ON DYNAMIC MEASURES OF RISK *

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Abstract

In the context of complete financial markets, we study *dynamic measures* for the *risk* associated with a given liability C at time $t = T$, of the form

$$\rho(x; C) := \sup_{\nu \in \mathcal{D}} \inf_{\pi(\cdot) \in \mathcal{A}(x)} \mathbf{E}_\nu \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ .$$

Here x is the initial capital available at time $t = 0$, $\mathcal{A}(x)$ the class of admissible portfolio strategies, $S_0(\cdot)$ the price of the risk-free instrument in the market, $\mathcal{P} = \{\mathbf{P}_\nu\}_{\nu \in \mathcal{D}}$ a suitable family of probability measures, and $[0, T]$ the temporal horizon during which all economic activity takes place. The classes $\mathcal{A}(x)$ and $\mathcal{D}$ are general enough to incorporate margin requirements, and uncertainty about the actual values of stock-appreciation rates, respectively. For this latter purpose we discuss, in addition to the above "max-min" approach, a related measure of risk in a "Bayesian" framework.

Risk-measures of this type were introduced by Artzner, Delbaen, Eber and Heath in a static setting, and were shown to possess certain desirable "coherence" properties.

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1 Introduction

It is well-known that in a financial market which is free of arbitrage opportunities and complete, any liability $C \geq 0$ can be hedged perfectly on a finite time-horizon $[0, T]$: starting with a large enough initial capital $x > 0$, and trading skillfully in the market, an agent can find portfolio rules that will allow his wealth $X^{x,\pi}(\cdot)$ to hedge the liability *without risk* at time $t = T$, that is

$$X^{x,\pi}(T) \geq C \quad \text{a.s.,} \qquad \text{for some portfolio } \pi(\cdot), \tag{1.1}$$

while maintaining solvency throughout $[0, T]$. The smallest amount of $x > 0$ that makes (1.1) possible, is given by the expected discounted value

$$C(0) := \mathbf{E}\left[\frac{C}{S_0(T)}\right] > 0 \tag{1.2}$$

of the liability under the (unique, risk-neutral) equivalent martingale measure $\mathbf{P}$, where $S_0(\cdot)$ is the price of the risk-free instrument. In fact, with $x = C(0)$ in (1.1), the corresponding "optimal hedging portfolio" $\pi(\cdot) \equiv \pi_C(\cdot)$ achieves exact replication of the liability: $X^{C(0),\pi_C}(T) = C$, a.s.

We discuss in this paper the predicament of an agent who is unable to comit at time $t = 0$ the entire amount $C(0)$ necessary for such perfect hedging. Then the liability C represents *genuine risk* for the agent, and the question is how to quantify this risk. Various ways for doing this have been proposed, and we refer to the excellent recent papers of Artzner, Delbaen, Eber & Heath (1996) and Föllmer & Leukert (1998) for discussion and overview. Motivated by the paper of Artzner et al. (1996), we propose measuring risk by the quantity

$$\inf_{\pi(\cdot)} \mathbf{E}_0 \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ \tag{1.3}$$

(such a measure of risk is also discussed briefly by Föllmer & Leukert (1998), section 2.4; see also Edirisinghe, Naik & Uppal (1993)). This is the *smallest expected discounted net-loss* that can be achieved by trading in the market; the expectation in (1.3) is under the original, "real-world" probability measure $\mathbf{P}_0$. Of course, the expression (1.3) vanishes for $x \geq C(0)$, as is clear from (1.1); and it becomes an interesting problem in *stochastic control,* to compute the quantity of (1.3) as well as the portfolio that attains the infimum, for any $x < C(0)$.

Suppose now that, in addition to the genuine risk that the liability C represents, the agent also faces some *uncertainty* regarding the model for the financial market itself. We capture such uncertainty by allowing for a *family* $\mathcal{P} = \{\mathbf{P}_\nu\}_{\nu \in \mathcal{D}}$ *of "real-world probability measures"*, equivalent to the risk-neutral measure $\mathbf{P}$, instead of just one (i.e., $\mathbf{P}_0$). For instance, each such measure may correspond to a different specification of the various stock-appreciation-rates. Thus, the "max-min" quantity

$$\underline{V}(x) := \sup_{\nu \in \mathcal{D}} \inf_{\pi(\cdot)} \mathbf{E}_\nu \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ \tag{1.4}$$

represents the maximal risk of the type (1.3) that the agent can encounter, when faced with the "worst possible scenario" $\nu \in \mathcal{D}$. Motivated again by Atzner et al. (1996), we propose (1.4) as a reasonable measure of risk in this situation. It was shown by these authors, in a static setting (that is, with $x = 0$, $\pi \equiv 0, S_0(\cdot) \equiv 1$) and on a finite probability space, that such measures of risk satisfy certain reasonable and desirable *coherence properties*, and are indeed characterized by them. In the special case $\mathbf{P} = \mathbf{P}_{\hat{\nu}}$ for some $\hat{\nu} \in \mathcal{D}$ (i.e., when the risk-neutral measure is included in the set of possible "real-world" measures), we show that the quantity $\underline{V}(x)$ is equal to

$$\overline{V}(x) := \inf_{\pi(\cdot)} \sup_{\nu \in \mathcal{D}} \mathbf{E}_\nu \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+, \tag{1.5}$$

the upper (min-max) value of a fictitious "stochastic game between the market and the agent". The saddle-point of the game is then shown to be the pair $(\hat{\pi}(\cdot), \hat{\nu})$, where $\hat{\pi}(\cdot)$ corresponds to the investment strategy that borrows the amount $C(0) - x$ from the bank at time zero, and then invests in the stock according to the "optimal hedging portfolio" $\pi_C(\cdot)$ for C.

We present in Section 2 the details of the model for the financial market, and for the dynamics of the agent's wealth $X^{x,\pi}(\cdot)$; this latter is flexible enough to allow for interesting *margin requirements*. Section 3 presents the general solution of the stochastic control problem (1.3), and several examples that allow explicit computation are treated in Section 4. With such computations in place, it is then straightforward to determine the smallest amount of initial capital that keeps the exposure to risk below a given, acceptable level. We discuss in Section 5 the stochastic game associated with (1.4) and (1.5). Finally, we present in Section 6 an alternative *Bayesian formulation* to the problem of measuring risk as least-expected-discounted-net-loss, in the presence of uncertainty about stock-appreciation-rates, along with examples for which computations are possible.

