# THE OPTIMAL STOPPING PROBLEM FOR A GENERAL AMERICAN PUT-OPTION

NICOLE EL KAROUI* AND IOANNIS KARATZAS†

**Abstract.** We derive the representation $E\left[\int_t^T e^{-\int_t^u r(s)ds}\left(K-\underline{M}(t,u)\right)^+ r(u)du + e^{-\int_t^T r(s)ds}\left(K\wedge S(T)-\underline{M}(t,T-)\right)^+ \Big| \mathcal{F}(t)\right]$ for the "early exercise premium" $V(t;K)-P_e(t;K)$ of the American put-option $V(t;K)=\operatorname{esssup}_{t\leq\tau\leq T}E\left[e^{-\int_t^\tau r(u)du}\left(K-S(\tau)\right)^+ \Big|\mathcal{F}(t)\right]$ on an asset with positive, continuous price process $S(\cdot)$). Here the supremum is ever all stopping times $\tau$ with values in $[t,T]$, $r(\cdot)\geq 0$ is the interest rate of the numéraire, $K>0$ is the "strike price" of the option, $P_e(t;K)=E\left[e^{-\int_t^T r(u)du}\left(K-S(T)\right)^+\big|\mathcal{F}(t)\right]$ is the value of the corresponding European put-option, $E$ denotes expectation under the so-called "risk-neutral equivalent martingale measure", and $\underline{M}(t,\theta)=\inf_{t\leq u\leq\theta}M(u)$, $t\leq\theta<T$ is the lower envelope of the "index process" $M(t)=\inf\left\{K>0/V(t;K)=K-S(t)\right\}$, $0\leq t\leq T$.

**1. Introduction and summary.** We offer in this paper a representation for the early exercise premium $V(t;K)-P_e(t;K)$ of an American put-option with given strike-price $K>0$, on a finite time-horizon $[0,T]$ and on an asset with arbitrary continuous, strictly positive process $S(\cdot)$. Here

$$(1.1)\qquad V(t;K)=\operatorname*{esssup}_{\substack{t\leq\tau\leq T\\ \tau\, stop.time}} E\left[e^{-\int_t^\tau r(u)du}\left(K-S(\tau)\right)^+\big|\mathcal{F}(t)\right]$$

is the value of the American put-option at time $t\epsilon[0,T]$,

$$(1.2)\qquad P_e(t;K)=E\left[e^{-\int_t^T r(u)du}\left(K-S(T)\right)^+\big|\mathcal{F}(t)\right]$$

the value of the corresponding European put-option, $r(\cdot)\geq 0$ is the interest rate process for the prevailing pure discount bound (numéraire) in the economy, and $E$ denotes expectation with respect to the so-called "risk-neutral" equivalent martingale measure. In terms of the Gittins-index-like process

$$(1.3)\qquad M(t)=\inf\left\{K>0/V(t;K)=K-S(t)\right\},\quad 0\leq t\leq T$$

* Laboratoire de Probabilités, Université Pierre et Marie Curie, 4, place Jussieu-Tour 56, 75252 Paris Cedex 05, France.

† Department of Statistics, 619 Mathematics Building, Columbia University, New York, N.Y 10027. Research supported in part by the National Science Foundation under Grant DMS-90-22188.

which gives at any time $t$ the smallest value of the strike-price that makes immediate exercise of the option profitable, and its lower envelope $\underline{M}(t,\theta) = \inf_{t\le u\le\theta} M(u)$, $t \le \theta < T$, our representation of the early exercise premium is

$$V(t;K) - P_e(t;K) = E\Big[\int_t^T e^{-\int_t^u r(s)ds}\big(K - \underline{M}(t,u)\big)^+ r(u)du + e^{-\int_t^u r(s)ds}\big(K\wedge S(T) - \underline{M}(t,T-)\big)^+ \Big|\mathcal{F}(t)\Big]. \tag{1.4}$$

It takes on the simpler, and more familiar, form

$$V(t;K) - P_e(t;K) = E\Big[\int_t^T e^{-\int_t^u r(s)ds} K 1_{\{M(u)\le K\}} r(u)du \Big|\mathcal{F}(t)\Big] \tag{1.5}$$

in a special case where $e^{-\int_0^t r(s)ds}S(t)$ is a $P$-martingale (cf. Remarks 5.4); and leads to the representation

$$V(t;K) = K - E\Big[\int_t^T e^{-\int_t^u r(s)ds}\big(K\wedge \underline{M}(t,u)\big) r(u)du + e^{-\int_t^T r(s)ds}\big(K\wedge S(T)\wedge \underline{M}(t,T-)\big)\Big|\mathcal{F}(t)\Big] \tag{1.6}$$

for the value of the American put-option as in (1.1).

The paper is organized as follows. Section 2 studies the optimal stopping problem of (1.1) in some detail, including an explicit representation for the right-hand derivative of the convex mapping $K \mapsto V(t;K)$, whereas section 4 introduces the index process $M(\cdot)$ of (1.3) via its lower envelope, as in El Karoui & Karatzas (1994). The connection of this optimal stopping problem with the pricing of the American put-option is made in section 3, using the by now standard framework of Bensoussan (1984). The representation (1.4) is then derived in section 5, using the formula for the derivative of $K \mapsto V(t;K)$ and properties of the lower envelope for the index process. Several consequences of the representation (1.4) are also discussed in section 5.

