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A NEW APPROACH TO THE SKOROHOD PROBLEM, AND ITS APPLICATIONS*

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Integration with respect to the initial position of a continuous process killed upon reaching a moving boundary, yields the paths of the solution to the *Skorohod problem* of reflecting the process along this boundary.

This idea is employed to show that direct integration of the optimal risk in a *stopping problem* for Brownian motion, yields the value function of the so-called *monotone follower* stochastic control problem and provides an explicit construction of its optimal process. Ideas from the theory of balayage for continuous semimartingales are employed, in order to find novel and useful representations for the value functions of these problems.

KEY WORDS: Processes killed or reflected at a boundary, the Skorohod problem, optimal stopping, excessive functions, dual predictable projections, balayage, principle of "smooth fit", singular stochastic control.

1. INTRODUCTION AND SUMMARY

We show that integration with respect to the initial position of a continuous process killed upon reaching a moving boundary, yields the paths of the solution to the *Skorohod problem* of reflecting the process along this boundary.

This simple observation is employed to show that direct integration of the *optimal stopping risk*

$$u(r,x)=\inf_{\sigma\leqq\tau-r} E\left[\int_0^{\sigma} h_x(r+t,x+W_t)\,dt+f(r+\sigma)1_{\{\sigma<\tau-r\}}+g'(x+W_{\tau-r})1_{\{\sigma=\tau-r\}}\right] \tag{1.1}$$

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for the Brownian motion W on the finite horizon $[0,\tau]$, leads ultimately to the relationships

$$\frac{\partial}{\partial x} V(r,x)=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right) Q(r,x,y)=u(r,x); \quad (r,x,y)\in[0,\tau]\times\mathscr{R}\times\mathscr{R}_{+} \tag{1.2}$$

for the value functions

$$V(r,x)=\inf_{\xi} E\left[\int_{0}^{\tau-r} h(r+t,x+W_t-\xi_t)\,dt + \int_{[0,\tau-r)} f(r+t)\,d\xi_t+g(x+W_{\tau-r}-\xi_{\tau-r})\right] \tag{1.3}$$

$$Q(r,x,y)=\inf_{\substack{\xi \\ \xi_{\tau-r}\leqq y}} E\left[\int_{0}^{\tau-r} h(r+t,x+W_t-\xi_t)\,dt + \int_{[0,\tau-r)} f(r+t)\,d\xi_t+g(x+W_{\tau-r}-\xi_{\tau-r})\right] \tag{1.4}$$

of the *monotone follower* stochastic control problem and its finite-fuel counterpart, respectively (ξ nondecreasing and adapted in (1.3), (1.4)). The methodology provides also a complete description of the optimal processes for the two problems (1.3) and (1.4), in terms of the optimal stopping boundary $\{s(r); 0\leqq r<\tau\}$ for the problem of (1.1), thus complementing and extending the results of the articles [10–11].

On the other hand, techniques analogous to those used in the theory of *balayage for semimartingales* establish the representation

$$u(r,x)=E\left[\int_{0}^{\tau-r} h_x(r+t,x+W_t)1_{\{x+W_t<s(r+t)\}}\,dt+g'(x+W_{\tau-r}) - \int_{0}^{\tau-r} f'(r+t)1_{\{x+W_t\geqq s(r+t)\}}\,dt\right] \tag{1.5}$$

and its corollary

$$V(r,x)=E\left[\int_{0}^{\tau-r} h(r+t,s(r+t)\wedge(x+W_t))\,dt+g(x+W_{\tau-r})\right.$$

$$- \int_0^{\tau-r} f'(r+t)(x+W_t-s(r+t))^+ \, dt \Bigg], \tag{1.6}$$

for the optimal risk u of (1.1) and the value function V of (1.3), respectively. It turns out that, in a certain sense, the representation (1.5) subsumes the celebrated "principle of smooth fit" for the optimal stopping problem. In its turn, the representation (1.6) leads to the computation

$$V(r, s(r)) = \int_r^{\tau} h(\theta, s(\theta)) \, d\theta - \int_{[r,\tau)} f(\theta) \, ds(\theta) + g(s(\tau)), \tag{1.7}$$

which expresses the value function along the moving boundary $\{s(r); 0 \leqq r < \tau\}$ as *the cost of a* (deterministic) *ride on this boundary.*

We present our approach to the Skorohod problem in Section 2 (Proposition 2.1). The optimal stopping problem of (1.1) is studied thoroughly in Section 3, whereas Section 4 reviews the monotone follower problems (1.3), (1.4). The identities of (1.2) are then derived in Section 5 using, in a very direct and elementary way, the methodology of Proposition 2.1, and Section 6 obtains the relations (1.6), (1.7). The representation (1.5) is one of the most important results of this paper; its (rather demanding) proof is carried out in an Appendix (Section 7).

2. THE SKOROHOD PROBLEM

On a complete probability space $(\Omega, \mathscr{F}, P)$ we consider a Brownian motion $W = \{W_t; 0 \leqq t \leqq \tau\}$ adapted to a filtration $\{\mathscr{F}_t\}$, which satisfies the usual conditions and $\mathscr{F}_\tau = \mathscr{F}$. Let $\mathscr{A}(\tau)$ denote the class of $\{\mathscr{F}_t\}$-adapted processes $\xi = \{\xi_t; 0 \leqq t \leqq \tau\}$, such that for $P-$a.e. $\omega \in \Omega$

i) $\xi_0(\omega) = 0$ and

ii) $t \mapsto \xi_t(\omega)$ is nondecreasing and left-continuous.

We shall also use the notation $\mathscr{A}(\tau, y) \triangleq \{\xi \in \mathscr{A}(\tau); \xi_\tau \leqq y, \text{a.s. } P\}$, for $0 < y < \infty$. We shall denote by ξ^+ the right-continuous modification of $\xi \in \mathscr{A}(\tau)$.

Given a continuous function $s: [0, \tau) \to \mathscr{R}$, a number $r \in [0, \tau)$, and an initial position $x \leqq s(r)$, the *Skorohod problem* is to find a continuous process $\xi \in \mathscr{A}(\tau - r)$, such that the resulting process

$$X_t = x + W_t - \xi_t; \qquad 0 \leqq t \leqq \tau - r \tag{2.1}$$

satisfies

$$X_t \leqq s(r+t); \qquad \forall \, t \in [0, \tau - r) \tag{2.2}$$

$$\int_0^{\tau-r} 1_{\{X_t < s(r+t)\}} \, d\xi_t = 0 \tag{2.3}$$

almost surely. In other words, one wants to apply a "downward push" to the Brownian path $x+W$ that would keep the resulting process X below the moving boundary $s(r+\cdot)$, and to do this in a minimal way (i.e., only when X is on the boundary).

The solution to this problem is well-known (cf. [14] or [12], p. 210), and is given by the process

$$\xi_t(r,x)=\begin{Bmatrix} 0 & ; & t=0 \\ \max[0, \sup_{0\leqq\theta\leqq t}\{x+W_\theta-s(r+\theta)\}]; & & 0<t\leqq\tau-r \end{Bmatrix}. \tag{2.4}$$

Equivalently, $\xi(r,x)$ is the smallest amongst the continuous processes $\xi\in\mathscr{A}(\tau-r)$ which satisfy (2.2). The resulting process

$$X_t(r,x)\triangleq x+W_t-\xi_t(r,x); \qquad 0\leqq t\leqq\tau-r \tag{2.5}$$

is the *Brownian motion started at x and reflected along the moving boundary* $s(r+\cdot)$.

Remark 2.1. If $x>s(r)$, the process $\xi(r,x)$ in (2.4) has an immediate jump of size $\xi_{0+}(r,x)=x-s(r)$, which brings $X(r,x)$ instantaneously on the boundary; from then on the situation is the same as before. □

Now consider the stopping time $\sigma(r,x)=\inf\{t\in[0,\tau-r);\ x+W_t\geqq s(r+t)\}\wedge(\tau-r)$, and define the *Brownian motion started at x and killed upon reaching the moving boundary* $s(r+\cdot)$:

$$K_t(r,x)\triangleq\begin{Bmatrix} x+W_t; & 0\leqq t<\sigma(r,x) \\ \Delta\ ; & \sigma(r,x)\leqq t<\tau-r \end{Bmatrix}. \tag{2.6}$$

Here, $\Delta=-\infty$ is the so-called "cemetery state". We are using throughout the convention $\inf\emptyset=\infty$.