2 The Market Model

We shall work throughout this paper within the context of a financial market $\mathcal{M}$ that consists of one *bank account* (risk-free instrument) and several *stocks* (risky instruments). The respective prices $S_0(\cdot)$ and $S_1(\cdot), \ldots, S_d(\cdot)$ of these financial instruments evolve according to the equations

$$\begin{aligned} dS_0(t) &= S_0(t)r(t)dt \;, \quad S_0(0) = 1 \\ dS_i(t) &= S_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_0^j(t)\right] \;, \quad S_i(0) = s_i > 0 \;;\; i = 1, \ldots, d. \end{aligned} \tag{2.1}$$

Here $W_0(\cdot) = (W_0^1(\cdot), \ldots, W_0^d(\cdot))'$ is a standard $d-$dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P_0})$, endowed with a filtration $\mathbf{F} = \{\mathcal{F}(t)\}_{0 \leq t \leq T}$; this filtration is the $\mathbf{P_0}$-augmentation of

$$\mathcal{F}^{W_0}(t) := \sigma(W_0(s);\ 0 \leq s \leq t) \;, \quad 0 \leq t \leq T,$$

the filtration generated by the Brownian motion $W_0(\cdot)$. The *coefficients* $r(\cdot)$ (interest-rate), $b(\cdot) = (b_1(\cdot), \ldots, b_d(\cdot))'$ (vector of stock-appreciation-rates) and $\sigma(\cdot) = \{\sigma_{ij}(\cdot)\}_{1 \leq i,j \leq d}$ (matrix of stock-volatilities) of the model $\mathcal{M}$, are all assumed to be progressively measurable with respect to $\mathbf{F}$. Furthermore, the matrix $\sigma(\cdot)$ is assumed to be invertible, and all processes $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$, $\sigma^{-1}(\cdot)$ are assumed to be bounded, uniformly in $(t, \omega) \in [0, T] \times \Omega$. In the special case of deterministic coefficients $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$, the filtration $\mathbf{F}$ coincides with the augmentation of $\mathcal{F}^S(t) = \sigma(S(u);\ 0 \leq u \leq t)$, $0 \leq t \leq T$, the filtration generated by the vector $S(\cdot) = (S_1(\cdot), \ldots, S_d(\cdot))'$ of price-processes.

Thanks to these assumptions, the "relative risk" process

$$\theta_0(t) := \sigma^{-1}(t)[b(t) - r(t)\tilde{\mathbf{1}}] \;, \quad 0 \leq t \leq T \tag{2.2}$$

where $\tilde{\mathbf{1}} = (1, \ldots, 1)' \in \mathbb{R}^d$, is itself bounded and $\mathbf{F}-$progressively measurable; thus

$$Z_0(t) := \exp\left[-\int_0^t \theta_0'(s)dW_0(s) - \frac{1}{2}\int_0^t ||\theta_0(s)||^2 ds\right], \; 0 \leq t \leq T \tag{2.3}$$

is a $\mathbf{P}_0-$martingale, and

$$\mathbf{P}(\Lambda) := \mathbf{E}_0[Z_0(T)1_\Lambda], \quad \Lambda \in \mathcal{F} \tag{2.4}$$

is a probability measure equivalent to $\mathbf{P_0}$. Under this so-called *risk-neutral equivalent martingale measure* $\mathbf{P}$, the discounted stock prices $\frac{S_1(\cdot)}{S_0(\cdot)}, \ldots, \frac{S_d(\cdot)}{S_0(\cdot)}$ become martingales, and the process

$$W(t) \;:=\; W_0(t) + \int_0^t \theta_0(s)ds \;, \quad 0 \leq t \leq T \tag{2.5}$$

becomes Brownian motion, by the Girsanov theorem. This is the standard setup of a *complete* financial market model $\mathcal{M}$; see, for example, Karatzas (1996).

In the context of the above market-model $\mathcal{M}$, consider an agent who starts out with initial capital x and can decide, at each time $t \in [0,T]$, which amount $\pi_i(t)$ to invest in each of the stocks $i = 1, \ldots, d$ without affecting their prices. With $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))'$ chosen, the agent places the amount $X(t) - \sum_{i=1}^d \pi_i(t)$ in the bank account, at time t; here $X(\cdot) \equiv X^{x,\pi}(\cdot)$ denotes his wealth process, which is thus seen to satisfy the equation

$$\begin{aligned} dX(t) &= \left[X(t) - \sum_{i=1}^{d} \pi_i(t)\right] r(t)dt + \sum_{i=1}^{d} \pi_i(t) \left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_0^j(t)\right] \\ &= r(t)X(t)dt + \pi'(t)\sigma(t)dW(t) \ ; \quad X(0) = x \ , \end{aligned}$$

or equivalently

$$d\left(\frac{X(t)}{S_0(t)}\right) = \frac{\pi'(t)}{S_0(t)}\sigma(t)dW(t) \ ; \quad X(0) = x. \tag{2.6}$$

Let us formalize these considerations as follows.

Definition 2.1 (i) A *portfolio process* $\pi : [0,T] \times \Omega \to \mathbb{R}^d$ is $\mathbf{F}-$progressively measurable and satisfies $\int_0^T ||\pi(t)||^2 dt < \infty$, a.s..

(ii) For a given portfolio process $\pi(\cdot)$, the process $X(\cdot) \equiv X^{x,\pi}(\cdot)$ defined by (2.6) is called the *wealth process* corresponding to portfolio $\pi(\cdot)$ and initial capital x.

(iii) Given a random variable $A \in \mathbf{L}^{1+\varepsilon}(\Omega, \mathcal{F}(T), \mathbf{P})$ for some $\varepsilon > 0$, a portfolio process $\pi(\cdot)$ is called *admissible for the initial capital* x, and we write $\pi(\cdot) \in \mathcal{A}(x)$, if

$$X^{x,\pi}(t) \geq S_0(t) \cdot \mathbf{E}\left[\frac{A}{S_0(T)} \,\middle|\, \mathcal{F}(t)\right] =: A(t), \qquad 0 \leq t \leq T \tag{2.7}$$

holds almost surely. Here $\mathbf{E}$ denotes expectation with respect to the probability measure $\mathbf{P}$ of (2.4).

□

Remark 2.1 From standard results on complete financial markets (e.g. Karatzas (1996), Chapter 1), the quantity

$$A(0) := \mathbf{E}\left[\frac{A}{S_0(T)}\right] \leq x \tag{2.8}$$

in (2.7) is the "Black-Scholes price" of the contingent claim A at time $t = 0$: the smallest value of the initial capital z, for which there exists a tame portfolio $\pi(\cdot)$ with $X^{z,\pi}(T) \geq A$, a.s. Similarly, $A(t)$ can be interpreted as the "price" of A at time t, for any given $t \in [0,T]$. The bound of (2.7) has the interpretation of a *margin requirement*: the value $X^{x,\pi}(\cdot)$ of the portfolio $\pi(\cdot)$ is never allowed to fall below the value $A(\cdot) \equiv X^{A(0),\pi_A}(\cdot)$ of the optimal hedging portfolio $\pi_A(\cdot)$ for the contingent claim A, with

$$\frac{A(t)}{S_0(t)} = \mathbf{E}\left[\frac{A}{S_0(T)}\middle|\mathcal{F}(t)\right] = A(0) + \int_0^t \frac{\pi_A'(u)}{S_0(u)}\sigma(u)dW(u) \ , \ 0 \le t \le T. \tag{2.9}$$

Remark 2.2 From (2.6), (2.9) it is clear that $\frac{X^{x,\pi}(\cdot)-A(\cdot)}{S_0(\cdot)}$ is a $\mathbf{P}-$local martingale; and from (2.7) we see that this process is nonnegative, thus also a $\mathbf{P}-$supermartingale. Since $\frac{A(\cdot)}{S_0(\cdot)}$ is clearly a $\mathbf{P}-$martingale from (2.9), we conclude that $\frac{X^{x,\pi}(\cdot)}{S_0(\cdot)}$ is a $\mathbf{P}-$supermartingale, and thus

$$\mathbf{E}\left[\frac{X^{x,\pi}(T)}{S_0(T)}\right] \le x, \quad \forall\ \pi(\cdot)\ \in\ \mathcal{A}(x)\ . \tag{2.10}$$