The paper was presaged by Jacka (1991), who obtained a special case of the representation (1.5), using very different methods. Our approach is fully probabilistic, and reminiscent of our recent work El Karoui & Karatzas (1994) on the continuous-time dynamic allocation or "multi-armed bandit" problem.

**2. The optimal stopping problem.** Consider a complete probability space $(\Omega, \mathcal{F}, P)$, and a filtration $\mathbf{F} = \{\mathcal{F}(t)\}_{0\le t\le T}$ of sub-$\sigma$-fields of $\mathcal{F}$ which satisfies the "usual conditions" of right-continuity and augmentation by $P$-negligible sets, and is quasi-left-continuous. With $T > 0$ a fixed real constant, and any $0 \leq v \leq u \leq T$, we denote by $\mathcal{S}_{v,u}$ the class of all stopping times of $\mathbf{F}$ with values in $[v, u]$. Let $r(\cdot)$, $S(\cdot)$ be two $\mathbf{F}$-progressively measurable processes with values in $[0, \infty)$ and $(0, \infty)$ respectively, and assume that $S(\cdot)$ has continuous paths with $P(S(t) = x) = 0$, $\forall\, t \epsilon (0, T]$, $x \epsilon \mathbb{R}$.

Our object of interest in this paper is the *family of optimal stopping problems*

$$V(t;K) := \operatorname{esssup}_{\tau \epsilon \mathcal{S}_{t,T}} E\Big[\exp\Big\{\int_t^T r(u)du\Big\}(K - S(\tau))^+ \big| \mathcal{F}(t)\Big], \quad 0 \leq t \leq T \tag{2.1}$$

parametrized by $K \epsilon [0, \infty)$. As we shall discuss in more detail in the next section, the interpretation for $S(\cdot)$ is that of the price-per-share of a certain asset, and for $V(t; K)$ that of the value of an *American put-option* on the asset (i.e., of a contract which confers to its holder to the right to sell one share of the asset at the specified "strike-price" $K$, and at any time during the interval $[t, T]$). In this context, the process

$$P_e(t;K) := E\Big[\exp\Big\{-\int_t^T r(u)du\Big\}(K - S(T))^+ \big| \mathcal{F}(t)\Big], \quad 0 \leq t \leq T \tag{2.2}$$

has the interpretation of the value of the corresponding *European put-option* (i.e., of a similar contract as before, but in which the holder can exercise his right only at the terminal time $T$).

We shall assume throughout that the process $P_e(\cdot\,; K)$ is strictly positive on $[0, T)$:

$$P_e(t;K) > 0; \quad \forall 0 \leq t < T,\ 0 < K < \infty. \tag{2.3}$$

It is also obvious that

$$V(t;K) \geq P_e(t;K) \vee \big(K - S(t)\big)^+; \quad \forall 0 \leq t \leq T,\ 0 < K < \infty. \tag{2.4}$$

From standard theory on optimal stopping (e.g. Fakeev (1970), Bismut & Skalli (1977), El Karoui (1981), Karatzas (1993)) we know that, with $K$ fixed and

$$Y(t;K) := e^{-\int_0^t r(u)du}\big(K - S(t)\big)^+; \quad 0 \leq t \leq T, \tag{2.5}$$

the process

$$\begin{aligned} Z(t;K) &:= e^{-\int_0^t r(u)du} V(t;K) \\ &= \operatorname{esssup}_{\tau \epsilon \mathcal{S}_{t,T}} E\big[Y(\tau;K) \big| \mathcal{F}(t)\big], \quad 0 \leq t \leq T \end{aligned} \tag{2.6}$$

is the *Snell envelope* of (i.e., the smallest supermartingale that dominates) $Y(\cdot\,;K)$. Clearly, $Z(T;K) = Y(T;K)$ a.s. The stopping time

$$\begin{aligned}\sigma_t(K) &:= \inf\{\theta\epsilon[t,T]/Z(\theta;K) = Y(\theta;K)\} \\ &= \inf\{\theta\epsilon[t,T]/V(\theta;K) = K - S(\theta)\} \wedge T\end{aligned} \tag{2.7}$$

*is optimal,* i.e., achieves the supremum in (2.1), (2.6), and

$$\text{the process } \{Z(\theta \wedge \sigma_t(K);K),\ \mathcal{F}(\theta)\}_{t \le \theta \le T} \text{ is a martingale.} \tag{2.8}$$

Furthermore, since $Y(\cdot\,;K)$ has continuous paths with values in $[0,K]$, the supermartingale $Z(\cdot\,;K)$ also takes values in $[0,K]$ and is *regular,* thus *quasi-left-continuous* thanks to the quasi-left-continuity of the filtration $\mathbf{F}$.

