random field G of (7.6). Recalling Assumptions 6.1, 6.2, and making use of the methodology developed in the proof of (6.19), (6.20), we consider the unique solution $(\mathcal{V},\Phi) \in C_{\mathbb{F}}\big([0,T];\mathbb{L}^2(\Omega;C^3(\mathbb{R}^+))\big)\times\mathbb{L}^2_{\mathbb{F}}\big(0,T;\mathbb{L}^2(\Omega;C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ of the Cauchy problem

$$\begin{aligned}-d\mathcal{V}(t,\eta) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2\mathcal{V}_{\eta\eta}(t,\eta) + \left(\frac{\mu'(t)}{\mu(t)} - r(t) - \frac{1}{2}\|\vartheta(t)\|^2\right)\mathcal{V}_\eta(t,\eta)\\ &- \vartheta^*(t)\Phi_\eta(t,\eta) + u(t, I(t,e^\eta))\Bigg]dt - \Phi^*(t,\eta)dW(t)\end{aligned} \tag{7.9}$$

for $\eta\in\mathbb{R},\ 0\le t<T$, and the terminal condition

$$\mathcal{V}(T,\eta) = 0, \quad \eta\in\mathbb{R}, \tag{7.10}$$

almost surely. As in (6.16), an application of Itô-Kunita-Wentzell formula, in conjunction with (6.10) and (7.9), yields

$$\begin{aligned} d\mathcal{V}(s,\log Y^{(t,y)}(s)) = &-u\left(s, I(s,Y^{(t,y)}(s))\right)ds \\ &- \left[\mathcal{V}_\eta(s,\log Y^{(t,y)}(s))\vartheta(s) - \Phi(s,\log Y^{(t,y)}(s))\right]^* dW(s), \end{aligned}$$

and by analogy with (6.17), leads to

$$\mathcal{V}(t,\log(y\mu(t))) = E_t\left[\int_t^T u\big(s, I(s,Y^{(t,y)}(s))\big)ds\right] = G(t,y). \tag{7.11}$$

We also introduce the random field $\Phi^G : [0,T]\times\mathbb{R}^+\times\Omega \to \mathbb{R}^d$ via

$$\Phi^G(t,y) \triangleq \Phi(t,\log(y\mu(t))), \tag{7.12}$$

and have the following result.

Lemma 7.1. *Adopting Assumptions 6.1 and 6.2, the pair of random fields (G,Φ^G), where G is given by (7.6) and Φ^G by (7.12), is of class $C_{\mathbb{F}}\big([0,T];\mathbb{L}^2(\Omega;C^3(\mathbb{R}^+))\big)\times \mathbb{L}^2_{\mathbb{F}}\big(0,T;\mathbb{L}^2(\Omega;C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ and the unique solution of the Cauchy problem*

$$\begin{aligned} -dG(t,y) = &\left[\frac{1}{2}\|\vartheta(t)\|^2y^2G_{yy}(t,y) - r(t)yG_y(t,y)\right. \\ &\left. -\vartheta^*(t)y\Phi^G_y(t,y) + u(t,I(t,y\mu(t)))\right]dt - \big(\Phi^G(t,y)\big)^* dW(t) \end{aligned} \tag{7.13}$$

on $[0,T)\times\mathbb{R}^+$, and the terminal condition

$$G(T,y) = 0 \quad \text{on } \mathbb{R}^+, \tag{7.14}$$

almost surely. Moreover, for every $(t,y)\in[0,T)\times\mathbb{R}^+$ we have almost surely:

$$G(t,y) - G(t,h) = y\mathcal{X}(t,y) - h\mathcal{X}(t,h) - \int_h^y \mathcal{X}(t,\xi)d\xi, \quad 0<h<y<\infty, \tag{7.15}$$

$$G_y(t,y) = y\mathcal{X}_y(t,y), \qquad G_{yy}(t,y) = \mathcal{X}_y(t,y) + y\mathcal{X}_{yy}(t,y). \tag{7.16}$$

Once again (cf. Remark 6.7), the additional smoothness of (G,Φ^G) will be essential in the formalization of explicit calculations, and the semimartingale decomposition of the process $G(\cdot,y)$, $y\in\mathbb{R}^+$, is realized by

$$\begin{aligned} G(t,y) = &\int_t^T \left[\frac{1}{2}\|\vartheta(s)\|^2y^2G_{yy}(s,y) - r(s)yG_y(s,y)\right. \\ &\left. -\vartheta^*(s)y\Phi^G_y(s,y) + u(s,I(s,y\mu(s)))\right]ds - \int_t^T \big(\Phi^G(s,y)\big)^* dW(s). \end{aligned}$$