Definition 2.2 If the process $\frac{X^{x,\pi}(\cdot)}{S_0(\cdot)}$ is not just a $\mathbf{P}-$supermartingale but also a $\mathbf{P}-$martingale (in other words, if (2.10) holds as equality), then we say that the portfolio $\pi(\cdot) \in \mathcal{A}(x)$ is *martingale-generating.*

□

Let us suppose now that, at time $t = T$, the agent faces total liabilities (net of targeted profits) described by a contingent claim C: a random variable in $\mathbf{L}^{1+\varepsilon}(\Omega, \mathcal{F}(T), \mathbf{P},)$ for some $\varepsilon > 0$, with

$$\mathbf{P}[C \ge A] = 1 \quad \text{and} \quad \mathbf{P}[C > A] > 0. \tag{2.11}$$

Starting with a given, fixed initial capital $x \ge A(0)$, and subject to the margin requirement of (2.7), the agent then tries to "cover his liability" at $t = T$ as well as he can. Of course, with initial capital $x \ge A(0)$ sufficiently large, the liability can be covered perfectly, *without risk* (see (2.14) below). Indeed, if we introduce the Black-Scholes price

$$C(0) := \mathbf{E}\left[\frac{C}{S_0(T)}\right] \tag{2.12}$$

$$C(t) := S_0(t)\mathbf{E}\left[\frac{C}{S_0(T)}\middle|\mathcal{F}(t)\right] = C(0) + \int_0^t \frac{\pi_C'(u)}{S_0(u)}\sigma(u)dW(u) \ , \ 0 \le t \le T \tag{2.13}$$

of C then, by analogy with Remark 2.1:

$$X^{x,\pi}(T) \ge C \ \text{ a.s. for some } \ \pi(\cdot) \in \mathcal{A}(x), \ \forall x \ge C(0). \tag{2.14}$$

In fact, the a.s. inequality of (2.14) holds as equality, if we take $x = C(0)$ and $\pi(\cdot) \equiv \pi_C(\cdot)$, the optimal hedging portfolio of the contingent claim C in (2.13).

Achieving a "hedge without risk" (i.e., the inequality of (2.14) with probability one) is no longer possible if $A(0) \leq x < C(0)$. In this case, we shall adopt the value function of the *stochastic control problem*

$$V_0(x) \equiv V_0(x; C) := \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_0 \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ \tag{2.15}$$

(least expected discounted net loss, over all admissible portfolios) as a reasonable - and in the terminology of Artzner et al. (1996), *coherent - measure of risk.*

We shall present the general solution of this problem in the next section, and then work out explicit computations for concrete contingent claims A, C in section 4. In section 5 we shall look at a whole family $\mathcal{P} = \{\mathbf{P}_\nu\}_{\nu\in\mathcal{D}}$ of possible "real-world probability measures", equivalent to $\mathbf{P}$, rather than at only one measure $\mathbf{P_0}$, and replace (2.15) by the following *stochastic game*

$$\rho(x; C) := \sup_{\nu\in\mathcal{D}} \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_\nu \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ . \tag{2.16}$$

This quantity is the *supremum of least expected discounted losses, under all possible probability measures* (or "scenarios") in $\mathcal{D}$. In the static hedging of $x = 0$, $\mathcal{A}(0)$ consisting of only $\pi(\cdot) \equiv \tilde{\mathbf{0}}$, $r(\cdot) \equiv 0$, and with a finite probability space, Artzner et al. (1996) characterized $\rho(C) = \rho(0; C) := \sup_{\nu\in\mathcal{D}} \mathbf{E}_\nu(C^+)$ as the only possible measure of risk with certain desirable *coherence properties.* Our motivation for studying the problem of (2.16) came from the paper of Artzner et al. (1996); see also Dembo (1997) for a related study of measures of risk based on scenarios.

In the present context, and for simplicity with $r(\cdot) \equiv 0$, $A = 0$ and $\mathcal{A}(x)$ replaced by the class of martingale-generating portfolios, the measure of risk in (2.16) satisfies the following properties (related to those of Artzner et al. (1996)):

$$\begin{aligned}
&(i) \quad \rho(x; C) \leq \|(C - x)^+\|_\infty = ess \sup_{\omega\in\Omega} (C(\omega) - x)^+ \\
&(ii) \quad \rho(x_1 + x_2; C_1 + C_2) \leq \rho(x_1; C_1) + \rho(x_2; C_2) \\
&(iii) \quad \rho(\lambda x; \lambda C) = \lambda \rho(x; C), \ \text{ for } \lambda \geq 0 \\
&(iv) \quad \left\{ \begin{array}{ll} x \mapsto \rho(x; C) & \text{is convex decreasing and} \\ x \mapsto x + \rho(x; C) & \text{is convex increasing, for fixed } C. \end{array} \right\}
\end{aligned} \tag{2.17}$$

Remark 2.3 Property (i) states that $\rho(x;C)$ cannot exceed the maximal possible net loss. The subadditivity property (ii) guarantees that an agent with initial capital $x = x_1 + x_2$, faced with a liability $C = C_1 + C_2$, is not motivated to set up two different accounts with initial holdings x_1, x_2 and with the hedging of the liabilities C_1, C_2 as their respective goals. According to Artzner et al. (1996), properties (ii) and (iii) cease to be appropriate when the size of the position C is so large as to influence risk directly (by making liquidation time depend on size). Property (iv) says that, as the initial capital x increases, both the risk $\rho(x;C)$ and the "exposure-to-risk-ratio" $\frac{\rho(x;C)}{x+\rho(x;C)}$, decrease.

Remark 2.4 A particularly interesting margin requirement, when one tries to hedge a contingent claim C at time $t = T$, is to have to satisfy the a.s. lower bound

$$X^{x,\pi}(t) \geq C(t) - kS_0(t) \quad \text{for all} \quad 0 \leq t \leq T \tag{2.18}$$

and some given, fixed $k > 0$. In other words, the value of the hedging portfolio $\pi(\cdot)$ is never allowed to fall below the current price $C(\cdot)$ of the contingent claim (as in (2.13)), by more than a fixed multiple of the price of the risk-free instrument. The requirement (2.18) can be cast in the form (2.7) by taking

$$A = C - kS_0(T). \tag{2.19}$$

3 One Probability Measure

Our focus in this section will be *the stochastic control problem of* (2.15). If $x \geq C(0)$, the property (2.14) shows that $V_0(x) = 0$; thus, we shall concentrate on initial capital x with $A(0) \leq x < C(0)$.

We shall employ the familiar tools of *convex duality*: starting with the convex function $R(z) = z^+$, consider its (random, $\mathcal{F}(T)-$measurable) Legendre-Fenchel transform

$$\tilde{R}(\zeta, \omega) := \min_{z \leq C(\omega) - A(\omega)} [z^+ - \zeta z] = \left\{ \begin{array}{ccc} (1-\zeta)[C(\omega) - A(\omega)] & ; & \zeta > 1 \\ 0 & ; & 0 < \zeta \leq 1 \end{array} \right\}. \tag{3.1}$$

Here the minimum is attained by any random variable of the form

$$I(\zeta, \omega) := \left\{ \begin{array}{ccc} C(\omega) - A(\omega) & ; & \zeta > 1 \\ 0 & ; & 0 < \zeta < 1 \\ U(\omega) & ; & \zeta = 1 \end{array} \right\} \tag{3.2}$$

where U is $\mathcal{F}(T)-$measurable and satisfies $0 \leq U \leq C - A$, a.s.