It will be shown that the paths of the reflected process $X(r,x)$ of (2.5), can be obtained from those of the killed process $K(r,x)$, by integrating in the spatial variable:

$$X_t(r,x)=\log\int_{-\infty}^{x}\exp\{K_t(r,z)\}\,dz; \qquad 0<t<\tau-r. \tag{2.7}$$

PROPOSITION 2.1 *For every absolutely continuous function* $\phi:\mathscr{R}\to\mathscr{R}$ *of the form* $\phi(x)=\int_{-\infty}^{x}\phi'(u)\,du$, *with* ϕ' *integrable on* $(-\infty,x]$ *for all* $x\in\mathscr{R}$ *and* $\phi'(-\infty)\triangleq 0$, *we have*

$$\phi(X_t(r,x))=\int_{-\infty}^{x}\phi'\{K_t(r,z)\}\,dz; \qquad 0<t<\tau-r. \tag{2.8}$$

In particular, (2.7) holds.

Proof For simplicity of notation, take $r=0$ and write $\xi(x)$, $\sigma(x)$ for $\xi(0,x)$, $\sigma(0,x)$. The rather obvious equivalences

$$\begin{aligned}\sigma(x-u)\leqq t &\Leftrightarrow x-u+W_\theta \geqq s(r+\theta), \quad \text{for some } 0\leqq\theta\leqq t\\ &\Leftrightarrow \sup_{0\leqq\theta\leqq t}\{x+W_\theta - s(r+\theta)\}\geqq u\\ &\Leftrightarrow \xi_t^+(x)\geqq u \end{aligned}\tag{2.9}$$

$$\sigma(x-u)<\tau \Leftrightarrow \xi_\tau(x)>u; \qquad 0\leqq u<\infty, \tag{2.10}$$

lead to

$$\sigma(x-u)=\inf\{0\leqq t<\tau;\ \xi_t^+(x)\geqq u\}\wedge\tau; \qquad 0<u<\infty. \tag{2.11}$$

In other words, the process $\{\sigma(x-u);\ 0<u<\infty\}$ is the left-continuous inverse of $\{\xi_t^+(x);\ 0\leqq t<\tau\}$.

Now let $\phi:\mathscr{R}\to\mathscr{R}$ be any absolutely continuous function; we obtain

$$\begin{aligned}\int_{-\infty}^{x}\phi'(z+W_t)1_{\{\sigma(z)\leqq t\}}\,dz &= \int_0^\infty \phi'(x-u+W_t)1_{\{\sigma(x-u)\leqq t\}}\,du\\ &= \int_0^\infty \phi'(x-u+W_t)1_{\{\xi_t^+(x)\geqq u\}}\,du\\ &= \phi(x+W_t)-\phi(x+W_t-\xi_t^+(x)); \qquad 0\leqq t<\tau \end{aligned}\tag{2.12}$$

from (2.9), and

$$\begin{aligned}\int_{-\infty}^{x}\phi'(z+W_\tau)1_{\{\sigma(z)<\tau\}}\,dz &= \int_0^\infty \phi'(x-u+W_\tau)1_{\{\xi_\tau(x)>u\}}\,du\\ &= \phi(x+W_\tau)-\phi(x+W_\tau-\xi_\tau(x)) \end{aligned}\tag{2.13}$$

from (2.10). For a function ϕ enjoying the properties of the theorem, (2.8) follows from (2.12) and the fact that $\xi(x)$ can have a jump only at $t=0$. □

Obviously, the identities (2.12), (2.13) and their corollaries (2.8), (2.7) hold for every process W with continuous paths, and not just for Brownian motion. On the other hand, suppose that W *is* Brownian motion, and consider the semigroups

$$P_t^r f \triangleq Ef(X_t(r,\cdot)), \quad Q_t^r f \triangleq Ef(K_t(r,\cdot)) = E[f(\cdot+W_t)1_{\{\sigma(r,\cdot)>t\}}]$$

of the two processes in (2.5), (2.6). For every absolutely continuous function $\phi:\mathscr{R}\to\mathscr{R}$ with compact support we obtain, by taking expectations in (2.8):

$$P_t^r\phi(x) = \int_{-\infty}^{x} Q_t^r\phi'(z)\,dz; \qquad 0<t<\tau-r. \tag{2.14}$$

In the presence of densities

$$P_t^r\phi(x) = \int_{-\infty}^{\infty} \phi(y)p_t^r(x,y)\,dy, \; Q_t^r\phi(x) = \int_{-\infty}^{\infty} \phi(y)q_t^r(x,y)\,dy$$

an integration by parts in (2.14) leads to the relation

$$p_t^r(x,y) = -\int_{-\infty}^{x} \frac{\partial}{\partial y} q_t^r(z,y)\,dz; \qquad 0<t<\tau-r. \tag{2.15}$$

To our knowledge, the representations (2.8), (2.15) are new.

3. THE OPTIMAL STOPPING PROBLEM

Throughout this paper, the functions $h:[0,\tau]\times\mathscr{R}\to[0,\infty)$ and $g:\mathscr{R}\to[0,\infty)$ will be of class $C^{0,1}$ and C^1, respectively, the function $f:[0,\tau]\to[0,\infty)$ will be continuous, and the growth condition

$$|h_x(t,x)|+|g'(x)|\leqq K\exp\{\mu|x|^\nu\}; \qquad \forall\,(r,x)\in[0,\tau]\times\mathscr{R} \tag{3.1}$$

will be assumed to hold for some positive constants K, μ and $\nu<2$.

For certain results we shall also need some, or all, of the following conditions:

$$\text{The functions } h(r,\cdot) \text{ and } g(\cdot) \text{ are convex.} \tag{3.2}$$

$$\text{The function } h(r,\cdot) \text{ is } \textit{strictly} \text{ convex.} \tag{3.3}$$

$$g'(x)\leqq f(\tau); \qquad \forall\,x\in\mathscr{R}. \tag{3.4}$$

One of the following holds:

$$0<c\leqq f(r)\leqq C; \qquad \forall\,0\leqq r\leqq\tau$$

$$c|x|^p-d\leqq h(r,x)\leqq C(1+|x|^p); \qquad \forall\,(r,x)\in[0,\tau]\times\mathscr{R}$$

$$c|x|^p-d\leqq g(x)\leqq C(1+|x|^p); \qquad \forall\,x\in\mathscr{R} \tag{3.5}$$

for some finite constants $0<c\leqq C$, $d\geqq 0$, $p>1$.

We shall study an optimal stopping problem for the Brownian motion $W=\{W_t,\mathscr{F}_t;0\leqq t\leqq\tau\}$ of Section 2, in which the nondecreasing functions $h_x(r,\cdot)$ and $g'(\cdot)$ represent the running cost and the terminal cost, respectively, and the function $f(r)$ represents the cost of stopping before the terminal time. More precisely, the *risk* (expected total cost) in this problem, corresponding to an $\{\mathscr{F}_t\}$-stopping time $\sigma\leqq\tau-r$, is given by

$$R(\sigma;r,x)=E\left[\int_0^{\sigma} h_x(r+t,x+W_t)\,dt+f(r+\sigma)1_{\{\sigma<\tau-r\}}+g'(x+W_{\tau-r})1_{\{\sigma=\tau-r\}}\right] \tag{3.6}$$

for every pair $(r,x)\in[0,\tau]\times\mathscr{R}$, and the *optimal risk* is defined as

$$u(r,x)=\inf_{\sigma\in\mathscr{S}_{0,\tau-r}} R(\sigma;r,x). \tag{3.7}$$

Here and in the sequel, we denote by $\mathscr{S}_{u,v}$ the class of $\{\mathscr{F}_t\}$-stopping times with values in $[u,v]$, $0\leqq u<v\leqq\tau$.