Proof of Lemma 7.1: Use (7.9), (7.10), (7.11) and (7.12) to check that the pair of random fields (G,Φ^G) has the asserted order of regularity and is the unique solution of the Cauchy problem (7.13), (7.14), almost surely. Repeat the computations in (5.34) concerning conditional expectations, subject to an initial time $t \neq 0$, to obtain (7.15); differentiation then yields (7.16). $\square$

We carry on our analysis with the subsequent lemma, which copes with the semimartingale decomposition of the random field $\mathcal{Y}$, defined in Lemma 6.6.

LEMMA 7.2. *Consider the hypotheses of Lemma 6.6. Then, there exists a pair of random fields* $(\Theta, \Sigma) \in \mathbb{L}_{\mathbb{F}}\big(0, T^{'}; C^1(\mathbb{R}^+)\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0, T^{'}; C^2(\mathbb{R}^+; \mathbb{R}^d)\big)$ *for each* $0 < T^{'} < T$, *such that*

$$-d\mathcal{Y}(t,x) = \Theta(t,x)dt - \Sigma^*(t,x)dW_0(t) \tag{7.17}$$

holds almost surely, for every $(t,x) \in [0,T) \times \mathbb{R}^+$. *In particular, these random fields are uniquely determined by the relationships:*

$$\begin{aligned}
&\frac{1}{2}\Big[\|\Sigma(t,x)\|^2 - \|\vartheta(t)\|^2\mathcal{Y}^2(t,x)\Big]\mathcal{X}_{yy}(t,\mathcal{Y}(t,x)) - \mu(t)I\big(t,\mu(t)\mathcal{Y}(t,x)\big)\\
&+ \Big[\big(r(t) - \|\vartheta(t)\|^2\big)\mathcal{Y}(t,x) + \vartheta^*(t)\Sigma(t,x) - \Theta(t,x)\Big]\mathcal{X}_y(t,\mathcal{Y}(t,x))\\
&+ r(t)x + \Big[\Sigma(t,x) + \vartheta(t)\mathcal{Y}(t,x)\Big]^*\Psi^{\mathcal{X}}_y(t,\mathcal{Y}(t,x)) + \vartheta^*(t)\Psi^{\mathcal{X}}(t,\mathcal{Y}(t,x)) = 0
\end{aligned} \tag{7.18}$$

and

$$\mathcal{X}_y(t,\mathcal{Y}(t,x))\Sigma(t,x) + \Psi^{\mathcal{X}}(t,\mathcal{Y}(t,x)) = 0. \tag{7.19}$$

Proof. Let $(t,x) \in [0,T) \times \mathbb{R}^+$. Invoking equation (6.19) for $\mathcal{X}$ and postulating the representation (7.17) for $\mathcal{Y}$, we may apply differentials and Proposition 6.4 on identity (6.21), and integrate over $[0,t]$, to compute

$$\begin{aligned}
&\int_0^t \Big\{\frac{1}{2}\Big[\|\Sigma(s,x)\|^2 - \|\vartheta(s)\|^2\mathcal{Y}^2(s,x)\Big]\mathcal{X}_{yy}(s,\mathcal{Y}(s,x)) - \mu(s)I\big(s,\mu(s)\mathcal{Y}(s,x)\big)\\
&\qquad + \Big[\big(r(s) - \|\vartheta(s)\|^2\big)\mathcal{Y}(s,x) + \vartheta^*(s)\Sigma(s,x) - \Theta(s,x)\Big]\mathcal{X}_y(s,\mathcal{Y}(s,x))\\
&\qquad + r(s)x + \Big[\Sigma(s,x) + \vartheta(s)\mathcal{Y}(s,x)\Big]^*\Psi^{\mathcal{X}}_y(s,\mathcal{Y}(s,x)) + \vartheta^*(s)\Psi^{\mathcal{X}}(s,\mathcal{Y}(s,x))\Big\}ds\\
&+ \int_0^t \Big\{\mathcal{X}_y(s,\mathcal{Y}(s,x))\Sigma(s,x) + \Psi^{\mathcal{X}}(s,\mathcal{Y}(s,x))\Big\}^* dW(s) = 0,
\end{aligned}$$

almost surely; (2.9) has also been used. Thus, the uniqueness for the decomposition of a continuous semimartingale [e.g. Karatzas & Shreve (1991), p 149] implies that both integrals of the above equation vanish. Differentiation of the Lebesgue integral implies (7.18), while the quadratic variation of the stochastic integral vanishes as well, leading to (7.19). The derived equations define uniquely the random fields Θ and Σ, assigning to them the claimed order of adaptivity, integrability and smoothness. $\square$