It follows from (3.1) that, for any initial capital $x \in [A(0), C(0))$ and any $\pi(\cdot) \in \mathcal{A}(x)$, $\zeta > 0$ we have

$$(C - X^{x,\pi}(T))^+ \geq \tilde{R}(\zeta Z_0(T)) + \zeta Z_0(T)(C - X^{x,\pi}(T)), \ a.s. \tag{3.3}$$

Thus, in conjunction with (2.4), (2.10) and (3.1) we obtain

$$\begin{aligned} \mathbf{E}_0 \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ &\geq \mathbf{E}_0 \left[\frac{\tilde{R}(\zeta Z_0(T))}{S_0(T)} \right] + \zeta \mathbf{E} \left[\frac{C - X^{x,\pi}(T)}{S_0(T)} \right] \\ &\geq \mathbf{E}_0 \left[\frac{\tilde{R}(\zeta Z_0(T))}{S_0(T)} \right] + \zeta (C(0) - x) \\ &= G_0(\zeta) + \zeta[(C(0) - x) - H_0(\zeta)] =: F_0(\zeta), \end{aligned} \tag{3.4}$$

where we have set

$$G_0(\zeta) := \mathbf{E}_0 \left[\frac{C - A}{S_0(T)} \ 1_{\{\zeta Z_0(T) \geq 1\}} \right] \ , \ 0 < \zeta \leq \infty \tag{3.5}$$

$$H_0(\zeta) := \mathbf{E} \left[\frac{C - A}{S_0(T)} \ 1_{\{\zeta Z_0(T) \geq 1\}} \right] \ , \ 0 < \zeta \leq \infty. \tag{3.6}$$

Both these functions are right-continuous and increasing, with $G_0(0+) = H_0(0+) = 0$ and

$$G_0(\infty) = \mathbf{E}_0 \left[\frac{C - A}{S_0(T)} \right] \ , \quad H_0(\infty) = \mathbf{E} \left[\frac{C - A}{S_0(T)} \right] = C(0) - A(0), \tag{3.7}$$

$$\zeta H_0(\zeta) - G_0(\zeta) = \int_0^\zeta H_0(u) du \ , \ \ 0 \leq \zeta < \infty.$$

In particular, the function $F_0(\cdot)$ of (3.4) is concave, and is given by

$$F_0(\zeta) = \zeta(C(0)-x) - \int_0^\zeta H_0(u)du\ , \quad 0<\zeta<\infty.$$

Remark 3.1 The inequalities of (3.4) hold, in fact, as equalities for some $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ and $\hat{\zeta} > 0$, *if and only if* we have

$$\mathbf{E}\left[\frac{X^{x,\hat{\pi}}(T)}{S_0(T)}\right] = x \tag{3.8}$$

(i.e., $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ is martingale-generating),

$$\inf\{\zeta>0\ /\ H_0(\zeta) \geq C(0)-x\} \leq \hat{\zeta} \leq \inf\{\zeta>0\ /\ H_0(\zeta) > C(0)-x\} \tag{3.9}$$

and

$$C - X^{x,\hat{\pi}}(T) = (C-A)1_{\{\hat{\zeta}Z_0(T)>1\}} + U1_{\{\hat{\zeta}Z_0(T)=1\}}\ ,\ a.s. \tag{3.10}$$

for some $\mathcal{F}(T)-$measurable random variable U that satisfies $0 \leq U \leq C-A$, a.s. In this case, $\hat{\pi}(\cdot)$ is optimal, since the lower bound of (3.4) is attained. Notice also that $F_0(\cdot)$ attains its maximum at the point $\hat{\zeta}$ of (3.9), and we have $A \leq X^{x,\hat{\pi}}(T) \leq C$, a.s.

Proposition 3.1 *For every $x \in [A(0), C(0))$ and $\hat{\zeta} \in (0,\infty]$ as in (3.9), there exists a random variable U with $0 \leq U \leq C-A$ such that*

$$\hat{X}(T) := C1_{\{\hat{\zeta}Z_0(T)\leq 1\}} + A1_{\{\hat{\zeta}Z_0(T)>1\}} - U1_{\{\hat{\zeta}Z_0(T)=1\}} \tag{3.11}$$

satisfies

$$\mathbf{E}\left[\frac{\hat{X}(T)}{S_0(T)}\right] = x. \tag{3.12}$$

Proof: From (3.6), (2.12) and $H_0(\hat{\zeta}) \geq C(0)-x$, we see that

$$\mathbf{E}\left[\frac{1}{S_0(T)}\left(C1_{\{\hat{\zeta}Z_0(T)<1\}} + A1_{\{\hat{\zeta}Z_0(T)\geq 1\}}\right)\right] = \mathbf{E}\left[\frac{C}{S_0(T)}\right] - H_0(\hat{\zeta}) \leq x.$$

It is then clear that (3.12) holds for the random variable $\hat{X}(T)$ of (3.11), if we show that

$$x \leq \mathbf{E}\left[\frac{1}{S_0(T)}\left(C1_{\{\hat{\zeta}Z_0(T)\leq 1\}} + A1_{\{\hat{\zeta}Z_0(T)>1\}}\right)\right]. \tag{3.13}$$

Since the function $F_0(\cdot)$ attains its maximum at $\hat{\zeta}$, for any $0<\varepsilon<\hat{\zeta}$ we have

$$\begin{aligned} 0 \quad \leq \quad \frac{F_0(\hat{\zeta}) - F_0(\hat{\zeta}-\varepsilon)}{\varepsilon} &= -x + \mathbf{E}\left[\frac{C}{S_0(T)}\right] - \mathbf{E}\left[\frac{C-A}{S_0(T)}1_{\{Z_0(T)\geq\frac{1}{\hat{\zeta}-\varepsilon}\}}\right] \\ &+ \frac{1}{\varepsilon}\mathbf{E}_0\left[\frac{C-A}{S_0(T)}(1-\hat{\zeta}Z_0(T))1_{\{\frac{1}{\hat{\zeta}}\leq Z_0(T)<\frac{1}{\hat{\zeta}-\varepsilon}\}}\right]. \end{aligned}$$

This last term is non-positive, and we obtain (3.13) by omitting it and letting $\varepsilon \downarrow 0$. □

Theorem 3.1 *For any given $x \in [A(0), C(0))$, $\hat{\zeta} \in (0, \infty]$ as in* (3.9) *and U as in Proposition 3.1, there exists a portfolio process $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ for which* (3.8) *and* (3.10) *hold, and which is optimal for the problem of* (2.15)*:*

$$V_0(x) = \mathbf{E}_0 \left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right)^+ = G_0(\hat{\zeta}). \tag{3.14}$$

Proof: Consider the random variable $\hat{X}(T)$ of (3.11) and the $\mathbf{P}-$martingale

$$\begin{aligned} \frac{\hat{X}(t)}{S_0(t)} &:= \mathbf{E}\left[\frac{\hat{X}(T)}{S_0(T)} \middle| \mathcal{F}(t) \right] \\ &= x + \int_0^t \frac{\hat{\pi}'(u)}{S_0(u)} \sigma(u) dW(u) \ , \ 0 \le t \le T \end{aligned} \tag{3.15}$$

in its representation as a stochastic integral with respect to $W(\cdot)$, for a suitable portfolio process $\hat{\pi}(\cdot)$ (see Karatzas (1996), Exercise 3.6, p.9). The process $\hat{X}(\cdot)$ defined by (3.15) clearly satisfies $\hat{X}(0) = x$, $\hat{X}(\cdot) \equiv X^{x,\hat{\pi}}(\cdot)$, as well as (3.10) and (3.8), by Proposition 3.1. Optimality of the portfolio process $\hat{\pi}(\cdot)$ is now a consequence of Remark 3.1.