In order to cast the optimal stopping problem of (3.7) into a more conventional form, let us introduce the functions

$$G(r,x)=E\left[\int_0^{\tau-r} h_x(r+t,x+W_t)\,dt+g'(x+W_{\tau-r})\right] \tag{3.8}$$

$$H(r,x)=E\left[\int_0^{\tau-r} h_x(r+t,x+W_t)\,dt+\{g'(x+W_{\tau-r})-f(r)\}1_{[0,\tau)}(r)\right], \tag{3.9}$$

which are continuous on $[0,\tau]\times\mathscr{R}$ and $[0,\tau)\times\mathscr{R}$, respectively. We have then from (3.6) and (3.7):

$$v(r,x)\triangleq G(r,x)-u(r,x)$$

$$=\sup_{\sigma\in\mathscr{S}_{0,\tau-r}} E\left[\int_{\sigma}^{\tau-r} h_x(r+t,x+W_t)\,dt+\{g'(x+W_{\tau-r})-f(r+\sigma)\}1_{\{\sigma<\tau-r\}}\right], \tag{3.10}$$

and by the strong Markov property the function of (3.10) is given as

$$v(r,x)=\sup_{\sigma\in\mathscr{S}_{0,\tau-r}} EH(r+\sigma,x+W_\sigma), \tag{3.11}$$

i.e., as the maximal reward in a problem of optimal stopping for Brownian motion, with payoff function $H(r,x)$. It is well known (e.g. [1]; [16], Lemma 4) that *v is continuous* on $[0,\tau)\times\mathscr{R}$ and that the stopping time

$$\sigma^{(\varepsilon)}(r,x)\triangleq\inf\{0<t<\tau-r;\ \ v(r+t,x+W_t)\leqq H(r+t,x+W_t)+\varepsilon\}\wedge(\tau-r)$$

$$=\inf\{0<t<\tau-r;\ \ u(r+t,x+W_t)+\varepsilon\geqq f(r+t)\}\wedge(\tau-r) \tag{3.12}$$

is ε-optimal, for every $\varepsilon>0$ and initial time-space pair $(r,x)\in[0,\tau)\times\mathscr{R}$. It is also well known (e.g. [5], [7], [8], [17]) that the RCLL (Right Continuous with Left-hand Limits) supermartingale

$$Z_t^{(r,x)} = v(r+t, x+W_t), \quad 0 \leqq t \leqq \tau - r \tag{3.13}$$

is the *Snell envelope* of the process

$$Y_t^{(r,x)} = H(r+t, x+W_t), \quad 0 \leqq t \leqq \tau - r \tag{3.14}$$

(i.e., the smallest RCLL supermartingale that majorizes $Y^{(r,x)}$), and that

$$Z_\sigma^{(r,x)} = \operatorname*{ess\,sup}_{\rho \in \mathscr{S}_{\sigma,\tau-r}} E[Y_\rho^{(r,x)} | \mathscr{F}_\sigma], \quad \text{a.s.} \tag{3.15}$$

holds for every $\sigma \in \mathscr{S}_{0,\tau-r}$.

Let us suppose now that the condition (3.2) holds; the stopping time of (3.12) can then be written equivalently as

$$\sigma^{(\varepsilon)}(r,x) = \inf\{0 < t < \tau - r;\quad x + W_t \geqq s^{(\varepsilon)}(r+t)\} \wedge (\tau - r), \tag{3.16}$$

for a suitable function $s^{(\varepsilon)}$ on $[0, \tau)$ given as

$$s^{(\varepsilon)}(r) \triangleq \inf\{x \in \mathscr{R};\quad u(r,x) + \varepsilon \geqq f(r)\} = \inf\{x \in \mathscr{R};\quad v(r,x) \leqq H(r,x) + \varepsilon\} \tag{3.17}$$

for $0 \leqq r < \tau$. If, in addition to (3.2), the condition (3.4) is valid, then the stopping time $\sigma^{(0)}(r,x)$ in the notation of (3.12) is actually *optimal* (cf. [16], Lemma 6; [7], Theorem 4; or [17], Theorem 6). This stopping time can also be cast in the form (3.16) as

$$\sigma_0(r,x) \triangleq \inf\{0 < t < \tau - r;\quad x + W_t \geqq s(r+t)\} \wedge (\tau - r), \tag{3.18}$$

where now, by analogy with (3.17),

$$s(r) \triangleq \inf\{x \in \mathscr{R}; u(r,x) = f(r)\} = \inf\{x \in \mathscr{R}; v(r,x) = H(r,x)\}; \quad 0 \leqq r \leqq \tau \tag{3.19}$$

is the optimal stopping boundary. For concreteness, we shall assume in certain parts of our development that

$$\begin{cases} \text{both intervals } \{x \in \mathscr{R};\quad u(r,x) < f(r)\} \quad \text{and} \quad \{x \in \mathscr{R};\quad u(r,x) = f(r)\} \\ \text{are nonempty, for every } 0 \leqq r < \tau. \end{cases} \tag{3.20}$$

The notation

$$\mathscr{C} \triangleq \{(r,x) \in [0,r) \times \mathscr{R};\quad u(r,x) < f(r)\} \tag{3.21}$$

will be employed for the optimal *continuation region* of this problem.

LEMMA 3.1 *Under the assumptions (3.2) and (3.20), the function $s(\cdot)$ of (3.19) is lower-semicontinuous on $[0, \tau)$.*

If (3.3) holds as well, then the function $s^\varepsilon(\cdot)$ of (3.17) is continuous on $[0, \tau)$, for every $\varepsilon > 0$ sufficiently small.

Proof Let us take a sequence $\{r_n\}_{n=1}^\infty \subseteq [0, \tau)$ with $\lim_{n\to\infty} r_n = r$ in $[0, \tau)$ and $\lim_{n\to\infty} s(r_n) = s^*$ in $\mathscr{R}$. From $u(r_n, s(r_n)) = f(r_n)$, $\forall n \geqq 1$ and the continuity of u on

$[0,\tau)\times\mathcal{R}$, we have $u(r,s^*)=f(r)$, whence $s(r)\leqq s^*$ and the lower-semicontinuity of $s(\cdot)$: $s(r)\leqq \lim_{n\to\infty} s(r_n)$.

For the second claim, fix $r\in[0,\tau)$ and take $y<x<s(r)$; the optimality of $\sigma_0(r,x)$ at (r,x) and the assumption (3.3) imply

$$u(r,x)-u(r,y)\geqq R(\sigma_0(r,x);r,x)-R(\sigma_0(r,x);r,y)$$

$$=E\int_0^{\sigma_0(r,x)}\{h_x(r+t,x+W_t)-h_x(r+t,y+W_t)\}\,dt$$

$$+E[\{g'(x+W_{\tau-r})-g'(y+W_{\tau-r})\}1_{\{\sigma_0(r,x)=\tau-r\}}]>0.$$

In other words, $u(r,\cdot)$ is strictly increasing on $(-\infty,s(r))$ for any given $r\in[0,\tau)$.

Then for sufficiently small $\varepsilon>0$, $s^\varepsilon(\cdot)$ is characterized by

$$u(r,s^\varepsilon(r))=f(r)-\varepsilon,\quad 0\leqq r<\tau. \tag{3.22}$$

But $\partial u/\partial r$, $\partial u/\partial x$ exist and are continuous in $\mathcal{C}$ (e.g. [16], Lemma 5), and we just showed that $\partial u/\partial x(r,x)>0$ in $\mathcal{C}$. From these observations, (3.22), and the implicit function theorem, it follows that $s^\varepsilon(\cdot)$ is continuous. □

In an Appendix (Section 7) we establish the following representation (3.23) for the maximal reward function $v(r,x)$ of (3.11), and show that it subsumes the "principle of smooth fit" (cf. Remark 7.7). The proof is rather lengthy; it uses techniques analogous to those employed in the theory of *balayage for semimartingales* as developed in [4], [6] and [18], as well as ideas from excursion theory.

THEOREM 3.2 *Suppose that conditions (3.2)–(3.4), (3.20) are satisfied, and that the function f is absolutely continuous. Then the function $v:[0,\tau]\times\mathcal{R}\to\mathcal{R}$ of (3.11) admits the representation*

$$v(r,x)=E\left[\int_0^{\tau-r}\{h_x(r+t,x+W_t)+f'(r+t)\}1_{\{x+W_t\geqq s(r+t)\}}\,dt\right]. \tag{3.23}$$

COROLLARY 3.3 *Under the conditions of Theorem 3.2, the optimal stopping risk $u=G-v$ of (3.7) can be represented as*

$$u(r,x)=E\left[\int_0^{\tau-r}h_x(r+t,x+W_t)1_{\{x+W_t<s(r+t)\}}\,dt+g'(x+W_{\tau-r})\right.$$

$$\left.-\int_0^{\tau-r}f'(r+t)1_{\{x+W_t\geqq s(r+t)\}}\,dt\right]. \tag{3.24}$$

Remark 3.4 Suppose that f is of class $C^1([0,\tau])$. Then the set $\{(r,x)\in[0,\tau)\times\mathcal{R};$

$h_x(r,x)+f'(r)<0\}$ is included in the continuation region $\mathscr{C}$ of (3.21); see [16], page 110, as well as Corollary 7.2. Therefore, the first interval in (3.20) is nonempty if $\{x\in\mathscr{R};\ h_x(r,x)+f'(r)<0\}\neq\emptyset$.

Conditions guaranteeing that the second interval in (3.20) is nonempty are provided in [16], Sections 3.1 and 3.2.