LEMMA 7.3. *Under the Assumptions 6.1, 6.2, the random fields* $\Psi^{\mathcal{X}}$ *and* Φ^G *of (6.18) and (7.12) accordingly, satisfy almost surely the relationship*

$$\Phi^G_y(t,y) - y\Psi^{\mathcal{X}}_y(t,y) = 0, \quad \forall\ (t,y) \in [0,T) \times \mathbb{R}^+. \tag{7.20}$$

Proof. Taking time-differentials, then integrating (7.16) over $[z,y]$, $0 < z < y < \infty$, we get

$$dG(t,y) - dG(t,z) = yd\mathcal{X}(t,y) - zd\mathcal{X}(t,z) - \int_z^y d\mathcal{X}(t,\lambda)d\lambda,$$

almost surely, for $0 \le t < T$. Now make the substitutions (6.19), (7.13) in the above formula, and equate the respective martingale parts (e.g. Karatzas & Shreve (1991), Problem 3.3.2) to end up with

$$\Phi^G(t,y) - \Phi^G(t,z) = y\Psi^{\mathcal{X}}(t,y) - z\Psi^{\mathcal{X}}(t,z) - \int_z^y \Psi^{\mathcal{X}}(t,\lambda)d\lambda. \tag{7.21}$$

Of course, (7.21) is valid only if the interchange of Lebesgue and Itô integrals

$$\int_z^y \int_0^t \Psi^{\mathcal{X}}(s,\lambda)dW(s)\ d\lambda = \int_0^t \int_z^y \Psi^{\mathcal{X}}(s,\lambda)d\lambda\ dW(s)$$

holds almost surely, for each $t \in [0,T)$. But this is true, due to the observation that $L(t,\cdot) = \int_z^{\cdot} \Psi^{\mathcal{X}}(t,\lambda)d\lambda$ is a C^2 random field on $[z,\infty)$, and Exercise 3.1.5 in Kunita (1990). Differentiating (7.21) we obtain (7.20). □

We are ready now to state the main result of this section.

Theorem 7.4. (Stochastic Hamilton-Jacobi-Bellman Equation): *Under Assumptions 6.1 and 6.2, the pair of random fields (V,Ξ), here the value random field $V(t,x,z)$ is given by (7.5), (7.7), and*

$$\Xi(t,x,z) \triangleq \Phi^G\big(t,\mathcal{Y}(t,x-w(t)z)\big) - \mathcal{Y}(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big), \tag{7.22}$$

is of class

$$C_{\mathbb{F}}\big(\{t\in[0,T];\ V(t,\cdot,\cdot)\in C^{3,3}(\mathcal{D}_t)\}\big)\times \mathbb{L}^2_{\mathbb{F}}\big(\{t\in[0,T);\ \Xi(t,\cdot,\cdot)\in C^{2,2}(\mathcal{D}_t;\mathbb{R}^d)\}\big).$$

Furthermore, this pair (V,Ξ) solves on $\big\{(t,x,z);\ t\in[0,T),\ (x,z)\in\mathcal{D}_t\big\}$ the stochastic Hamilton-Jacobi-Bellman *partial differential equation of dynamic programming*

$$\begin{aligned}
-dV(t,x,z) = \operatorname*{ess\,sup}_{\substack{0\le c<\infty\\ \pi\in\mathbb{R}^d}}\bigg\{&\frac{1}{2}\|\sigma^*(t)\pi\|^2x^2V_{xx}(t,x,z)\\
&+\Big[r(t)x - c + \pi^*\sigma(t)\vartheta(t)x\Big]V_x(t,x,z) + \Big[\delta(t)c-\alpha(t)z\Big]V_z(t,x,z)\\
&+\pi^*\sigma(t)x\Xi_x(t,x,z) + u(t,c-z)\bigg\}dt - \Xi(t,x,z)dW(t)
\end{aligned} \tag{7.23}$$

with the boundary conditions (7.7) and (7.8), almost surely. Furthermore, the pair of random fields $(\Pi(t,x,z),C(t,x,z))$ of (6.25), (6.26) provides the optimal values for the maximization in (7.23).