Here are some basic properties of the random fields $(t,K,\omega) \mapsto \sigma_t(K,\omega)$, $(t,K,\omega) \mapsto V(t;K,\omega)$, considered in their measurable versions.

LEMMA 2.1. *For every $t\epsilon[0,T)$, the mapping*

**(i)** $K \mapsto V(t;K)$ *is convex, increasing, null at $K=0$, and strictly positive,*

**(ii)** $K \mapsto K - V(t;K)$ *is concave, increasing, null at $K=0$, and dominated by $K \wedge S(t)$,*

**(iii)** $K \mapsto \sigma_t(K)$ *is decreasing, right-continuous, with $\sigma_t(0+) = T$, almost surely.*

*Proof.*

**(i)** The convexity and increase follow from the facts that the mapping $K \mapsto (K-x)^+$ has these properties, and that we are then taking supremum over the class $\mathcal{S}_{t,T}$ of stopping times.

**(ii)** We have from (2.1)

$$\begin{aligned} K - V(t;K) &= \operatorname{essinf}_{\tau\epsilon\mathcal{S}_{t,T}} \\ &E\Big[K(1 - e^{-\int_t^\tau r(u)du}) + e^{-\int_t^\tau r(u)du}(K \wedge S(\tau))\,\big|\mathcal{F}(t)\Big] \\ &\le K \wedge S(t)\,. \end{aligned} \tag{2.9}$$

The two functions of $K$ inside the expectation are linear and concave, respectively, and both are increasing; since we are taking an infimum (over the class $\mathcal{S}_{t,T}$), these properties persist.

**(iii)** Introduce the nonnegative random field

$$\begin{aligned}\varphi(t;K) &:= V(t;K) - K + S(t)\,, \\ &0 \le t \le T\,,\quad K\epsilon(0,\infty)\end{aligned} \tag{2.10}$$

in terms of which we can re-write (2.7) as $\sigma_t(K) = \inf\{\theta\epsilon[t,T]/\varphi(\theta;K) = 0\} \wedge T$. For any fixed $t\epsilon[0,T)$, the mapping $K \mapsto \varphi(t;K)$ is continuous and decreasing (from (ii)); thus if $\{K_n\}_{n\epsilon\mathbf{N}} \subset (K,\infty)$ is a strictly decreasing sequence with $K = \lim_{n\to\infty} K_n$, we have $0 \le \varphi(\sigma_t(K_2);K_1) \le \varphi(\sigma_t(K_2);K_2) = 0$ so that $\sigma_t(K_1) \le \sigma_t(K_2)$, a.s. Therefore, $\sigma_* := \lim_{n\to\infty} \uparrow \sigma_t(K_n)$ exists and $\sigma_* \le \sigma_t(K)$.

More generally, $\varphi(\sigma_t(K_\ell); K_m) = 0$ for $\ell > m$; now let $\ell \uparrow \infty$ to obtain, from the quasi-left-continuity of $Z(\cdot\,; K)$ (and thus of $V(\cdot\,, K), \varphi(\cdot\,; K)$ as well): $\varphi(\sigma_*; K_m) = 0$, $\forall m \epsilon \mathbf{N}$. Finally, let $m \to \infty$ and exploit the continuity of $\varphi(t;\,\cdot)$, to obtain $\varphi(\sigma_*; K) = 0$, a.s. It follows that $\sigma_t(K) \le \sigma_*$, and thus $K \mapsto \sigma_t(K)$ is right-continuous. □

THEOREM 2.2. *For every $t \epsilon [0, T)$, the convex mapping $K \mapsto V(t, K)$ has right-hand derivative given by*

$$\begin{aligned} \frac{\partial^+}{\partial K} & V(t; K) \\ &= E\big[R^t_{\sigma_t(K)} 1_{\{\sigma_t(K)<T\}} + R^t_T 1_{\{S(T)\le K,\ \sigma_t(K)=T\}} \big| \mathcal{F}(t)\big] \\ &= E\big[R^t_{\sigma_t(K)} - R^t_T 1_{\{\sigma_t(K)=T, S(T)>K\}} \big| \mathcal{F}(t)\big]\,, \end{aligned} \tag{2.11}$$

*with the notation*

$$R^t_\theta := \exp\Big\{ - \int_t^\theta r(s)ds \Big\}, \quad 0 \le t \le \theta \le T. \tag{2.12}$$

*Proof.* Fix $(t, K) \epsilon [0, T) \times (0, \infty)$, and for any given $\varepsilon > 0$ denote $\sigma^\varepsilon \equiv \sigma_t(K+\varepsilon)$, $\sigma^0 \equiv \sigma_t(K)$. Of course $\sigma^\varepsilon \le \sigma^0$ a.s., and thus $Z(\cdot \wedge \sigma^\varepsilon; K)$ is a martingale (from (2.8)); therefore, $V(t; K) = E\big[R^t_{\sigma^\varepsilon} \cdot V(\sigma^\varepsilon; K)\big|\mathcal{F}(t)\big]$. On the other hand, $V(t; K+\varepsilon) = E\big[R^t_{\sigma^\varepsilon} \cdot \big(K + \varepsilon - S(\sigma^\varepsilon)\big)^+ \big| \mathcal{F}(t)\big]$ from the optimality of $\sigma^\varepsilon$ at $(t, K+\varepsilon)$, whence

$$\begin{aligned} V(t; K+\varepsilon) - V(t; K) &= \\ & E\Big[R^t_{\sigma^\varepsilon}\Big\{\big(K+\varepsilon - S(\sigma^\varepsilon)\big)^+ - V(\sigma^\varepsilon; K)\Big\}\Big|\mathcal{F}(t)\Big]\,. \end{aligned} \tag{2.13}$$