Remark 3.5 Suppose that, in addition to satisfying conditions (3.2)–(3.4), the functions h, g and f are three times continuously differentiable and satisfy, together with their derivatives, polynomial growth conditions as $|x|$ tends to infinity. Suppose also that

$$f(t)\geqq 1, \qquad \forall\, 0\leqq t\leqq\tau$$

$$\lim_{x\to-\infty} g'(x)<f(\tau)$$

$$h_{xx}(t,x)\geqq\alpha[1+|h_{tx}(t,x)-f''(t)|], \qquad \forall\,(t,x)\in[0,\tau]\times\mathscr{R}$$

hold, for some $\alpha>0$. Then it is shown in [15], using analytical methods, that the function $s\colon[0,\tau)\to\mathscr{R}$ is locally Lipschitz continuous, that the condition (3.20) holds, and that the function $u(r,x)$ of (1.1) is of class $C^{1,1}([0,\tau)\times\mathscr{R})$ and satisfies a polynomial growth condition in $|x|$.

4. THE MONOTONE FOLLOWER PROBLEM

With the notation of (2.1) and the assumptions of Section 3 on the cost functions h, f and g, the *Monotone Follower* stochastic control problem consists of minimizing the expected cost

$$J(\xi;r,x)=E\left[\int_0^{\tau-r} h(r+t,X_t)\,dt+\int_{[0,\tau-r)} f(r+t)\,d\xi_t+g(X_{\tau-r})\right] \tag{4.1}$$

over $\xi\in\mathscr{A}(\tau-r)$, for any given $r\in[0,\tau)$ and $x\in\mathscr{R}$. We shall denote by

$$V(r,x)=\inf_{\xi\in\mathscr{A}(\tau-r)} J(\xi;r,x) \tag{4.2}$$

the value function of this problem, and by

$$Q(r,x,y)=\inf_{\xi\in\mathscr{A}(\tau-r,y)} J(\xi;r,x); \qquad 0<y<\infty \tag{4.3}$$

the value function of its *finite-fuel* counterpart (i.e., subject to the constraint $\xi_{\tau-r}\leqq y$, a.s.).

We shall single out the "cost of doing nothing" in the monotone follower problem of (4.2), i.e., the function

$$P(r,x) \triangleq J(0;r,x)$$

$$= E\left[\int_0^{\tau-r} h(r+t, x+W_t)\,dt + g(x+W_{\tau-r})\right]. \tag{4.4}$$

It is not hard to see, using the Feynman–Kac Theorem 4.4.2 in [12], that $P(r,\cdot)$ is an anti-derivative of the function $G(r,\cdot)$ in (3.8):

$$\frac{\partial}{\partial x} P(r,x) = G(r,x). \tag{4.5}$$

The problems (4.2), (4.3) were studied in the articles [11] and [10], respectively. Under the assumptions (3.2), (3.4) and (3.5), it was shown there that for every $(r,x) \in [0,\tau) \times \mathscr{R}$:

i) there exists an optimal process $\xi^*(r,x) \in \mathscr{A}(\tau - r, x)$ for the problem of (4.2),

ii) the "truncated version" of this process, namely

$$\xi_t^*(r,x,y) = \begin{Bmatrix} \xi_t^*(r,x); & 0 \leqq t \leqq T(y) \\ y \quad ; & T(y) < t \leqq \tau - r \end{Bmatrix}, \tag{4.6}$$

with $T(y) = \inf\{0 \leqq t < \tau - r; \xi_t^*(r,x) \geqq y\} \wedge (\tau - r)$, is optimal for the finite-fuel problem of (4.3), and

iii) the identities

$$\frac{\partial}{\partial x} V(r,x) = u(r,x) \tag{4.7}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) Q(r,x,y) = u(r,x) \tag{4.8}$$

hold.

The methodology of both articles [11], [10] had the control problems of (4.2), (4.3), respectively, as its starting point, and used a probabilistic technique of "switching of paths" at appropriate random times as a means of comparing expected costs at nearby points, differentiating the functions $V(r,\cdot)$, and $Q(r,\cdot,\cdot)$, and obtaining the identities (4.7) and (4.8).

In the next section we shall use the approach of Section 2 (in particular, the identities (2.11)–(2.13)), to *integrate* directly the optimal stopping risk $u(r,\cdot)$ and to provide an alternative derivation of the above results; cf. Theorems 5.5, 5.7 and Remark 5.6. A useful stochastic representation for the value function $V(r,x)$ will be derived in Section 6, this time by integrating both sides of (3.23) in the spatial argument; cf. Theorem 6.1 and Remark 6.2.

5. FIRST INTEGRATION

For an arbitrary process $\xi \in \mathscr{A}(\tau)$, let us introduce the left-continuous inverse

$$T(u) \triangleq \inf\{0 \leqq t < \tau; \xi_t^+ \geqq u\} \wedge \tau; \qquad 0 \leqq u < \infty \tag{5.1}$$

and notice the a.s. equivalences

$$T(u) \leqq t \Leftrightarrow \xi_t^+ \geqq u; \qquad 0 \leqq t < \tau, 0 < u < \infty \tag{5.2}$$

$$T(u) = \tau \Leftrightarrow \xi_\tau \leqq u; \qquad 0 \leqq u < \infty \tag{5.3}$$

which generalize (2.9), (2.10). In particular, (5.2) leads via a monotone class argument to the change-of-variable formula

$$\int_0^\infty f(T(u)) 1_{\{T(u)<\tau\}}\, du = \int_{[0,\tau)} f(t)\, d\xi_t. \qquad \text{a.s. P} \tag{5.4}$$

for every Borel-measurable function $f: [0,\tau] \to [0,\infty)$.

Lemma 5.1 *For any $\xi \in \mathscr{A}(\tau)$ and $x \in \mathscr{R}$, we have*

$$\int_{-\infty}^{x} \left[\int_{T(x-z)}^{\tau} h_x(t, z+W_t)\, dt + \{g'(z+W_\tau) - f(T(x-z))\} 1_{\{T(x-z)<\tau\}} \right] dz$$

$$= \int_0^\tau \{h(t, x+W_t) - h(t, x+W_t-\xi_t)\}\, dt$$

$$- \int_{[0,\tau)} f(t)\, d\xi_t + \{g(x+W_\tau) - g(x+W_\tau - \xi_\tau)\}, \tag{5.5}$$

almost surely. In particular,

$$P(0,x) - J(\xi; 0, x) \leqq \int_{-\infty}^{x} v(0,z)\, dz; \qquad \forall\, \xi \in \mathscr{A}(\tau), x \in \mathscr{R}. \tag{5.6}$$

Proof From (5.4) we get

$$\int_{-\infty}^{x} f(T(x-z)) 1_{\{T(x-z)<\tau\}}\, dz = \int_0^\infty f(T(u)) 1_{\{T(u)<\tau\}}\, du$$

$$= \int_{[0,\tau)} f(t)\, d\xi_t, \qquad \text{a.s. P.} \tag{5.7}$$

Similarly, we obtain from (5.2) and the Fubini theorem, by analogy with (2.12):

$$\int_{-\infty}^{x} \int_{T(x-z)}^{\tau} h_x(t, z+W_t)\, dt\, dz = \int_{-\infty}^{x} \int_{0}^{\tau} h_x(t, z+W_t) 1_{\{T(x-z) \leqq t\}}\, dt\, dz$$

$$= \int_{0}^{\tau} \int_{-\infty}^{x} h_x(t, z+W_t) 1_{\{\xi_t^+ \geqq x-z\}}\, dz\, dt$$

$$= \int_{0}^{\tau} \{h(t, x+W_t) - h(t, x+W_t - \xi_t)\}\, dt, \quad \text{a.s. P}, \tag{5.8}$$

because $\xi(\omega)$, $\xi^+(\omega)$ differ on a set which is at most countable, for **P**-a.e. $\omega \in \Omega$. Finally, with the help of (5.3) we have, by analogy with (2.13):

$$\int_{-\infty}^{x} g'(z+W_\tau) 1_{\{T(x-z)<\tau\}}\, dz = \int_{-\infty}^{x} g'(z+W_\tau) 1_{\{\xi_\tau > x-z\}}\, dz$$

$$= g(x+W_\tau) - g(x+W_\tau - \xi_\tau), \qquad \text{a.s. P}. \tag{5.9}$$

Putting these identities together we obtain (5.5), and (5.6) follows by taking expectations and using the Fubini theorem. □