Proof. Differentiation of (6.21), (7.5) and (7.22), in combination with (7.16) and (7.20), leads almost surely to

$$\begin{aligned}
&\mathcal{X}_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\mathcal{Y}_x(t,x-w(t)z) = 1,\\
&V_x(t,x,z) = \mathcal{Y}(t,x-w(t)z),\quad V_z(t,x,z) = -w(t)\mathcal{Y}(t,x-w(t)z),\\
&V_{xx}(t,x,z) = \mathcal{Y}_x(t,x-w(t)z),\ \Xi_x(t,x,z) = -\mathcal{Y}_x(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)
\end{aligned}$$

for $(x,z)\in\mathcal{D}_t,\ 0\le t<T$. Using these formulae and (6.7), we may rewrite the right-hand side of (7.23) as

$$\Big[r(t)x\mathcal{Y}(t,x-w(t)z) + \alpha(t)w(t)z\mathcal{Y}(t,x-w(t)z)$$

$$
\begin{aligned}
&+ \operatorname*{ess\,sup}_{0\le c<\infty} \big\{ u(t,c-z) - c\,\mu(t)\mathcal{Y}(t,x-w(t)z) \big\} \\
&+ \operatorname*{ess\,sup}_{\pi\in\mathbb{R}^d} \bigg\{ \frac{1}{2}\|\sigma^*(t)\pi\|^2 x^2 \mathcal{Y}_x(t,x-w(t)z) + \pi^*\sigma(t)x\Big[\vartheta(t)\mathcal{Y}(t,x-w(t)z) \\
&\qquad\qquad - \mathcal{Y}_x(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big]\bigg\}\Bigg]dt \\
&-\Big[\Phi^G\big(t,\mathcal{Y}(t,x-w(t)z)\big) - \mathcal{Y}(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big]dW(t).
\end{aligned}
$$

The strict concavity of both expressions to be maximized allows us to differentiate and solve the resulting equations, in order to attain the optimal values of c and π. These values turn out to coincide with (6.25) and (6.26), respectively. Substituting them now into the later expression, we are led to

$$
\begin{aligned}
&\Big[r(t)x\mathcal{Y}(t,x-w(t)z) + \alpha(t)w(t)z\mathcal{Y}(t,x-w(t)z) \\
&\quad + u\Big(t, I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big)\Big) \\
&\quad - \mu(t)\mathcal{Y}(t,x-w(t)z)\Big[z + I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big)\Big] \\
&\quad - \frac{1}{2\mathcal{Y}_x(t,x-w(t)z)}\Big\|\vartheta(t)\mathcal{Y}(t,x-w(t)z) \\
&\qquad\qquad - \mathcal{Y}_x(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big\|^2\Big]dt \\
&-\Big[\Phi^G\big(t,\mathcal{Y}(t,x-w(t)z)\big) - \mathcal{Y}(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big]^* dW(t).
\end{aligned}
\tag{7.24}
$$

On the other hand, employing differentials on (7.5), we have that

$$
dV(t,x,z) = dG\big(t,\mathcal{Y}(t,x-w(t)z)\big) - w'(t)z\mathcal{Y}_x(t,x-w(t)z)G_y\big(t,\mathcal{Y}(t,x-w(t)z)\big).
$$

Also, couple (7.17) with (2.9) to derive the alternative representation of $\mathcal{Y}$:

$$
d\mathcal{Y}(t,x) = \big[\vartheta^*(t)\Sigma(t,x) - \Theta(t,x)\big]dt + \Sigma^*(t,x)dW(t).
$$

Then, a straightforward application of Itô-Kunita-Wentzell formula, involving (7.13), yields that the left-hand side of (7.23) is equal to

$$
\begin{aligned}
&-\Bigg[\frac{1}{2}\Big[\|\Sigma(t,x-w(t)z)\|^2 - \|\vartheta(t)\|^2\mathcal{Y}^2(t,x-w(t)z)\Big]G_{yy}\big(t,\mathcal{Y}(t,x-w(t)z)\big) \\
&\quad + \Big[r(t)\mathcal{Y}(t,x-w(t)z) + \vartheta^*(t)\Sigma(t,x-w(t)z) \\
&\qquad\qquad - \Theta(t,x-w(t)z)\Big]G_y\big(t,\mathcal{Y}(t,x-w(t)z)\big) \\
&\quad + \Big[\Sigma(t,x-w(t)z) + \vartheta(t)\mathcal{Y}(t,x-w(t)z)\Big]^*\Phi^G_y\big(t,\mathcal{Y}(t,x-w(t)z)\big) \\
&\quad - u\Big(t, I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big)\Big)\Bigg]dt \\
&-\Big[G_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Sigma(t,x-w(t)z) + \Phi^G\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big]^* dW(t),
\end{aligned}
$$