But on $\{\sigma^\varepsilon < T\}$, we have: $K+\varepsilon - S(\sigma^\varepsilon) = V(\sigma^\varepsilon; K+\varepsilon) > 0$, $V(\sigma^\varepsilon; K) > K - S(\sigma^\varepsilon)$, thus $\big(K+\varepsilon - S(\sigma^\varepsilon)\big)^+ - V(\sigma^\varepsilon; K) < \varepsilon$. Furthermore, on the event $\{\sigma^\varepsilon = T\}$ we have $V(\sigma^\varepsilon; K) = \big(K - S(T)\big)^+$, and so

$$\begin{aligned} \big(K+\varepsilon - S(\sigma^\varepsilon)\big)^+ &- V(\sigma^\varepsilon; K) = \\ & \big(K+\varepsilon - S(T)\big)^+ - \big(K - S(T)\big)^+ \le \varepsilon 1_{\{K+\varepsilon \ge S(T)\}}\,. \end{aligned}$$

Back into (2.13), these observations lead to the upper bound

$$\begin{aligned} & \frac{V(t; K+\varepsilon) - V(T; K)}{\varepsilon} \\ & \qquad \le E\big[R^t_{\sigma^\varepsilon} 1_{\{\sigma^\varepsilon < T\}} + R^t_T 1_{\{\sigma^0 = T\} \cap \{S(T) \le K+\varepsilon\}} \big| \mathcal{F}(t)\big] \\ & \qquad \le E\big[R^t_{\sigma^0} 1_{\{\sigma^0 < T\}} + R^t_T 1_{\{\sigma^0 = T\} \cap \{S(T) \le K\}} \big| \mathcal{F}(t)\big] + \\ & \qquad + E\big[(R^t_{\sigma^\varepsilon} - R^t_{\sigma^0}) 1_{\{\sigma^0 < T\}} + 1_{\{\sigma^\varepsilon < \sigma^0 = T\}} \\ & \qquad\quad + 1_{\{K < S(T) \le K+\varepsilon\}} \big| \mathcal{F}(t)\big]\,. \end{aligned}$$

But now $\lim_{\varepsilon \downarrow 0} \uparrow \sigma^\varepsilon = \sigma^0$ a.s., and from the (conditional) monotone and bounded convergence theorems, we obtain that the last conditional expectation goes to zero as $\varepsilon \downarrow 0$, whence

$$\overline{\lim}_{\varepsilon \downarrow 0} \frac{V(t;K+\varepsilon) - V(t;K)}{\varepsilon} \leq E\big[R^t_{\sigma^0} 1_{\{\sigma^0 < T\}} + R^t_T 1_{\{\sigma^0 = T,\ S(T) \leq K\}} \big| \mathcal{F}(t)\big]\,. \tag{2.14}$$

To obtain a lower bound, recall the supermartingale property of $Z(\cdot\,;K+\varepsilon)$, which gives $V(t;K+\varepsilon) \geq E\big[R^t_{\sigma^0} V(\sigma^0;K+\varepsilon)\big|\mathcal{F}(t)\big]$, and in conjunction with $V(t;K) = E\big[R^t_{\sigma^0}(K - S(\sigma^0))^+\big|\mathcal{F}(t)\big]$ get

$$\begin{aligned} V(t;K+\varepsilon) - V(t;K) &\\ &\geq E\big[R^t_{\sigma^0}\{V(\sigma^0;K+\varepsilon) - (K - S(\sigma^0))^+\}\big|\mathcal{F}(t)\big] \\ &\geq E\big[R^t_{\sigma^0}\{(K+\varepsilon - S(\sigma^0))^+ - (K - S(\sigma^0))^+\}\big|\mathcal{F}(t)\big]\,. \end{aligned} \tag{2.15}$$

Now on $\{\sigma^0 < T\}$, we have $K - S(\sigma^0) = V(\sigma^0;K) > 0$, whence

$$(K+\varepsilon - S(\sigma^0))^+ - (K - S(\sigma^0))^+ = \varepsilon\,.$$

On the other hand, on $\{\sigma^0 = T\}$ the last expression in braces in (2.15) becomes

$$\begin{aligned} (K+\varepsilon - S(T))^+ &- (K - S(T))^+ \\ &= \{(K+\varepsilon - S(T)) - (K - S(T))\}1_{\{S(T)\leq K\}} \\ &\qquad + (K+\varepsilon - S(T))^+ 1_{\{S(T) > K\}} \\ &\geq \varepsilon 1_{\{S(T) \leq K\}}\,. \end{aligned}$$