Remark 5.2 It should be noted at this point that (5.5), as well as its consequences (5.6) and

$$P(0,x) - V(0,x) \leqq \int_{-\infty}^{x} v(0,z)\, dz, \tag{5.10}$$

have been obtained under minimal assumptions on the functions h, f, g; in particular, *none of* (3.2)–(3.5), (3.20) *has been utilized.* However, we shall need some of these conditions in order to show that (5.10) actually holds as an *equality.*

Remark 5.3 Repeating the proof of Lemma 5.1, one can show that for every $y \in (0, \infty)$ the equation

$$\int_{x-y}^{x} E\left[\int_{T(x-z)}^{\tau} h_x(t, z+W_t)\, dt + \{g'(z+W_\tau) - f(T(x-z))\} 1_{\{T(x-z)<\tau\}}\right] dz$$

$$= P(0,x) - J(\xi; 0, x) \tag{5.11}$$

and its consequence

$$P(0,x) - J(\xi; 0, x) \leqq \int_{x-y}^{x} v(0,z)\, dz \tag{5.12}$$

hold for every $\xi \in \mathscr{A}(\tau, y)$ and $x \in \mathscr{R}$. For (5.11), notice that $T(x-z) = \tau$ holds if $z \leqq x-y$, thanks to (5.3). □

In order to obtain inequalities in the opposite direction of (5.10), (5.12), let us consider a continuous function $s:[0,\tau)\to\mathscr{R}$ and the corresponding process of entrance times

$$\sigma(x)=\inf\{0\leqq t<\tau; x+W_t\geqq s(t)\}\wedge\tau; \qquad x\in\mathscr{R}$$

which has a.s. nonincreasing and right-continuous paths. Recall also the proof of Proposition 2.1 and the notation employed there.

LEMMA 5.4 *For every fixed $x\in\mathscr{R}$, we have*

$$P(0,x)-J(\xi(x);0,x)=\int_{-\infty}^{x} E\left[\int_{\sigma(z)}^{\tau} h_x(t,z+W_t)\,dt + \{g'(z+W_\tau)-f(\sigma(z))\}1_{\{\sigma(z)<\tau\}}\right]dz. \tag{5.13}$$

Furthermore, for any $y\in(0,\infty)$ and with $\xi(x,y)$ obtained from the process $\xi(x)$ as in (4.6), we have

$$P(0,x)-J(\xi(x,y);0,x)=\int_{x-y}^{x} E\left[\int_{\sigma(z)}^{\tau} h_x(t,z+W_t)\,dt + \{g'(z+W_\tau)-f(\sigma(z))\}1_{\{\sigma(z)<\tau\}}\right]dz. \tag{5.14}$$

Proof Follows directly from (2.12) and (2.13), just as in the proof of Lemma 5.1 and Remark 5.3.

THEOREM 5.5 *Under the conditions (3.2), (3.3) and (3.20), the identities*

$$V(r,x)=P(r,x)-\int_{-\infty}^{x} v(r,z)\,dz \tag{5.15}$$

and

$$\begin{aligned}Q(r,x,y)&=P(r,x)-\int_{x-y}^{x} v(r,z)\,dz\\ &=P(r,x-y)-V(r,x-y)+V(r,x)\end{aligned} \tag{5.16}$$

hold for every $(r,x,y)\in[0,\infty)\times\mathscr{R}\times(0,\infty)$. In particular, (4.7) and (4.8) hold.

Proof It is quite plausible, and not hard to show, that

$$\lim_{y\to\infty} Q(r,x,y) = V(r,x) \tag{5.17}$$

holds; cf. Section 8 (Appendix). For simplicity of notation, we shall establish (5.15), (5.16) only for $r=0$. Let us recall the notation of (3.16), (3.17) for ε-optimal stopping times $\sigma^{(\varepsilon)}$ and their corresponding moving boundaries $s^{(\varepsilon)}$ (Lemma 3.1), and read Lemma 5.4 with $(s, \sigma, \xi(x), \xi(x,y))$ replaced by their counterparts $(s^{(\varepsilon)}, \sigma^{(\varepsilon)}, \xi^{(\varepsilon)}(x), \xi^{(\varepsilon)}(x,y))$. We obtain from (5.14):

$$P(0,x) - Q(0,x,y) \geqq P(0,x) - J(\xi^{\varepsilon}(x,y); 0, x) \geqq \int_{x-y}^{x} v(0,z)\,dz - \varepsilon y$$

for every $\varepsilon > 0$, whence

$$P(0,x) - Q(0,x,y) \geqq \int_{x-y}^{x} v(0,z)\,dz.$$

In conjunction with (5.12), this establishes the first identity in (5.16) for $r=0$; the identity (5.15) follows then easily thanks to (5.17) upon letting $y\to\infty$, and leads immediately to the second identity in (5.16). For the last claim, we just differentiate in (5.15), (5.16) with respect to x, and recall (4.5) as well as (3.10).

Remark 5.6 The relations (4.7), (4.8) were established in [10, 11] under the assumption that there exists an optimal process for the problem (4.2).

THEOREM 5.7 *Suppose that (3.2)–(3.4) and (3.20) hold, and that the function $s(\cdot)$ of (3.19) is continuous on $[0,\tau)$. Then the process $\xi(r,x)$ of (2.4) is optimal for $V(r,x)$, and its truncated version as in (4.6) is optimal for $Q(r,x,y)$.*

Proof Again, we discuss only the case $r=0$. We recall from (3.18) the notation $\sigma(x) = \sigma_0(0,x)$ for the optimal stopping time of the problem (3.7), and obtain from (5.13):

$$P(0,x) - J(\xi(0,x); 0, x) = \int_{-\infty}^{x} v(0,z)\,dz.$$

The conclusion $J(\xi(0,x); 0, x) = V(0,x)$ follows from (5.15). The second claim is proved similarly, using (5.14) and (5.16).

6. SECOND INTEGRATION

Let us integrate now both sides of the expression

$$v(r,z) = E\int_0^{\tau-r} [h_x(r+t, z+W_t) + f'(r+t)]1_{\{z+W_t \geq s(r+t)\}}\,dt \tag{3.23}$$

with respect to the spatial argument z.

THEOREM 6.1 *Suppose that (3.2), (3.4), (3.20) are satisfied, and that f is absolutely continuous. Then the value function of the monotone follower problem (4.2) admits the representation*

$$V(r,x)=E\left[\int_0^{\tau-r} h(r+t, s(r+t)\wedge(x+W_t))\,dt+g(x+W_{\tau-r})\right.$$

$$\left.-\int_0^{\tau-r} f'(r+t)(x+W_t-s(r+t))^+\,dt\right]. \tag{6.1}$$

In particular, if the function $s(\cdot)$ is of bounded variation:

$$V(r,s(r))=\int_r^\tau h(\theta,s(\theta))\,d\theta-\int_r^\tau f(\theta)\,ds(\theta)+g(s(\tau)). \tag{6.2}$$

Proof Let us integrate both sides of (3.23) over $(-\infty,x)$; from the straightforward a.s. identity

$$\int_{-\infty}^{x} h_x(r+t,z+W_t)1_{\{z+W_t\geq s(r+t)\}}\,dz$$

$$=h(r+t,x+W_t)-h(r+t,s(r+t)\wedge(x+W_t)),$$

the Fubini theorem, and Theorem (3.2), the result is

$$\int_{-\infty}^{x} v(r,z)\,dz=E\left[\int_0^{\tau-r}\{h(r+t,x+W_t)-h(r+t,s(r+t)\wedge(x+W_t))\}\,dt\right.$$

$$\left.+\int_0^{\tau-r} f'(r+t)(x+W_t-s(r+t))^+\,dt\right]$$

$$=P(r,x)-N(r,x);\qquad \forall\, x\in\mathscr{R} \tag{6.3}$$

where

$$N(r,x)\triangleq E\left[\int_0^{\tau-r} h(r+t,s(r+t)\wedge(x+W_t))\,dt+g(x+W_{\tau-r})\right.$$

$$\left.-\int_0^{\tau-r} f'(r+t)(x+W_t-s(r+t))^+\,dt\right]. \tag{6.4}$$

From (5.15) and (6.3) we conclude that $N\equiv V$, and obtain the representation (6.1); thus, for $x\geq s(r)$ we have, thanks to (4.7):

$$V(r,s(r))=V(r,x)-\int_{s(r)}^{x} u(r,z)\,dz$$

$$=N(r,x)-(x-s(r))\cdot f(r)$$

$$=E\Bigg[\int_0^{\tau-r} h(r+t,s(r+t)\wedge(x+W_t))\,dt+g(s(\tau)\wedge(x+W_{\tau-r}))$$

$$+f(\tau)\{(x+W_{\tau-r}-s(\tau))^+-(x-s(r))\}$$

$$-\int_0^{\tau-r} f'(r+t)\{(x+W_t-s(r+t))^+-(x-s(r))\}\,dt\Bigg]. \tag{6.5}$$

The expression on the right-hand side of (6.5) is thus independent of x, as long as $x\geqq s(r)$. If we let $x\uparrow\infty$ and appeal to the Dominated Convergence Theorem, we obtain the value of the function V along the moving boundary $s(\cdot)$:

$$V(r,s(r))=E\Bigg[\int_0^{\tau-r} h(r+t,s(r+t))\,dt-\int_0^{\tau-r} f'(r+t)\{W_t-s(r+t)+s(r)\}\,dt$$

$$+g(s(\tau))+f(\tau)\{W_{\tau-r}-s(\tau)+s(r)\}\Bigg],$$

whence

$$V(r,s(r))=\int_r^{\tau} h(\theta,s(\theta))\,d\theta+\int_r^{\tau} f'(\theta)s(\theta)\,d\theta-f(\tau)s(\tau)+f(r)s(r)+g(s(\tau)). \tag{6.6}$$