□

According to (3.15), the optimal portfolio $\hat{\pi}(\cdot)$ of (3.14) coincides with the hedging portfolio for the contingent claim $\hat{X}(T)$ of (3.11); in the special case $A = 0$ and $\mathbf{P}_0[\hat{\zeta} Z_0(T) = 1] = 0$, this latter is just C "knocked out" on the event $\{\hat{\zeta} Z_0(T) > 1\}$. Note that for $x = A(0)$, the conditions (3.8), (3.9) are satisfied by $\hat{\zeta} = \infty$, $X^{x,\hat{\pi}}(T) = A$, and the optimal portfolio $\hat{\pi}(\cdot)$ of Theorem 3.1 coincides with $\hat{\pi}_A(\cdot)$, the hedging portfolio for the contingent claim A in (2.9).

Proposition 3.2 *Suppose that $Z_0(T) > 0$ is a (non-random) constant, namely $Z_0(T) = 1$ and $\theta_0(\cdot) \equiv 0$. Then we have*

$$V_0(x) = \mathbf{E}_0 \left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right)^+ = C(0) - x, \quad \textit{for} \quad A(0) \le x < C(0), \tag{3.16}$$

where $\hat{\pi}(\cdot)$ is any martingale-generating portfolio in $\mathcal{A}(x)$ that satisfies

$$A \le X^{x,\hat{\pi}}(T) \le C, \ a.s. \tag{3.17}$$

Proof: In this context, the functions of (3.6), (3.5) become $H_0(\zeta) = G_0(\zeta) = 0$ for $0 < \zeta < \frac{1}{Z_0(T)} = 1$, and $H_0(\zeta) = C(0) - A(0) = Z_0(T) G_0(\zeta) = G_0(\zeta)$ for $\zeta \ge \frac{1}{Z_0(T)} = 1$. Thus $\hat{\zeta} = 1$ in (3.9), and writing (3.4) with $\zeta = 1$, we obtain

$$\mathbf{E}_0 \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ \ge F_0(1) = C(0) - x \ , \quad \forall \pi(\cdot) \in \mathcal{A}(x). \tag{3.18}$$

But for any martingale-generating portfolio $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ that satisfies (3.17), we have

$$\mathbf{E}_0 \left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right)^+ = \mathbf{E} \left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right) = C(0) - x,$$

and (3.16) follows from this in conjunction with (3.18). □

Note that one possible choice for the strategy $\hat{\pi}(\cdot)$ attaining the minimal risk $C(0) - x$ in the case of constant $Z_0(T)$, is to borrow the amount $C(0) - x$ from the bank at $t = 0$, and then use the hedging portfolio $\pi_C(\cdot)$ for C in the stock. This strategy $\hat{\pi}(\cdot)$ results in final wealth $X^{x,\hat{\pi}}(T) = (x - C(0))S_0(T) + C$, but is admissible if and only if this latter quantity dominates A almost surely; indeed, $(x - C(0))S_0(T) + C \geq A$ leads to $X^{x,\hat{\pi}}(t) \equiv (x - C(0))S_0(t) + C(t) \geq A(t)$, $\forall\ 0 \leq t \leq T$ a.s., in conjunction with (2.9), (2.13).

Example 3.1 *Maximizing the Probability of Perfect Hedge.* Consider the case $A = 0$, $C = S_0(T)$. We have

$$H_0(\zeta) = \mathbf{P}[\zeta Z_0(T) \geq 1], \quad G_0(\zeta) = \mathbf{P_0}[\zeta Z_0(T) \geq 1]$$

and

$$V_0(x) := \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_0 \left(1 - \frac{X^{x,\pi}(T)}{S_0(T)} \right)^+ = G_0(\hat{\zeta}), \ 0 < x < 1 \tag{3.19}$$

where $\hat{\zeta}$ is the smallest positive number that satisfies $H_0(\hat{\zeta}) = 1 - x$. The optimal portfolio $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ and wealth $\hat{X}(\cdot) \equiv X^{x,\hat{\pi}}(\cdot)$ processes for this problem, are given by

$$\frac{\hat{X}(t)}{S_0(t)} = \mathbf{P}[\hat{\zeta} Z_0(T) < 1 | \mathcal{F}(t)] = x + \int_0^t \frac{\hat{\pi}'(u)}{S_0(u)} \sigma(u) dW(u), \quad 0 \leq t \leq T \tag{3.20}$$

and are also optimal for the problem

$$\sup_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{P}_0[X^{x,\pi}(T) \geq C] \tag{3.21}$$

of maximizing the probability of a perfect hedge (Kulldorff (1993), Heath (1993); see also Föllmer & Leukert (1998) for a more general study of the problem). This is easily seen if one applies the duality method of the present section to the function $z \mapsto 1_{[1,\infty)}(z)$, instead of the function $z \mapsto z^+$. However, the two problems do not share the same optimal policies in general, for non-constant discounted value $\frac{C}{S_0(T)}$ of the contingent claim; see Spivak (1998).

4 Examples: Constant coefficients, one stock

Consider now the case $d = 1$, with constant $\sigma(\cdot) \equiv \sigma > 0$, $r(\cdot) \equiv r \in \mathbb{R}$ in the model of (2.1) so that $\theta_0(t) = (b(t) - r)/\sigma$. Suppose also that $A = 0$ in (2.7) and that the contingent claim C, in the problem of (2.15), is given as a function

$$C = g(S(T)) \tag{4.1}$$

of the stock-price at $t = T$, for some $g : (0, \infty) \to [0, \infty)$ which is continuous and piecewise continuously differentiable. In this case the stock-price satisfies

$$S(T) = S(t)e^{\sigma(W(T)-W(t)+\mu(T-t))},\ 0 \le t \le T \ \text{ and } \ S(0) = s > 0, \tag{4.2}$$

where

$$\mu := r/\sigma - \sigma/2. \tag{4.3}$$

Moreover, the functions of (3.6), (3.5) take the form

$$H_0(\zeta) = e^{-rT}\mathbf{E}\left[g\left(se^{\sigma(W(T)+\mu T)}\right) \cdot 1_{\{\int_0^T \theta_0(s)dW(s) \le \log\zeta + \frac{1}{2}\int_0^T \theta_0^2(s)ds\}}\right], \tag{4.4}$$

$$G_0(\zeta) = e^{-rT}\mathbf{E}_0\left[g\left(se^{\sigma(W_0(T)+\mu T+\int_0^T \theta_0(s)ds}\right) \cdot 1_{\{\int_0^T \theta_0(s)dW_0(s) \le \log\zeta - \frac{1}{2}\int_0^T \theta_0^2(s)ds\}}\right]. \tag{4.5}$$