Back into (2.15), these considerations give

$$\underline{\lim}_{\varepsilon \downarrow 0} \frac{V(t;K+\varepsilon) - V(t;K)}{\varepsilon} \geq E\big[R^t_{\sigma^0} 1_{\{\sigma^0 < T\}} + R^t_T 1_{\{\sigma^0 = T,\ S(T) \leq K\}} \big| \mathcal{F}(t)\big]\,,$$

and therefore also (2.11) in conjunction with (2.14). □

**3. The American put-option.** Suppose now that the filtration $\mathbf{F}$ is the augmentation of the natural filtration generated by a $d$-dimensional standard Brownian motion $W = (W_1, \ldots, W_d)'$ on some complete probability space $(\Omega, \mathcal{F}, P_0)$. Consider a financial market $\mathcal{M}$ with $d+1$ instruments (assets), one *bond* with price $S_0(\cdot)$ governed by

$$dS_0(t) = S_0(t) r(t) dt, \quad S_0(0) = 1\,, \tag{3.1}$$

and $d$ *stocks*, with prices-per-share $S_i(\cdot)$ which satisfy the stochastic equations

$$dS_i(t) = S_i(t)\big[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t) dW_j(t)\big]\,; \quad i = 1, \ldots, d\,,\ \ 0 \leq t \leq T \tag{3.2}$$

driven by the Brownian motion $W$. Here $r(\cdot)$, $b(\cdot) = \{b_i(\cdot)\}_{i=1}^{d}$, $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{i,j=1}^{d}$ are all bounded, $\mathbf{F}$-progressively measurable processes, and we suppose that the same properties hold for $\theta(t) = \sigma^{-1}(t)\big[b(t) - r(t)\underset{\sim}{1}\big]$. We are also assuming here that $\sigma(t)$ is invertible for all $0 \le t \le T$, almost surely. Then

$$(3.3) \quad Z(t) = \exp\Big[-\int_0^t \theta'(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\Big], \quad 0 \le t \le T$$

is an $(\mathbf{F}, P_0)$-martingale, and thus $P(A) := E_0[Z(T)1_A]$ defines a new probability measure on $(\Omega, \mathcal{F})$ which is *equivalent* to $P_0$. Under this new measure $P$, the process $\tilde{W}(t) = W(t) - \int_0^t \theta(s)ds$, $0 \le t \le T$ is standard Brownian motion and the discounted stock price processes $\frac{S_i(\cdot)}{S_0(\cdot)}$, $i = 1, \ldots, d$ are *martingales* —whence the terminology "risk-neutral equivalent martingale measure" for $P$.

Consider now an arbitrary "asset" in this market $\mathcal{M}$, with price-per-share process $S(\cdot)$ satisfying the conditions of section 2; in particular, we can take $S(\cdot) \equiv S_i(\cdot)$ for some $i = 1, \ldots, d$ but this is not necessary. Suppose that, at time $t = 0$, you sign a contract with another "agent", which gives you the right (but not the obligation) to sell to the agent one share of the asset, at the contractually specified price $K \epsilon (0, \infty)$ and at *any* time $\rho$ in $[0, T]$. Such a contract is called an *American put-option* with horizon $T$ and "strike price" $K$. (The corresponding contract with only one possible exercise time, namely $\rho = T$, is called a *European* put-option). The signing of such a contract effectively commits the agent to make to you a payment of $\big(K - S(\rho)\big)^+$ at the exercise time $\rho$. *What is the "fair price", or "value" of the contract, that you should be charged at* $t = 0$*?*

The agent can of course invest in the instruments of the market $\mathcal{M}$, by committing an initial capital $x > 0$ and then selecting a *portfolio* process $\pi = (\pi_1, \ldots, \pi_d)'$ and an increasing *cumulative consumption* process $C$ with $C(0) = 0$ (both $\mathbf{F}$-progressively measurable with $C(T) + \int_0^T \| \pi(t) \|^2 dt < \infty$, a.s.). His wealth-process $X(\cdot) = X^{x,\pi,C}(\cdot)$ is then determined by

$$(3.4) \quad \begin{aligned} dX(t) &= \sum_{i=1}^{d} \pi_i(t)\big[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\big] \\ &\quad + \big(X(t) - \sum_{i=1}^{d} \pi_i(t)\big)r(t)dt - dC(t), \quad X(0) = x\,. \end{aligned}$$

The agent should strive to cover his obligation by selecting $x$ and $(\pi, C)$ in such a way that

$$(3.5) \qquad X^{x,\pi,c}(t) \ge \big(K - S(t)\big)^+, \quad \forall 0 \le t \le T$$

holds almost surely. Now the fair price $F_A$ for the American put-option, should be the smallest initial capital that allows the agent to achieve this, i.e.,

$$F_A := \inf\{x > 0/\exists(\pi, C) \text{ s.t. (3.5) holds a.s.}\}. \tag{3.6}$$