The expression (6.2) follows from (6.6), in case $s(\cdot)$ is of bounded variation. (In this case, (6.2) acquires the significance of the cost corresponding to a "deterministic ride along the moving boundary $s(\theta)$, $r\leqq\theta\leqq\tau$".) □

Remark 6.2 Equating (6.6) with the expression (6.1) evaluated at $x=s(r)$, we obtain the integral equation

$$E\Bigg[\int_0^{\tau-r} h(r+t,s(r+t)\wedge(s(r)+W_t))\,dt+g(s(r)+W_{\tau-r})$$

$$-\int_0^{\tau-r} f'(r+t)(s(r)+W_t-s(r+t))^+\,dt\Bigg]$$

$$=\int_r^\tau h(\theta,s(\theta))\,d\theta+\int_r^\tau f'(\theta)s(\theta)\,d\theta-f(\tau)s(\tau)+f(r)s(r)+g(s(\tau))$$

for the free-boundary function $s(\cdot)$. It would be interesting to study questions of existence and uniqueness of solution for the equation (6.7), as an alternative to the integral equations of [16] which require a lot more smoothness.

7. APPENDIX 1: A NEW REPRESENTATION FOR THE OPTIMAL STOPPING RISK

We want to justify here the representation

$$v(r,x)=E\left[\int_0^{\tau-r}[h_x(r+t,x+W_t)+f'(r+t)]1_{\{x+W_t\geqq s(r+t)\}}\,dt\right] \tag{3.23}$$

for the function of (3.10), (3.11), *under the conditions of Theorem 3.2.*

In the notation of (3.11), (3.9) and (3.19), let us write this function as

$$v(r,x)=\sup_{\sigma\in\mathscr{S}_{0,\tau-r}}E\left[\int_\sigma^{\tau-r}\{h_x(r+t,x+W_t)+f'(r+t)\}\,dt+\{g'(x+W_{\tau-r})-f(\tau)\}1_{\{\sigma<\tau-r\}}\right]$$

$$=E[C^{(r,x)}_{\tau-r}-C^{(r,x)}_{\sigma_0(r,x)}], \tag{7.1}$$

where $\sigma_0(r,x)$ is the optimal stopping time of (3.18) and

$$C^{(r,x)}_t\triangleq\int_0^t\{h_x(r+\theta,x+W_\theta)+f'(r+\theta)\}\,d\theta+\{g'(x+W_{\tau-r})-f(\tau)\}1_{\{0<\tau-r\leqq t\}}. \tag{7.2}$$

This process is absolutely continuous on $[0,\tau-r)$, with a possible jump at $t=\tau-r$; it is quite easy to see that it is also predictable ([3], T 31 on p. 85).

Now for every fixed $t\in[0,\tau-r]$, introduce the stopping time

$$\sigma_t(r,x)\triangleq\inf\{\theta\in(t,\tau-r];\quad x+W_\theta\geqq s(r+\theta)\}\wedge(\tau-r) \tag{7.3}$$

and observe that

i) from the a.s. continuity of W and the lower-semicontinuity of $s(\cdot)$ (Lemma 3.1) it follows that: $x+W_{\sigma_t(r,x)}\geqq s(r+\sigma_t(r,x))$, a.s. on $\{\sigma_t(r,x)<\tau-r\}$,

ii) for every fixed (t,r), the mapping $x\mapsto\sigma_t(r,x)$ is a.s. decreasing and right-continuous, and

iii) for fixed (r,x), the mapping $t\mapsto\sigma_t(r,x)$ is a.s. increasing and right-continuous.

If one considers the *non-adapted* process of bounded variation

$$\tilde{C}_t^{(r,x)} \triangleq C_{\sigma_t(r,x)}^{(r,x)} - C_{\sigma_0(r,x)}^{(r,x)}, \tag{7.4}$$

then (7.1) takes the form

$$v(r,x) = E[\tilde{C}_{\tau-r}^{(r,x)}] \tag{7.5}$$

of a potential associated with $\tilde{C}^{(r,x)}$. But an excessive function such as $v(r,x)$ is also the potential associated with an *adapted, nondecreasing* process $D^{(r,x)}$ (see [2], Section IV.4 for general theory concerning the representation of excessive functions as potentials of appropriate additive functionals). The basic result of this section is the following theorem, which provides an explicit characterization of the process $D^{(r,x)}$ in question.

THEOREM 7.1 *The dual predictable projection $D^{(r,x)}$ of the process $\tilde{C}^{(r,x)}$ in (7.4) is* nondecreasing, *and admits the representation*

$$D_t^{(r,x)} = \int_0^t \{h_x(r+\theta, x+W_\theta) + f'(r+\theta)\} 1_{\{x+W_\theta \geqq s(r+\theta)\}} \, d\theta. \tag{7.6}$$

COROLLARY 7.2 *From the nondecreasing nature of $D^{(r,x)}$ and its representation (7.6), it follows readily that, if f is of class $C^1([0,\tau])$, the set $\{(r,x) \in [0,\tau) \times \mathscr{R};\ h_x(r,x) + f'(r) < 0\}$ is included in the continuation region $\mathscr{C}$ of (3.21). On the other hand, from (7.5) and the nature of $D^{(r,x)}$ as the dual predictable projection of $\tilde{C}^{(r,x)}$, it develops that*

$$v(r,x) = E[D_{\tau-r}^{(r,x)}],$$

which is the desired representation (3.23). □

In the (french!) terminology of the general theory of processes, $D^{(r,x)}$ is called the "*balayée prévisible*" *of* $C^{(r,x)}$. In fact, our methodology for establishing Theorem 7.1 is inspired by the theory of "balayage" for semimartingales, as developed in [4], [6] and [18].

The existence of $D^{(r,x)}$ is established easily; indeed, it suffices to write the process of (3.13) in the form

$$Z_t^{(r,x)} = E[C_{\tau-r}^{(r,x)} - C_{\sigma_t(r,x)}^{(r,x)} | \mathscr{F}_t] = E[\tilde{C}_{\tau-r}^{(r,x)} - \tilde{C}_t^{(r,x)} | \mathscr{F}_t]. \tag{7.7}$$

In order to derive this representation we use (7.1), the strong Markov property, and the fact that $\{Z_{\theta \wedge \sigma_t(r,x)}^{(r,x)}, \mathscr{F}_\theta;\ t \leqq \theta \leqq \tau - r\}$ is a martingale for every fixed $(t,x) \in [0, \tau - r) \times \mathscr{R}$. Now the process $Z^{(r,x)}$ of (3.13) is a RCLL supermartingale of class $D[0, \tau - r]$, and as such it admits the Doob–Meyer decomposition

$$Z_t^{(r,x)} = E[D_{\tau-r}^{(r,x)} | \mathscr{F}_t] - D_t^{(r,x)} \tag{7.8}$$

for some *predictable nondecreasing* process $D^{(r,x)}$ with RCLL paths (cf. [12], Section 1.4). Comparing the right-hand sides of (7.7), (7.8) we conclude that $D^{(r,x)}$ is the dual predictable projection of $\tilde{C}^{(r,x)}$.