Case A: Constant $\theta_0 = (b - r)/\sigma > 0$. In this case an easy computation shows

$$H_0(\zeta) = Q_+\left(\frac{\log\zeta + \frac{1}{2}\theta_0^2 T}{\theta_0\sqrt{T}}, T, s\ ;\ \mu\right)$$

$$G_0(\zeta) = Q_+\left(\frac{\log\zeta - \frac{1}{2}\theta_0^2 T}{\theta_0\sqrt{T}}, T, s\ ;\ \theta_0 + \mu\right),$$

where we have set

$$Q_+(u, \tau, s\ ;\ \mu) := e^{-r\tau}\int_{-\infty}^{u} g(se^{\sigma(z\sqrt{\tau}+\mu\tau)})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz. \tag{4.6}$$

Clearly, the function $u \mapsto Q_+(u; \mu) \equiv Q_+(u, T, s; \mu)$ is absolutely continuous and increasing, with $Q_+(-\infty; \mu) = 0$ and $Q_+(\infty; \mu) = C(0)$. We denote its left-continuous inverse by $Q_+^{-1}(\cdot; \mu)$ and obtain

$$\frac{\log\mathcal{Y}_0(x) + \frac{1}{2}\theta_0^2 T}{\theta_0\sqrt{T}} = Q_+^{-1}(C(0) - x\ ;\ \mu) \tag{4.7}$$

as well as

$$\begin{aligned} V_0(x) &= G_0(\mathcal{Y}_0(x)) = Q_+\left(\frac{\log\mathcal{Y}_0(x) - \frac{1}{2}\theta_0^2 T}{\theta_0\sqrt{T}}\ ;\ \theta_0 + \mu\right) \\ &= Q_+\left(Q_+^{-1}(C(0) - x; \mu) - \theta_0\sqrt{T}; \theta_0 + \mu\right), \end{aligned} \tag{4.8}$$

where we have set $\mathcal{Y}_0(x) = \inf\{\zeta > 0 / H_0(\zeta) \geq C(0) - x\}$ for the quantity of (3.9).

Case B: Constant $\theta_0 = (b-r)/\sigma < 0$. The above results remain true in this case too, if we replace Q_+ by

$$Q_-(u,\tau,s;\mu) := e^{-r\tau}\int_u^\infty g(se^{\sigma(z\sqrt{\tau}+\mu\tau)})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz = Q_+(\infty,\tau,s;\mu) - Q_+(u,\tau,s;\mu); \quad (4.9)$$

in particular,

$$V_0(x) = G_0(\mathcal{Y}_0(x)) = Q_-\left(Q_-^{-1}(C(0)-x),\mu) - \theta_0\sqrt{T}\ ;\ \theta_0+\mu\right). \quad (4.10)$$

Remark 4.1 Note that as we let $\theta_0 \downarrow 0$ in (4.8), or $\theta_0 \uparrow 0$ in (4.10), we recover the value function

$$V_0(x) = Q_\pm\left(Q_\pm^{-1}(C(0)-x\ ;\ \mu);\mu\right) = C(0) - x\ ,\quad A(0) \leq x < C(0)$$

of (3.16) for the case $\theta_0(\cdot) \equiv 0$.

□

Now let us try to *compute the optimal portfolio* $\hat{\pi}(\cdot)$ of (3.14), as well as its associated wealth-process $\hat{X}(\cdot) \equiv X^{x,\hat{\pi}}(\cdot)$. We use as our starting point the expression of (3.15), now in the form

$$\begin{aligned}\hat{X}(t) &= e^{-r(T-t)}\mathbf{E}\left[g\left(S(t)e^{(\sigma(W(T)-W(t)+\mu(T-t))}\right)\ 1_{\{\theta_0(W(T)-W(t))\leq \log \mathcal{Y}_0(x)-\theta_0 W(t)+\frac{T}{2}\theta_0^2\}}\middle|\mathcal{F}(t)\right]\\ &= \begin{cases} Q_-\left(U_+(t), T-t, S(t);\mu\right) & ,\quad \text{if}\ \ \theta_0 > 0 \\ Q_+\left(U_-(t), T-t, S(t);\mu\right) & ,\quad \text{if}\ \ \theta_0 < 0\end{cases}\end{aligned} \quad (4.11)$$

for $0 \leq t \leq T$. Here, the process

$$U_\pm(t) := \frac{\sqrt{T}Q_\pm^{-1}(C(0)-x;\mu) - W(t)}{\sqrt{T-t}} \quad (4.12)$$

satisfies the linear stochastic differential equation

$$dU(t) = \frac{U(t)}{2(T-t)}dt - \frac{dW(t)}{\sqrt{T-t}},\quad U_\pm(0) = Q_\pm^{-1}(C(0)-x;\mu). \quad (4.13)$$

Now we can apply Itô's rule to the $\mathbf{P}-$martingale $e^{-rt}\hat{X}(t)$, and obtain

$$d(e^{-rt}\hat{X}(t)) = \sigma e^{-rt}\left(s\frac{\partial Q_\mp}{\partial s} - \frac{1}{\sigma\sqrt{T-t}}\frac{\partial Q_\mp}{\partial u}\right)(U_\pm(t), T-t, S(t);\mu)\cdot dW(t) \quad (4.14)$$

since the bounded-variation term vanishes. A comparison of this expression with $d(e^{-rt}\hat{X}(t)) = \sigma e^{-rt}\hat{\pi}(t)dW(t)$ of (3.15) yields

$$\hat{\pi}(t) = \left(s\frac{\partial Q_\mp}{\partial s} - \frac{1}{\sigma\sqrt{T-t}}\frac{\partial Q_\mp}{\partial u} \right)(U_\pm(t), T-t, S(t); \mu). \tag{4.15}$$

But

$$\begin{aligned} \frac{\partial Q_\pm}{\partial u} &= \pm\, e^{-r\tau} g(se^{\sigma(u\sqrt{\tau}+\mu\tau)})\frac{e^{-u^2/2}}{\sqrt{2\pi}} \\ s\frac{\partial Q_+}{\partial s} &= e^{-r\tau}\int_{-\infty}^{u} f(se^{\sigma(z\sqrt{\tau}+\mu\tau)})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz \\ s\frac{\partial Q_-}{\partial s} &= e^{-r\tau}\int_{u}^{\infty} f(se^{\sigma(z\sqrt{\tau}+\mu\tau)})\frac{e^{z^2/2}}{\sqrt{2\pi}}dz \end{aligned}$$

where $f(s) := sg'(s)$. Finally, substituting into (4.15) we obtain the optimal portfolio in the form

$$\hat{\pi}(t) = \begin{Bmatrix} \Pi_+(U_+(t), T-t, S(t)) & , & \text{if } \theta_0 > 0 \\ \Pi_-(U_-(t), T-t, S(t)) & , & \text{if } \theta_0 < 0 \end{Bmatrix} \tag{4.16}$$

for $0 \le t < T$, where

$$\Pi_+(u,\tau,s) := e^{-r\tau}\left[\int_u^\infty f(se^{\sigma(z\sqrt{\tau}+\mu\tau)})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz + \frac{e^{-u^2/2}}{\sigma\sqrt{2\pi\tau}}g(se^{\sigma(u\sqrt{\tau}+\mu\tau)})\right], \tag{4.17}$$

$$\Pi_-(u,\tau,s) := e^{-r\tau}\left[\int_{-\infty}^u f(se^{\sigma(z\sqrt{\tau}+\mu\tau)})\frac{e^{-z^2/2}}{\sqrt{2\pi}}dz - \frac{e^{-u^2/2}}{\sigma\sqrt{2\pi\tau}}g(se^{\sigma(u\sqrt{\tau}+\mu\tau)})\right]. \tag{4.18}$$