It was shown by Bensoussan (1984), Karatzas (1988, 1989) (see also Karatzas & Shreve (1994)) that

$$F_A = V(0;K) \ \text{ as in (2.1)}; \tag{3.7}$$

in other words, the valuation of this contract is given in terms of the optimal stopping problem of (2.1). It is a far more straightforward matter to see that the value

$$F_E := \inf\{x > 0/\exists(\pi, C) \text{ s.t. } X^{x,\pi,C}(T) \geq (K - S(T))^+, \text{ a.s.}\} \tag{3.8}$$

of the corresponding European put-option is given by the famous Black & Scholes formula

$$F_E = P_e(0;K) \ \text{ as in (2.2).} \tag{3.9}$$

The nonnegative number $F_A - F_E = V(0;K) - P_e(0;K)$ has then an obvious interpretation as *early exercise premium* for the American put-option. Furthermore, $\sigma_0(K) = \inf\{\theta\epsilon[0,T)/V(\theta;K) = K - S(\theta)\} \wedge T$ as in (2.7) has the interpretation of *optimal exercise time* $\rho$ for this problem.

Similarly, $V(t;K)$ (resp. $P_e(t;K)$) can be interpreted as the *value* for the American (respectively, European) put-option, and $V(t;K) - P_e(t;K)$ as an "early exercise premium", *at any time* $t\epsilon[0,T]$; see Karatzas & Shreve (1994) for the particulars of this interpretation.

*Remarks 3.1:* Marc Romano (Université de Paris-Dauphine) observes that the third expression in (2.11) leads to the bounds

$$0 \leq E[R^0_{\sigma_0(K)}] - \frac{\partial^+}{\partial K} V(0;K) \leq E(R^0_T) \xrightarrow[T\to\infty]{} 0, \tag{3.10}$$

where the last property holds, for example, if $\int_0^\infty r(u)du = \infty$, a.s.

**4. Index processes.** For every $t\epsilon[0,T)$, the set $\{K > 0/V(t;K) = K - S(t)\}$ is an interval of the form $[M(t), \infty)$, where $M(t)$ is an $\mathcal{F}(t)$-measurable random variable that satisfies

$$M(t) = \inf\{K > 0/V(t;K) = K - S(t)\} \geq S(t), \ \text{ a.s.} \tag{4.1}$$

We also define

$$M(T) := S(T). \tag{4.2}$$

Now let

$$\underline{M}(t,\theta) := \begin{cases} \sup\{K>0/\sigma_t(K)>\theta\} = \inf\{K>0/\sigma_t(K)\le\theta\}, & t\le\theta<T \\ \underline{M}(t,T-)\wedge S(T), & \theta = T \end{cases} \tag{4.3}$$

be the right-continuous inverse of the decreasing mapping $K \mapsto \sigma_t(K)$. We have the properties

$$\sigma_t(K)>\theta \iff \underline{M}(t,\theta)>K \iff M(u)>K,\ \ \forall\, u\epsilon[t,\theta] \tag{4.4}$$

for $0\le t\le\theta<T$, so that $\underline{M}(t,\theta) = inf_{t\le u\le\theta}M(u)$ is the "lower envelope" of $M(\cdot)$ on $[t,\theta]$. It is also clear that $\sigma_t(K) = \inf\{\theta\epsilon[t,T)/M(\theta)\le K\}\wedge T$.

*Remarks 4.1:* For every $t\epsilon[0,T)$, we have from (4.4):

$$\begin{aligned} &\bigcap_{\varepsilon>0}\{\sigma_t(K+\varepsilon)<T\} \\ &\qquad = \bigcap_{\varepsilon>0}\ \bigcup_{T-t\le\alpha\le T}\{\sigma_t(K+\varepsilon)\le T-\alpha\} \\ &\qquad = \bigcap_{\varepsilon}\bigcup_{\alpha}\{\underline{M}(t,T-\alpha)\le K+\varepsilon\} \\ &\qquad = \bigcap_{\varepsilon>0}\{\underline{M}(t,T-)\le K+\varepsilon\} = \{\underline{M}(t,T-)\le K\}. \end{aligned}$$

Therefore,

$$\underline{M}(t,T-) = \inf\{K>0/\sigma_t(K)<T\}$$

as well as

$$\begin{aligned} &\{\underline{M}(t,T-)\le K\} \\ &\qquad = \{\sigma_t(K)<T\}\cup\{\sigma_t(K)=T;\sigma_t(K+\varepsilon)<T,\quad \forall\varepsilon>0\}. \end{aligned}$$

We deduce

$$\{\underline{M}(t,T-)\le K<S(T)\} = \{\sigma_t(K)<T,\ S(T)>K\}, \tag{4.5}$$

because on $\{\sigma_t(K+\varepsilon)<T\}$ we have $S(\sigma_t(K+\varepsilon)) = K+\varepsilon - V(\sigma_t(K+\varepsilon);K+\varepsilon) \le K+\varepsilon$ and thus on $\cap_{\varepsilon>0}\{\sigma_t(K)=T,\ \sigma_t(K+\varepsilon)<T\}$ : $S(T) = S(\sigma_t(K)) = S(\lim_{\varepsilon\downarrow 0}\sigma_t(K+\varepsilon)) = \lim_{\varepsilon\downarrow 0}S(\sigma_t(K+\varepsilon)) \le K$. $\diamond$

The processes $M(\cdot)$ and $\underline{M}(t,\cdot)$ have obvious similarities with the *Gittins index process* and its lower envelope, which have proved very useful in the study of dynamic allocation (or "multi-armed bandit") problems; see, for example, El Karoui & Karatzas (1994). Clearly, in the context of section 3, $M(t)$ can be interpreted as the smallest value of the strike-price $K>0$ that makes immediate exercise of the American put-option profitable at time $t$.