In order to actually *calculate* $D^{(r,x)}$, we need to study the process $\tilde{C}^{(r,x)}$ of (7.4) a bit more closely. Let us take a realization $\omega \in \Omega$ for which the Brownian path $t \mapsto W_t(\omega)$ is continuous, and introduce the set

$$\mathscr{H}(\omega) \triangleq \{\sigma_0(\omega) \leqq u < \tau - r; \qquad x + W_u(\omega) \geqq s(r+u)\} \tag{7.9}$$

and the function

$$l_t(\omega) \triangleq \sup\{s \in [0,t); \quad s \in \mathscr{H}(\omega)\}, \qquad 0 \leqq t \leqq \tau - r. \tag{7.10}$$

Thanks to the lower-semicontinuity of $s(\cdot)$ (Lemma 3.1) the set $\mathscr{H}(\omega)$ is closed in $[\sigma_0(\omega), \tau - r)$; on the other hand, the function of (7.10) is the left-continuous inverse of the right-continuous function $\{\sigma_t(\omega),\ 0 \leqq t \leqq \tau - r\}$, and the sets $\{t; l_t(\omega) = t\}$ and $\mathscr{H}(\omega)$ can differ by at most countably many points. The complement $\mathscr{H}^c(\omega)$ of the closed set $\mathscr{H}(\omega)$ is the union of a countable collection of disjoint open intervals, and we shall denote by $\mathscr{E}^{\rightarrow}(\omega)$ $(\mathscr{E}^{\rightarrow}_{\pi}(\omega))$ the collection of their left-endpoints (respectively, the subset of $\mathscr{E}^{\rightarrow}(\omega)$ which consists of points that are not totally isolated). Now for any given $\varepsilon > 0$, there exists a finite number $N_\varepsilon(\omega)$ of such intervals whose length exceeds ε; let us denote by $(L^\varepsilon_n, R^\varepsilon_n)$ the nth of them, and recall from Dellacherie ([13], p. 126) that

$$S^\varepsilon_n \triangleq \varepsilon + L^\varepsilon_n, \quad R^\varepsilon_n \equiv \sigma_{S^\varepsilon_n} \quad \text{are stopping times.} \tag{7.11}$$

With this notation, and dropping from now on the dependence on (r,x) and ω, we obtain the decomposition

$$\tilde{C}_t = \int_{(\sigma_0, \sigma_t]} dC_u = \int 1_{\{0 \lessdot l_u \leqq t\}} dC_u = I_t + \tilde{J}_t \tag{7.12}$$

for the process of (7.4), where

$$\begin{aligned} I_t &\triangleq \int 1_{\{0 \lessdot l_u \leqq t\}} 1_{\{l_u = u\}} dC_u \\ &= \int_{(\sigma_0, \sigma_t]} [h_x(r+u, x+W_u) + f'(r+u)] 1_{\{l_u = u\}} du + \Delta I_{\tau - r} 1_{\{0 \lessdot \tau - r \leqq t\}} \\ &= \int_0^t [h_x(r+u, x+W_u) + f'(r+u)] 1_{\{x + W_u \geqq s(r+u)\}} du + \Delta I_{\tau - r} 1_{\{0 \lessdot \tau - r \leqq t\}}, \end{aligned} \tag{7.13}$$

$$\Delta I_{\tau - r} \triangleq [g'(x + W_{\tau - r}) - f(\tau)] 1_{\{l_{\tau - r} = \tau - r\}}, \tag{7.14}$$

and

$$\tilde{J}_t \triangleq \int 1_{\{0<l_u\leq t\}} 1_{\{l_u<u\}}\, dC_u = \sum_{\substack{g\in\mathscr{E}_\pi^{\rightarrow}\\ 0<g\leq t}} [C_{\sigma_g}-C_g]. \tag{7.15}$$

The process I of (7.13) is predictable, absolutely continuous on $[0,\tau-r)$, and has a possible jump size $\Delta I_{\tau-r}$ at $t=\tau-r$. Its absolutely continuous part

$$I_t^{ac} \triangleq \int_0^t [h_x(r+u,x+W_u)+f'(r+u)]1_{\{x+W_u\geq s(r+u)\}}\, du$$

can be written equivalently as

$$I_t^{ac} \triangleq \int_0^t [h_x(r+u,x+W_u)+f'(r+u)]1_{\{x+W_u>\check{s}(r+u)\}}\, du \tag{7.16}$$

with

$$\check{s}(r) \triangleq \overline{\lim_{\delta\downarrow 0}}\, s(r+\delta), \quad 0\leq r<\tau, \tag{7.17}$$

because the lower-semicontinuous function $s(\cdot)$ can have at most countably many points of discontinuity (e.g. [13], p. 492).

On the other hand, the Monotone Convergence Theorem allows us to approximate the process $\tilde{J}$ of (7.15) by

$$\tilde{J}_t^{(\varepsilon)} \triangleq \sum_{n=1}^{N_\varepsilon} 1_{\{L_n^\varepsilon\leq t\}}[C_{R_n^\varepsilon}-C_{S_n^\varepsilon}]. \tag{7.18}$$

Indeed, we have

$$\begin{aligned}\tilde{J}_t &= \lim_{\varepsilon\downarrow 0}\int 1_{\{\varepsilon+l_u<u\}}1_{\{0<l_u\leq t\}}\, dC_u\\ &= \lim_{\varepsilon\downarrow 0}\sum_{n\geq 1} 1_{\{L_n^\varepsilon\leq t\}}[C_{\sigma_{S_n^\varepsilon}}-C_{S_n^\varepsilon}]\\ &= \lim_{\varepsilon\downarrow 0}\tilde{J}_t^{(\varepsilon)}, \quad \text{almost surely}\end{aligned}$$

and $\lim_{\varepsilon\downarrow 0} E|\tilde{J}_t^{(\varepsilon)}-\tilde{J}_t|=0$, for every fixed $t\in[0,\tau-r]$.

LEMMA 7.3 *The dual predictable projection of* $\tilde{J}^{(\varepsilon)}$ *in (7.18) is the same as that of the* nonincreasing, *pure jump process*

$$J_t^{(\varepsilon)} \triangleq \sum_{n=1}^{N_\varepsilon} 1_{\{L_n^\varepsilon\leq t\}}[H(r+S_n^\varepsilon,x+W_{S_n^\varepsilon})-v(r+S_n^\varepsilon,x+W_{S_n^\varepsilon})]. \tag{7.19}$$

Proof The process $\tilde{J}^{(\varepsilon)}$ of (7.18) has the same dual predictable projection, as the process

$$\sum_n 1_{\{L_n^\varepsilon \leq t\}} E[C_{R_n^\varepsilon} - C_{S_n^\varepsilon} | \mathscr{F}_{S_n^\varepsilon}]$$

$$= \sum_n 1_{\{L_n^\varepsilon \leq t\}} [H(r+S_n^\varepsilon, x+W_{S_n^\varepsilon}) - E\{H(r+R_n^\varepsilon, x+W_{R_n^\varepsilon}) | \mathscr{F}_{S_n^\varepsilon}\}]$$

$$= \sum_n 1_{\{L_n^\varepsilon \leq t\}} [H(r+S_n^\varepsilon, x+W_{S_n^\varepsilon}) - E\{v(r+R_n^\varepsilon, x+W_{R_n^\varepsilon}) | \mathscr{F}_{S_n^\varepsilon}\}] = J_t^{(\varepsilon)},$$

because of the strong Markov property and the fact that v is harmonic in the continuation region $\mathscr{C}$ of (3.21); cf. [16], Lemma 5. □

PROPOSITION 7.4 *The dual predictable projection J of the process $\tilde{J}$ in (7.15) is a.s. nonincreasing and flat away from the set $Z(\omega) \triangleq \{0 \leq u < \tau - r; x + W_u(\omega) \leq \check{s}(r+u)\}$, for a.e. $\omega \in \Omega$.*

Proof It is seen from (7.15) that $\tilde{J}$ itself is flat off the set $\mathscr{E}^{\rightarrow}(\omega)$, and thus, by Lemma 7.5 below, also flat off the set $Z(\omega)$. But then this is also true for its dual predictable projection J, because Z is predictable. On the other hand, for every $0 \leq s < t < \tau - r$ and $A \in \mathscr{F}_s$, we have

$$E[1_A(J_{\tau-r} - J_s)] = E[1_A(\tilde{J}_{\tau-r} - \tilde{J}_s)] = \lim_{\varepsilon \downarrow 0} E[1_A(\tilde{J}_{\tau-r}^{(\varepsilon)} - \tilde{J}_s^{(\varepsilon)})]$$

$$= \lim_{\varepsilon \downarrow 0} E[1_A(J_{\tau-r}^{(\varepsilon)} - J_s^{(\varepsilon)})]$$

$$\leq \lim_{\varepsilon \downarrow 0} E[1_A(J_{\tau-r}^{(\varepsilon)} - J_t^{(\varepsilon)})] = E[1_A(J_{\tau-r} - J_t)].$$