Remark 4.2 (i) In this one-dimensional, constant-coefficient framework, the expressions (4.16) and (4.11) for the optimal portfolio and its associated wealth-process *do not depend on the appreciation rate b of the stock, except through the sign of $b-r$.* This feature is attractive from a practical point of view, given the difficulties in estimating stock appreciation rates.

(ii) The above analysis also gives that the functions $Q_\pm(u,\tau,s) \equiv Q_\pm(u,\tau,s;\mu)$ of (4.6), (4.9) satisfy the linear parabolic PDE

$$\frac{\partial Q}{\partial \tau} = \frac{1}{2}\left[\sigma^2 s^2\frac{\partial^2 Q}{\partial s^2} + \frac{1}{\tau}\frac{\partial^2 Q}{\partial u^2} - \frac{2\sigma s}{\sqrt{\tau}}\frac{\partial^2 Q}{\partial u \partial s}\right] + rs\frac{\partial Q}{\partial s} + \frac{u}{2\tau}\frac{\partial Q}{\partial u} - rQ\ . \tag{4.19}$$

Example 4.2 *Constant claim $C \equiv c > 0$, $A = 0$.* In this case, we have the straightforward computations $Q_\pm(u,\tau) = ce^{-r\tau}\Phi(\pm u)$, $Q_\pm^{-1}(\zeta,\tau) = \pm\Phi^{-1}(\frac{\zeta e^{r\tau}}{c})$, $C(0) = ce^{-rT}$ and

$$U_\pm(t) = -\frac{W(t) \pm \sqrt{T}\Phi^{-1}\left(\frac{x}{c}e^{rT}\right)}{\sqrt{T-t}}.$$

In particular,

$$H_0(\zeta) = ce^{-rT}\Phi\left(\frac{\log\zeta + \frac{1}{2}\theta_0^2 T}{|\theta_0|\sqrt{T}}\right)$$

if $\theta_0 \neq 0$. Thus, (4.8) and (4.10) yield

$$V_0(x) = ce^{-rT}\left[1 - \Phi\left(\Phi^{-1}\left(\frac{xe^{rT}}{c}\right) + |\theta_0|\sqrt{T}\right)\right] \tag{4.20}$$

for $0 < x < ce^{-rT}$. Moreover, from (4.11), (4.16) it is not hard to verify the expressions

$$\hat{X}(t) = Q_{\mp}(U_{\pm}(t), T-t) = ce^{-r(T-t)}\,\Phi\left(\frac{\pm W(t) + \sqrt{T}\Phi^{-1}\left(\frac{xe^{rT}}{c}\right)}{\sqrt{T-t}}\right), \tag{4.21}$$

$$\hat{\pi}(t) = \pm\frac{ce^{-r(T-t)}}{\sigma\sqrt{T-t}}\varphi(U_{\pm}(t)) = \pm\frac{ce^{-r(T-t)}}{\sigma\sqrt{T-t}}\left(\varphi\circ\Phi^{-1}\right)\left(\frac{\hat{X}(t)}{c}e^{r(T-t)}\right) \tag{4.22}$$

for the optimal wealth and portfolio processes, respectively, on $0 \le t < T$. We have denoted by $\pm$ the sign of $\theta_0 = (b-r)/\sigma$, as well as

$$\Phi(u) = \int_{-\infty}^{u}\varphi(z)dz\ ,\quad \varphi(u) = \frac{e^{-u^2/2}}{\sqrt{2\pi}}\ . \tag{4.23}$$

Consider the case $r = 0$, $c = 1$; then $1 - V_0(x)$, $\hat{\pi}(\cdot)$, $\hat{X}(\cdot)$ coincide with the value function, optimal portfolio and corresponding wealth process, respectively, for the problem (3.21) of maximizing the probability $\mathbf{P}_0[X^{x,\pi}(T) \ge 1]$ of perfect hedge, for $0 < x < 1$. These were obtained by Kulldorff (1993) and Heath (1993).

Example 4.3 *European call-option* $C = (S(T) - q)^+$ for some $q > 0$, $A = 0$. In this case $g(s) = (s-q)^+$, $f(s) = s1_{(q,\infty)}(s)$ in the formulae of (4.6), (4.9) and (4.17), (4.18). With the notation

$$\alpha(\lambda) := \frac{1}{\sigma\sqrt{\tau}}\log\left(\frac{s}{q}\right) + \lambda\sqrt{\tau}$$

these now become

$$\begin{aligned} Q_-(u,\tau,s;\mu) &= e^{-r\tau}\int_{u\vee(-\alpha(\mu))}^{\infty}(se^{\sigma(z\sqrt{\tau}+\mu\tau)} - q)\varphi(z)dz \\ &= s\Phi\left((\sigma\sqrt{\tau} - u)\wedge\alpha(\mu+\sigma)\right) - qe^{-r\tau}\Phi\left((-u)\wedge\alpha(\mu)\right), \end{aligned}$$

$$\begin{aligned} Q_+(u,\tau,s;\mu) &= Q_-(-\infty,\tau,s;\mu) - Q_-(u,\tau,s;\mu) \\ &= s\left(\Phi\left(\alpha(\mu+\sigma)\right) - \Phi\left((\sigma\sqrt{\tau} - u)\right)\right)^+ - qe^{-r\tau}\left(\Phi\left(\alpha(\mu)\right) - \Phi(-u)\right)^+ \end{aligned}$$

as well as

$$\Pi_+(u,\tau,s) = s\ \Phi\left((\sigma\sqrt{\tau} - u)\wedge\alpha(\mu+\sigma)\right) + \frac{\varphi(u)}{\sigma\sqrt{\tau}}\left(se^{\sigma(u\sqrt{\tau} - \frac{\sigma}{2}\tau)} - qe^{-r\tau}\right)^+$$

$$\Pi_-(u,\tau,s) = s\left(\Phi\left(\alpha(\mu+\sigma)\right) - \Phi(\sigma\sqrt{\tau}-u)\right)^+ - \frac{\varphi(u)}{\sigma\sqrt{\tau}}\left(se^{\sigma(u\sqrt{\tau}-\frac{\sigma}{2}\tau)} - qe^{-r\tau}\right)^+ .$$