**5. Early exercise premium in terms of indices.** We can present now our main result, the expression (5.1) for the *early exercise premium* $V(t;K) - P_e(t;K)$ for the American option of section 3 in terms of the lower envelope $\underline{M}(t,\cdot)$ of the index process as in (4.3). This leads to a similar representation (5.3) for the value $V(t;K)$ itself.

THEOREM 5.1. *In terms of the lower envelope $\underline{M}(t,\cdot)$ of the index process $M(\cdot)$ in (4.2), (4.3), we have for every $t\epsilon[0,T)$ the following representation of the "early exercise premium":*

$$V(t;K) - P_e(t;K) = E\Big[\int_t^T R_u^t(K - \underline{M}(t,u))^+ r(u)du + R_T^t((K \wedge S(T)) - \underline{M}(t,T-))^+ \Big| \mathcal{F}(t)\Big]. \tag{5.1}$$

*Proof.* It is quite easy to compute $\frac{\partial^+}{\partial K} P_e(t;K) = E[R_T^t 1_{\{S(T)\le K\}}|\mathcal{F}(t)]$ from (2.2). Therefore, we obtain in conjunction with (2.11):

$$\begin{aligned} \frac{\partial^+}{\partial K}\Big[V(t;K) - P_e(t;K)\Big] &= E\Big[1_{\{\sigma_t(K)<T\}}\big(R^t_{\sigma_t(K)} - R_T^t 1_{\{S(T)\le K\}}\big)\Big|\mathcal{F}(t)\Big] \\ &= E\Big[\big(R^t_{\sigma_t(K)} - R_T^t\big) + R_T^t 1_{\{S(T)>K\}\cap\{\sigma_t(K)<T\}}\Big|\mathcal{F}(t)\Big] \\ &= E\Big[\int_t^T R_u^t 1_{\{\underline{M}(t,u)\le K\}} r(u)du + R_T^t 1_{\{\underline{M}(t,T-)\le K<S(T)\}} \Big|\mathcal{F}(t)\Big], \end{aligned} \tag{5.2}$$

using (4.5), (4.4) and its corollary $R^t_{\sigma_t(K)} - R_T^t = \int_t^T R_u^t 1_{\{\sigma_t(K)\le u\}} r(u)du = \int_t^T R_u^t 1_{\{\underline{M}(t,u)\le K\}} r(u)du$.

Integrating with respect to $K$ in (5.2) over $[0,K]$, and using $V(t;0) = P_e(t;0) = 0$, we obtain (5.1) from the conditional Fubini theorem. □

COROLLARY 5.2. *For every $t\epsilon[0,T)$, $K\epsilon(0,\infty)$ we have the a.s. representation*

$$V(t;K) = K - E\Big[\int_t^T R_u^t(K \wedge \underline{M}(t,u))r(u)du + R_T^t(K \wedge \underline{M}(t,T))\Big|\mathcal{F}(t)\Big]; \tag{5.3}$$

*in particular,*

$$\lim_{K\to\infty}[K - V(t;K)] = E\Big[\int_t^T R_u^t \underline{M}(t,u)r(u)du + R_T^t \underline{M}(t,T)\Big|\mathcal{F}(t)\Big]. \tag{5.4}$$

*Proof.* We can write (2.2) equivalently as

$$K - P_e(t;K) = E\Big[\int_t^T R_u^t K r(u)du + R_T^t\big(K \wedge S(T)\big)\Big|\mathcal{F}(t)\Big].$$

Subtracting (5.1) memberwise from this equality, and recalling the definition of (4.3) for $t = T$, we obtain (5.3); (5.4) follows then by Monotone Convergence. □

*Remarks 5.3: The process*

$$(5.5)\quad R_t^0\big[\underline{M}(0,t) - V(t,\underline{M}(0,t))\big] + \int_0^t R_u^0 \underline{M}(0,u)r(u)du, \quad 0 \le t < T$$

*is an* **F**-*martingale.* Indeed, (5.3) gives

$$\underline{M}(0,t) - V\big(t,\underline{M}(0,t)\big) = E\Big[\int_t^T R_u^t \underline{M}(0,u)r(u)du + R_T^t \underline{M}(0,T)\Big|\mathcal{F}(t)\Big],$$

and the martingale property follows directly from this.