This shows that the process $\{M_t \triangleq E[J_{\tau-r} | \mathscr{F}_t] - J_t;\ 0 \leq t \leq \tau - r\}$ is a submartingale; we may assume that this process has RCLL paths, because the function $t \mapsto EJ_t = E\tilde{J}_t$ is right-continuous; cf. [12], p. 16. From the fact that J is a predictable process of bounded variation, and the uniqueness part of the Doob–Meyer decomposition, we conclude that J is nonincreasing. □

LEMMA 7.5 *For a.e. $\omega \in \Omega$, the sets $\mathscr{E}^{\rightarrow}(\omega)$ and $\{0 \leq u < \tau - r;\ x + W_u(\omega) > \check{s}(r+u)\}$ are disjoint.*

Proof Take any $g \in \mathscr{E}^{\rightarrow}(\omega)$. Thanks to the continuity of $W(\omega)$, and the fact that $g + \delta$ is in $\mathscr{H}^c(\omega)$ for all $\delta > 0$ sufficiently small, we obtain

$$x + W_g(\omega) = \lim_{\delta \downarrow 0} (x + W_{g+\delta}(\omega)) \leq \overline{\lim_{\delta \downarrow 0}}\, s(r+g+\delta) = \check{s}(r+g),$$

according to the notation of (7.17). □

We can now conclude the

Proof of Theorem 7.1 Taking dual predictable projections in (7.12), we obtain $D = I + J$. It develops that $I = D + (-J)$ is nondecreasing, because so are both D and $-J$. This means, in particular, that the random variable $\Delta I_{\tau-r}$ of (7.14) is a.s.

nonnegative and this, in conjunction with the assumption (3.4), yields $\Delta I_{\tau-r}=0$, a.s. Consequently $I^{ac}=D+(-J)$ is nondecreasing, and $-J$ is absolutely continuous with respect to I^{ac}. But from (7.16), I^{ac} is flat away from $\{0\leqq u<\tau-r;\ x+W_u>\check{s}(r+u)\}$, and J is flat away from Z, and these two sets are disjoint. It follows that J is evanescent and thus D is given by (7.6), as claimed. □

COROLLARY 7.6 *For every* $(r,x)\in[0,\tau]\times\mathscr{R}$, *the process*

$$M_t^{(r,x)}\triangleq u(r+t,x+W_t)+\int_0^t h_x(r+\theta,x+W_\theta)1_{\{x+W_\theta<s(r+\theta)\}}\,d\theta$$

$$-\int_0^t f'(r+\theta)1_{\{x+W_\theta\geqq s(r+\theta)\}}\,d\theta;\quad 0\leqq t\leqq\tau-r \tag{7.20}$$

is a martingale. In particular, (3.24) *holds.*

Proof From (7.8), (3.13) and (7.6) it develops that

$$v(r+t,x+W_t)+\int_0^t [h_x(r+\theta,x+W_\theta)+f'(r+\theta)]1_{\{x+W_\theta>s(r+\theta)\}}\,d\theta;\quad 0\leqq t\leqq\tau-r$$

is a martingale. Now recall the definitions (3.10) and (3.8); the latter implies that $G(r+t,x+W_t)+\int_0^t h_x(r+\theta,x+W_\theta)\,d\theta$ is a martingale. □

Remark 7.7 In the region $\{(r,x)\in[0,\tau)\times\mathscr{R};\ x\geqq s(r)\}$ we have $v=H$; now H is continuous on $[0,\tau]\times\mathscr{R}$ and of class $C^{1,2}$ on $[0,\tau)\times\mathscr{R}$. In particular, $v(r,\cdot)$ is differentiable from the right at the point $x=s(r)$:

$$v_x(r,s(r)+)=H_x(r,s(r));\qquad \forall\, r\in[0,\tau).$$

On the other hand, v is harmonic in the continuation region $\mathscr{C}$ of (3.24); suppose that the left-derivative $v_x(r,s(r)-)$ also exists at every $r\in[0,\tau)$. Then from the fact that $v(r,x)$ is the potential of an *absolutely continuous* process $D^{(r,x)}$, we conclude that the jump

$$v_x(r,s(r)+)-v_x(r,s(r)-)$$

cannot but be equal to zero; for otherwise it would charge the Brownian local time, giving rise to a singular component in $D^{(r,x)}$. The conclusion

$$v_x(r,x)\text{ is continuous at } x=s(r) \tag{7.21}$$

is the celebrated *principle of "smooth fit"* in optimal stopping.

An equivalent derivation of the principle of smooth fit (inspired by [9]) runs as follows: write the representation (3.23) in the form

$$v(r,x)=\int_0^{\tau-r}\int_{s(r+t)}^{\infty}[h_x(r+t,\xi)+f'(r+t)]p(t;x,\xi)\,d\xi \tag{7.22}$$

with

$$p(t;x,\xi)=(2\pi t)^{-1/2}\exp[-(x-\xi)^2/2t],$$

and assume that h is actually of class $C^{0,2}$ with second derivative $h_{xx}(r,\cdot)$ satisfying a polynomial growth condition of the type (3.1). Differentiating in (7.22) one obtains.

$$v_x(r,x)=\int_0^{\tau-r}[h_x(r+t,s(r+t))+f'(r+t)]p(t;x,s(r+t))\,dt$$

$$+\int_0^{\tau-r}\int_{s(r+t)}^{\infty}h_{xx}(r+t,\xi)p(t;x,\xi)\,d\xi\,dt,$$

an expression which is continuous in x; in particular, (7.21) holds.

8. APPENDIX 2

We shall establish here the identity (5.17). Let us start by noticing that the function $Q(r,x,\cdot)$ is decreasing and dominates $V(r,x)$; therefore,

$$\lim_{y\to\infty} Q(r,x,y)\geqq V(r,x). \tag{8.1}$$

Take now an arbitrary $\eta\in\mathscr{A}(\tau)$ with $J(\eta;0,x)<\infty$, and create $\eta(y)\in\mathscr{A}(\tau,y)$ as in (4.6), with $T(y)=\inf\{0\leqq t<\tau;\ \eta_t\geqq y\}\wedge\tau$. We obtain

$$Q(0,x,y)-J(\eta;0,x)\leqq J(\eta(y);0,x)-J(\eta;0,x)$$

$$=E\Bigg[\int_{T(y)}^{\tau}\{h(t,x-y+W_t)-h(t,X_t)\}\,dt-\int_{(T(y),\tau)}f(t)\,d\eta_t$$

$$-(\eta_{T(y)+}-y)f(T(y))1_{\{T(y)<\tau\}}+\{g(x-y+W_\tau)-g(X_\tau)\}1_{\{T(y)<\tau\}}\Bigg]$$

$$\leqq I_1(y)+I_2(y),$$

where $X_t=x+W_t-\eta_t$ and

$$I_1(y)\triangleq E\int_0^{\tau}\{h(t,x-y+W_t)-h(t,X_t)\}1_{(T(y),\tau)}(t)\,dt$$

$$I_2(y)\triangleq E[\{g(x-y+W_\tau)-g(X_\tau)\}1_{\{T(y)<\tau\}}].$$

But on $\{T(y)<t<\tau\}$ we have $X_t\leqq x-y+W_t\leqq x+W_t$, and the nonnegativity and convexity of $h(t,\cdot)$ imply

$$h(t, x-y+W_t) \leqq h(t, x+W_t) + h(t, X_t).$$

Consequently,

$$|I_1(y)| \leqq E\int_0^\tau \{h(t, x+W_t) + 2h(t, X_t)\} 1_{(T(y), \tau)}(t)\, dt$$

$$\leqq P(0, x) + 2 \cdot J(\eta; 0, x) < \infty.$$

But $T(y) \uparrow \tau$ as $y \to \infty$, a.s. P, and the dominated convergence theorem gives $\lim_{y\to\infty} I_1(y) = 0$. Similarly, $\lim_{y\to\infty} I_2(y) = 0$, and so for every $\eta \in \mathscr{A}(\tau)$ with $J(\eta; 0, x) < \infty$:

$$\lim_{y\to\infty} Q(0, x, y) \leqq J(\eta; 0, x).$$

But now we can take the infimum over such η, to obtain

$$\lim_{y\to\infty} Q(0, x, y) \leqq V(0, x). \tag{8.2}$$

The relation (5.17) follows from (8.1) and (8.2), at least for $r=0$; the general case is similar.

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