which via (7.16) becomes

$$
\begin{aligned}
&-\bigg[\frac{1}{2}\|\Sigma(t,x-w(t)z)\|^2\mathcal{X}_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\\
&\quad+\mathcal{Y}(t,x-w(t)z)\bigg\{\frac{1}{2}\Big[\|\Sigma(t,x-w(t)z)\|^2\\
&\qquad\qquad-\|\vartheta(t)\|^2\mathcal{Y}^2(t,x-w(t)z)\Big]\mathcal{X}_{yy}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\\
&\qquad+\Big[r(t)\mathcal{Y}(t,x-w(t)z)-\frac{1}{2}\|\vartheta(t)\|^2\mathcal{Y}(t,x-w(t)z)\\
&\qquad\quad+\vartheta^*(t)\Sigma(t,x-w(t)z)-\Theta(t,x-w(t)z)\Big]\mathcal{X}_y\big(t,\mathcal{Y}(t,x-w(t)z)\big)\bigg\}\\
&\quad+\Big[\Sigma(t,x-w(t)z)+\vartheta(t)\mathcal{Y}(t,x-w(t)z)\Big]^*\Phi_y^G\big(t,\mathcal{Y}(t,x-w(t)z)\big)\\
&\quad-u\Big(t,I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big)\Big)\bigg]dt\\
&-\Big[\Phi^G\big(t,\mathcal{Y}(t,x-w(t)z)\big)-\mathcal{Y}(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big]^*dW(t).
\end{aligned}
$$

Finally, Lemmata 7.2 and 7.3 transform the latter to

$$
\begin{aligned}
&-\bigg[-r(t)\big[x-w(t)z\big]\mathcal{Y}(t,x-w(t)z)-u\Big(t,I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big)\Big)\\
&\quad+\mu(t)I\big(t,\mu(t)\mathcal{Y}(t,x-w(t)z)\big)+\frac{1}{2}\|\vartheta(t)\|^2\frac{\mathcal{Y}^2(t,x-w(t)z)}{\mathcal{Y}_x(t,x-w(t)z)}\\
&\quad-\vartheta^*(t)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)+\frac{1}{2}\|\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\|^2\mathcal{Y}_x(t,x-w(t)z)\bigg]dt\\
&-\Big[\Phi^G\big(t,\mathcal{Y}(t,x-w(t)z)\big)-\mathcal{Y}(t,x-w(t)z)\Psi^{\mathcal{X}}\big(t,\mathcal{Y}(t,x-w(t)z)\big)\Big]^*dW(t).
\end{aligned}
$$

Expanding the norm in (7.24) and recalling (6.8), we conclude that both sides of (7.23) coincide almost surely. □

Remark 7.5. Carrying out the maximization according to the proof of Theorem 7.4, the equation (7.23) takes the more conventional form

$$
\begin{aligned}
-dV_t(t,x,z)=\mathcal{H}\Big(&V_{xx}(t,x,z),V_x(t,x,z),V_z(t,x,z),\boldsymbol{\Xi}_x(t,x,z),t,x,z\Big)dt\\
&-\boldsymbol{\Xi}(t,x,z)dW(t),
\end{aligned}
\tag{7.25}
$$

where

$$
\begin{aligned}
\mathcal{H}(\mathrm{A},p,q,\mathrm{B},t,x,z)\triangleq&-\frac{1}{2\mathrm{A}}\|\vartheta(t)p+\mathrm{B}\|^2+\Big[r(t)x-z-I(t,p-\delta(t)q)\Big]p\\
&+\Big[(\delta(t)-\alpha(t))z+\delta(t)I(t,p-\delta(t)q)\Big]q+u\big(t,I(t,p-\delta(t)q)\big)
\end{aligned}
$$

for $\mathrm{A}<0,\ p>0,\ q<0$ and $\mathrm{B}\in\mathbb{R}$. Notice that we have obtained a closed-form solution of the *strongly nonlinear* stochastic Hamilton-Jacobi-Bellman equation (7.25), by solving instead the two *linear* equations (6.19), (7.13) subject to the appropriate initial and regularity conditions, and then performing the composition (7.5).