*Remarks 5.4:* In the *special setting of section 3 with* $d = 1$, $r > 0$, $\sigma = \sigma_{11} > 0$ *real constants* and $S(\cdot) \equiv S_1(\cdot)$, (5.1) takes the simpler form

$$(5.6)\quad \begin{aligned} V(t;K) - P_e(t;K) &= E\Big[\int_t^T R_u^t K 1_{\{M(u) \le K\}} r(u)du\Big|\mathcal{F}(t)\Big] \\ &= E\Big[\int_t^T R_u^t K 1_{\{V(u;K)=K-S(u)\}} r(u)du\Big|\mathcal{F}(t)\Big]. \end{aligned}$$

This last expression was obtained by S. Jacka (1991). In fact, for this special case it can be easily verified that

$$(5.7)\quad V(t;K) = u\big(T-t, S(t);K\big), \quad M(t) = \frac{S(t)}{b_1(T-t)}; \quad 0 \le t < T$$

where $(\theta, x, K) \mapsto u(\theta, x; K) : (0,\infty)^3 \to (0,\infty)$ and $(\theta, K) \mapsto b_K(\theta) : (0,\infty)^2 \to (0,\infty)$ are suitable functions with the scaling properties $u(\theta, x; K) = K u(\theta, \frac{x}{K}; 1)$, $b_k(\theta) = K b_1(\theta)$. Furthermore, $b_1(\cdot)$ is then continuous and decreasing with $b_1(0+) = 1$; it is the optimal exercise boundary for the American put-option corresponding to $K = 1$, in the sense that the optimal stopping time of (2.7) with $t = 0$, $K = 1$ takes the form $\sigma_0(1) = \inf\{t \epsilon [0,T) / S(t) \le b_1(T-t)\}$; see Jacka (1991), Myneni (1992) or Karatzas

& Shreve (1994) for details. In this special case, the formulae (5.2), (5.1) become, respectively,

$$\frac{\partial^+}{\partial K}\big[V(t;K)-P_e(t;K)\big] = E\Big[\int_t^T 1_{\{\inf_{t\le\theta\le u}\frac{S(\theta)}{b_1(T-\theta)}\le K\}}\, re^{-r(u-t)}du + e^{-r(T-t)}1_{\{\inf_{t\le\theta<T}\frac{S(\theta)}{b_1(T-\theta)}\le K<S(T)\}}\Big|\mathcal{F}(t)\Big] \tag{5.8}$$

$$V(t;K)-P_e(t;K) = E\Big[\int_t^T re^{-r(u-t)}\Big(K-\inf_{t\le\theta\le u}\frac{S(\theta)}{b_1(T-\theta)}\Big)^+du + e^{-r(T-t)}\Big(K\wedge S(T)-\inf_{t\le\theta<T}\frac{S(\theta)}{b_1(T-\theta)}\Big)^+\Big|\mathcal{F}(t)\Big] \tag{5.9}$$

This last representation is of some interest, as it involves compound European options of the path-dependent (or "look-back") type. It follows readily from (5.4) that

$$\lim_{K\to\infty}\big[K-V(t;K)\big] = E\Big[\int_t^T re^{-r(u-t)}\inf_{t\le\theta\le u}\Big(\frac{S(\theta)}{b_1(T-\theta)}\Big)du + e^{-r(T-t)}\inf_{t\le\theta\le T}\Big(\frac{S(\theta)}{b_1(T-\theta)}\Big)\Big|\mathcal{F}(t)\Big]. \tag{5.10}$$

## REFERENCES

[1] A.BENSOUSSAN, *On the theory of option pricing.*, Acta Appl. Math. 2 (1984), 139–158.

[2] J.M.BISMUT, B. SKALLI, *Temps d' arrêt optimal, théorie générale de processus et processus de Markov*, Z. Wahrsch. verw. Gebiete 39 (1977), 301–313.

[3] N.EL KAROUI, I.KARATZAS, *Dynamic allocation problems in continuous time*, Annals of Applied Probability, to appear (1994).

[4] N.EL KAROUI, *Les aspects probabilistes du contrôle stochastique*, Lecture Notes in Mathematics 876 (1981), 73–238 Springer-Verlag, Berlin.

[5] A.G.FAKEEV, *Optimal stopping rules for processes with continuous parameter*, Theory Probab. Appal. 15 (1970), 324–331.

[6] S.D.JACKA, *Optimal stopping and the American put*, Math. Finance 1 (2) (1991), 1–14.

[7] I.KARATZAS, *On the pricing of American options*, Appl. Math. Optimiz. 17 (1988), 37–60.

[8] I.KARATZAS, *Optimization problems in the theory of continuous trading*, SIAM J. Control & Optimization 27 (1989), 1221–1259.

[9] I.KARATZAS, *Lectures on the probabilistic aspects of optimal stopping*, Class notes, Columbia University, Spring (1993).

[10] I.KARATZAS, S.E.SHREVE, *Monograph on mathematical finance*, In preparation (1994).

[11] R.MYNENI, *The pricing of the American option*, Annals of Applied Probability 2 (1992), 1–23.