Remark 7.6. Theorem 7.4 provides a rare illustration of the Peng (1992) approach to stochastic Hamilton-Jacobi-Bellman equations. More precisely, it formulates the nonlinear stochastic partial differential equation satisfied by the value random field of the stochastic optimal control problem (7.4). To our knowledge, this is the first concrete illustration of BSPDE's in a stochastic control context beyond the classical linear/quadratic regulator worked out in Peng (1992).

As a consequence, (7.23) provides a *necessary condition* that must be satisfied by the value random field V of (7.4). On the contrary, due to the absence of an appropriate growth condition for V as each component of $(x,z) \in \mathcal{D}_t$ increases to infinity, (7.23) fails to be also sufficient; in other words, we cannot claim directly that V is the unique solution of (7.23) with boundary conditions (7.7), (7.8). We decide though to treat this matter by establishing a *necessary and sufficient condition* for the *convex dual* of V, defined as

$$\widetilde{V}(t,y) \triangleq \operatorname*{ess\,sup}_{(x,z)\in\mathcal{D}_t} \big\{ V(t,x,z) - \big(x - w(t)z\big)y \big\}, \quad y \in \mathbb{R}, \tag{7.26}$$

by analogy with (5.25). Doing so, we avoid investigating the solvability of the *nonlinear* stochastic partial differential equation (7.23), since it turns out that $\widetilde{V}$ is equivalently characterized as the unique solution of a *linear* parabolic backward stochastic partial differential equation (cf. (7.32)) and V can be easily recovered by inverting the above Legendre-Fenchel transformation to have almost surely

$$V(t,x,z) = \operatorname*{ess\,inf}_{y\in\mathbb{R}} \big\{ \widetilde{V}(t,y) + \big(x - w(t)z\big)y \big\}, \quad (x,z) \in \mathcal{D}_t.$$

We formalize these considerations as follows.

Theorem 7.7. (Convex Dual of $V(t,\cdot)$): *Considering Assumptions 6.1, 6.2, and a given $t \in [0,T)$, $V(t,\cdot,\cdot)$ is a generalized utility function, as defined in Theorem 5.8, almost surely; also,*

$$V_x(t,x,z) = \mathcal{Y}(t, x - w(t)z), \quad \forall\ (x,z) \in \mathcal{D}_t, \tag{7.27}$$

$$V_z(t,x,z) = -w(t)\mathcal{Y}(t, x - w(t)z), \quad \forall\ (x,z) \in \mathcal{D}_t. \tag{7.28}$$

Furthermore, for $(t,y) \in [0,T] \times \mathbb{R}^+$, we have

$$\widetilde{V}(t,y) = G(t,y) - y\mathcal{X}(t,y) = E_t\left[\int_t^T \tilde{u}(s, yY^{(t,1)}(s))ds\right], \tag{7.29}$$

$$\widetilde{V}_y(t,y) = -\mathcal{X}(t,y), \tag{7.30}$$

almost surely. Finally, the pair of random fields $(\widetilde{V}, \Lambda)$, where

$$\Lambda(t,y) \triangleq \Phi^G(t,y) - y\Psi^{\mathcal{X}}(t,y), \quad (t,y) \in [0,T] \times \mathbb{R}^+, \tag{7.31}$$

belongs to $C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big) \times \mathbb{L}^2_{\mathbb{F}}\big(0,T; \mathbb{L}^2(\Omega; C^2(\mathbb{R}^+;\mathbb{R}^d))\big)$ and is the unique solution of the following Cauchy problem for the linear *BSPDE*

$$\begin{aligned} -d\widetilde{V}(t,y) = \Bigg[&\frac{1}{2}\|\vartheta(t)\|^2 y^2 \widetilde{V}_{yy}(t,y) - r(t)y\widetilde{V}_y(t,y) - \vartheta^*(t)y\Lambda_y(t,y) \\ &+ \tilde{u}(t, y\mu(t)) \Bigg] dt - \Lambda^*(t,y)dW(t) \qquad \text{on } [0,T) \times \mathbb{R}^+, \end{aligned} \tag{7.32}$$

(7.33) $\widetilde{V}(T,y) = 0 \quad on\ \mathbb{R}^+.$

Merging now (7.22) and (7.31), we notice that the random fields Ξ and Λ of the martingale parts of V and $\widetilde{V}$, respectively, are related via the a.s. expression

$$\Xi(t,x,z) = \Lambda\big(t, \mathcal{Y}(t, x - w(t)z)\big), \quad t \in [0,T), \quad (x,z) \in \mathcal{D}_t. \tag{7.34}$$

Proof of Theorem 7.7: Setting claim (5.26) aside, the first two parts of this result represent the dynamic, stochastic counterpart of Theorem 5.8. Thus, all the respective assertions, including (7.27)–(7.30), can be proved through a similar methodology, keeping in mind the new feature of conditional expectation. From Lemmata 6.6, 7.1, (7.29) and (7.31), it is easy to verify the stated regularity for the pair $(\widetilde{V}, \Lambda)$, while the equations (7.32) and (7.33) are direct implications of (7.29), (7.13), (3.4), (6.19) and (7.14) with (6.20). □

Remark 7.8. In a Markovian framework, and with nonrandom model coefficients (cf. Remark 6.10), the unique solutions (6.13), (7.6), (7.4) and (7.26) of the stochastic partial differential equations of Lemmata 6.6, 7.1, and Theorems 7.4, 7.7, respectively, are deterministic functions. In particular, the stochastic integrals in these equations vanish, reducing them to deterministic ones.

The example that follows illustrates the use of Theorem 7.7 as an alternative method for characterizing, even computing, the value random field and the stochastic feedback formulas of the optimal portfolio/consumption pair.

Example 7.9. *(Logarithmic utility).* Take $u(t,x) = \log x$, $\forall\ (t,x) \in [0,T] \times \mathbb{R}^+$; thus, $I(t,y) = 1/y$, $\tilde{u}(t,y) = -\log y - 1$ for $(t,y) \in [0,T] \times \mathbb{R}^+$.
Case 1: *Deterministic coefficients.* The Cauchy problem (7.32) takes now the form

$$\widetilde{V}_t(t,y) + \frac{1}{2}\|\vartheta(t)\|^2 y^2 \widetilde{V}_{yy}(t,y) - r(t) y \widetilde{V}_y(t,y) = -\tilde{u}(t, y\mu(t)) \ \text{ on } [0,T) \times \mathbb{R}^+. \tag{7.35}$$

Motivated by the non-homogeneous term of (7.35), we seek appropriate functions $\nu, m : [0,T] \to \mathbb{R}$ such that

$$\tilde{v}(t,y) \triangleq -\nu(t)\log(y\mu(t)) - m(t) \tag{7.36}$$

satisfies (7.35), (7.33). Indeed, this is the case if and only if

$$\nu(t) = T - t, \qquad m(t) = \int_t^T \left[1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right] ds, \tag{7.37}$$

for $0 \le t \le T$, and then $\tilde{v} \in C([0,T] \times \mathbb{R}^+) \cap C^{1,3}([0,T) \times \mathbb{R}^+)$. From Theorem 7.7, $\tilde{v}$ is the unique solution of the Cauchy problem (7.35), (7.33), thus $\widetilde{V} \equiv \tilde{v}$,

$$\mathcal{X}(t,y) = \frac{\nu(t)}{y}\ , \quad G(t,y) = \nu(t)\Big[1 - \log(y\mu(t))\Big] - m(t), \quad (t,y) \in [0,T] \times \mathbb{R}^+.$$

Therefore,

$$\mathcal{Y}(t,x) = \frac{\nu(t)}{x},\ x \in \mathbb{R}^+, \quad V(t,x,z) = \nu(t)\log\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right) + \nu(t) - m(t),\ (x,z) \in \mathcal{D}_t,$$

and the feedback formulae (6.29), (6.30) for the optimal consumption and portfolio are given, for every $0 \le t < T$, by

$$C(t,x,z) = z + \frac{x - w(t)z}{\nu(t)\mu(t)} \quad \text{and} \quad \Pi(t,x,z) = (\sigma^*(t))^{-1}\vartheta(t)\frac{x - w(t)z}{x}, \quad (x,z) \in \mathcal{D}_t.$$

Case 2: *Random coefficients.* Our goal is to find an $\mathbb{F}$-*adapted* pair of random fields that satisfies (7.32), (7.33). By analogy with (7.36) – (7.37), we introduce in this case the $\mathbb{F}$-adapted random field

$$\tilde{\mathbf{v}}(t,y) \triangleq -\nu(t)\log(y\mu(t)) - \mathbf{m}(t)$$

for $(t,y) \in [0,T] \times \mathbb{R}^+$, with $\nu(t) = T - t$ and

$$\mathbf{m}(t) = E_t\left[\int_t^T \left\{1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right\} ds\right].$$

Moreover, the completeness of the market stipulates the existence of an $\mathbb{R}^d$-valued, $\mathbb{F}$-progressively measurable, square-integrable process $\ell(\cdot)$, such that the Brownian martingale

$$\mathbf{M}(t) = E_t\left[\int_0^T \left\{1 - (T-s)\left(\frac{1}{2}\|\vartheta(s)\|^2 + r(s) - \frac{\mu'(s)}{\mu(s)}\right)\right\} ds\right]$$

has the representation

$$\mathbf{M}(t) = \mathbf{M}(0) + \int_0^t \ell^*(s)dW(s), \quad 0 \le t \le T.$$

It is verified directly that the pair $(\tilde{\mathbf{v}}, \ell)$, where $\tilde{\mathbf{v}} \in C_{\mathbb{F}}\big([0,T]; \mathbb{L}^2(\Omega; C^3(\mathbb{R}^+))\big)$, satisfies (7.32), (7.33). Therefore, Theorem 7.7 implies that $(\tilde{\mathbf{v}}, \ell)$ agrees with $(\widetilde{V}, \Lambda)$, and

$$\mathcal{X}(t,y) = \frac{\nu(t)}{y}, \quad G(t,y) = \nu(t)\big[1 - \log(y\mu(t))\big] - \mathbf{m}(t), \quad (t,y) \in [0,T] \times \mathbb{R}^+.$$

Consequently, for $0 \le t < T$, it transpires that $\mathcal{Y}(t,x) = \nu(t)/x$, $x \in \mathbb{R}^+$ and

$$V(t,x,z) \;=\; \nu(t)\log\left(\frac{x - w(t)z}{\nu(t)\mu(t)}\right) + \nu(t) - \mathbf{m}(t), \quad (x,z) \in \mathcal{D}_t\,.$$

For this special choice of utility preference, $\mathcal{X}$ (and so $\mathcal{Y}$) is deterministic, and the feedback formulas (6.25), (6.26) for the optimal consumption and portfolio decisions are the same as those of the previous case.

Remark 7.10. Within the Markovian context stipulated by nonrandom coefficients, Detemple & Zapatero (1992) obtain a closed form representation for the optimal portfolio via an application of the Clark (1970) formula; this reduces to "feedback form" for the logarithmic utility function. This feedback formula now becomes a special case of (6.30) (cf. Example 7.9, Case 1) that was established in Theorem 6.10 for any arbitrary utility function.

In the case $\delta(\cdot) = \alpha(\cdot) = 0$ and $z = 0$, namely, without habit formation in the market model, we have that $\mu(\cdot) = 1$ from (6.7), whence our analysis remains valid for a *random* interest rate process $r(\cdot)$ as well. Then, this paper generalizes the role of dynamic programming and partial differential equations in classical utility optimization, explored in Karatzas, Lehoczky & Shreve (1987) for the special case of deterministic coefficients.

8. Conclusion. In this paper we explored various aspects of portfolio-consumption optimization under the presence of addictive habits in complete financial markets. The effective state space of the optimal wealth and standard of living processes was identified as a random wedge, and the investor's value function was found to exhibit properties similar to those of a utility function. Of particular interest is the interplay between the dynamic programming principles and the stochastic partial differential equation theory that led to the characterization of the value random field as a solution of a (highly non-linear) Hamilton-Jacobi-Bellman backward stochastic partial differential equation. In fact, the convex dual of the value random field turned out to be the unique solution of a parabolic backward stochastic partial differential equation. A byproduct of this analysis was an additional representation for the optimal investment-consumption policies on the current level of the optimal wealth and standard of living processes.

The existence of an optimal portfolio/consumption pair in an *incomplete market* (that is, when the number of stocks is strictly smaller than the dimension of the driving Brownian motion), is an open question. Following the duality methodology deployed by Karatzas, Lehoczky, Shreve & Xu (1991), one can complete the market with fictitious stocks by parametrizing a certain family of continuous exponential local martingales, which includes $Z(\cdot)$ of (2.6) and gives rise to an analogous class of state-price density processes. An associated dual optimization problem can be defined in terms of the respective parametrized "adjusted" state-price density processes, such that a possible minimizer induces a null demand for the imaginary stocks. But in the context of habit formation, the dual functional fails to be convex with respect to the dual parameter, and new methodologies are needed to handle the problem.

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