Example 4.4 *Margin Requirement, Arbitrary Contingent Claim.* Consider now an arbitrary contingent claim C, and take $A := C - kS_0(T)$ as in (2.19). Thus, for $C(0) - k < x < C(0)$, we are trying to minimize the expected discounted loss

$$\mathbf{E}[e^{-rT}(C - X^{x,\pi}(T))^+]$$

over portfolio processes $\pi(\cdot)$ that satisfy the "margin requirement"

$$X^{x,\pi}(t) \geq A(t) = C(t) - kS_0(t), \quad \forall\ 0 \leq t \leq T \tag{4.24}$$

almost surely. However, with $y := x + k - C(0) \in (0,k)$, $c := ke^{rT}$, $p(\cdot) := \pi(\cdot) - \pi_C(\cdot)$ and

$$Y^{y,p}(\cdot) := X^{x,\pi}(\cdot) + kS_0(\cdot) - C(\cdot),$$

this is the same as *minimizing*

$$\mathbf{E}_0[e^{-rT}(c - Y^{y,p}(T))^+]$$

subject to the new margin requirement

$$Y^{y,p}(t) \geq 0\ , \forall\ 0 \leq t \leq T$$

almost surely. Thus we are in the setting of Example 4.2 with $c = ke^{rT}$, which gives

$$\begin{aligned} V_0(x) &= \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_0\left[e^{-rT}(C - X^{x,\pi}(T))^+\right] = k\left[1 - \Phi\left(\Phi^{-1}\left(\frac{y}{k}\right) + |\theta_0|\sqrt{T}\right)\right] \\ &= k\left[\Phi\left(\Phi^{-1}\left(\frac{C(0)-x}{k}\right) - |\theta_0|\sqrt{T}\right)\right] \end{aligned} \tag{4.25}$$

for the value function, and

$$\begin{aligned} \hat{p}(t) &= \pm\frac{ke^{rt}}{\sigma\sqrt{T-t}}\,\varphi\left(\frac{\pm W(t) + \sqrt{T}\Phi^{-1}\left(\frac{y}{k}\right)}{\sqrt{T-t}}\right) \\ &= \pm\frac{ke^{rt}}{\sigma\sqrt{T-t}}\left(\varphi\circ\Phi^{-1}\right)\left(\frac{\hat{Y}(t)}{ke^{rt}}\right) \\ \hat{Y}(t) &= ke^{rt}\cdot\Phi\left(\frac{\pm W(t) + \sqrt{T}\Phi^{-1}\left(\frac{y}{k}\right)}{\sqrt{T-t}}\right) \equiv Y^{y,\hat{p}}(t)\ , \quad 0 \leq t < T \end{aligned} \tag{4.26}$$

for the auxiliary portfolio and wealth processes $\hat{p}(\cdot)$, $\hat{Z}(\cdot)$. Thus, the optimal processes for our original problem of (4.25) can be expressed as

$$\begin{aligned} \hat{\pi}(t) = \pi_C(t) + \hat{p}(t) &= \pi_C(t) \pm \frac{ke^{rt}}{\sigma\sqrt{T-t}}\,\varphi\left(\frac{\sqrt{T}\Phi^{-1}\left(\frac{C(0)-x}{k}\right) \mp W(t)}{\sqrt{T-t}}\right) \\ &= \pi_C(t) \pm \frac{ke^{rt}}{\sigma\sqrt{T-t}}\,\varphi\left(\Phi^{-1}\left(\frac{C(t)-\hat{X}(t)}{ke^{rt}}\right)\right)\,, \end{aligned} \tag{4.27}$$

$$\begin{aligned} \hat{X}(t) &= \hat{Y}(t) + C(t) - ke^{rt} \\ &= C(t) - ke^{rt}\cdot\Phi\left(\frac{\sqrt{T}\Phi^{-1}\left(\frac{C(0)-x}{k}\right) \mp W(t)}{\sqrt{T-t}}\right) \equiv X^{x,\hat{\pi}}(t). \end{aligned} \tag{4.28}$$

For any given $0 < \varepsilon < 1$, the number

$$\bar{x}(\varepsilon) = C(0) - k\cdot\Phi\left(\Phi^{-1}(\varepsilon) + |\theta_0|\sqrt{T}\right) \tag{4.29}$$

provides the smallest value of initial capital $x \in (C(0)-k, C(0))$ for which the risk $V_0(x)$ does not exceed εk; and the unique solution $x_* \equiv x_*(\varepsilon)$ of the equation

$$\Phi^{-1}\left(\frac{C(0)-x_*}{k}\right) - \Phi^{-1}\left(\frac{\varepsilon}{1-\varepsilon}\frac{x_*}{k}\right) = |\theta_0|\sqrt{T} \tag{4.30}$$

provides the smallest value of initial capital $x \in (C(0)-k, C(0))$ such that the "exposure-to-risk" ratio $\frac{V_0(x)}{x+V_0(x)}$ does not exceed ε.

5 A family of probability measures

Let us now try to modify appropriately our model $\mathcal{M}$ of (2.1), in order to incorporate some degree of uncertainty about the stock appreciation rates. One possible way to do this, is by means of random but bounded perturbations $\nu_i(\cdot)$ with values in $[-N_i, N_i]$, where $N_i \in [0,\infty)$ is a known maximal possible deviation of the actual appreciation rate for the i^{th} stock from the value $b_i(\cdot)$, $i = 1, \ldots, d$.

More formally, let us denote by $\mathcal{D}$ the space of such random perturbations; it consists of all $\mathbf{F}-$progressively measurable vector processes $\nu(\cdot) = (\nu_1(\cdot), \ldots, \nu_d(\cdot))'$ with values in the rectangle $\times_{i=1}^d [-N_i, N_i]$. For every $\nu(\cdot) \in \mathcal{D}$, we introduce the exponential $\mathbf{P_0}-$martingale

$$L_\nu(t) := \exp\left[-\int_0^t (\sigma^{-1}(s)\nu(s))' dW_0(s) - \frac{1}{2}\int_0^t ||\sigma^{-1}(s)\nu(s)||^2 ds\right];\ 0 \le t \le T \tag{5.1}$$

and the probability measure

$$\mathbf{P}_\nu(\Lambda) := \mathbf{E}_0[L_\nu(T) 1_\Lambda] \tag{5.2}$$

on $\mathcal{F}(T)$, under which the process

$$W_\nu(t) \ := \ W_0(t) - \int_0^t (\sigma^{-1}(s)\nu(s))' ds\ , \quad 0 \le t \le T \tag{5.3}$$

is Brownian motion. This way, under the new probability measure $\mathbf{P}_\nu$, the model $\mathcal{M}$ of (2.1) becomes

$$\begin{aligned} dS_0(t) &= S_0(t) r(t) dt, \quad S_0(0) = 1 \\ dS_i(t) &= S_i(t)\left[(b_i(t) + \nu_i(t))dt + \sum_{j=1}^d \sigma_{ij}(t) dW_\nu^j(t)\right], \\ S_i(0) &= s_i \in (0,\infty)\ ;\quad i = 1, \ldots, d\ . \end{aligned} \tag{5.4}$$

The resulting modified model $\mathcal{M}_\nu$ resembles that of (2.1); now $W_\nu(\cdot)$ plays the role of the driving Brownian motion (under $\mathbf{P}_\nu$), but the stock appreciation rates have been modified and incorporate the *random fluctuations* $\nu_i(\cdot)$, $|\nu_i(\cdot)| \le N_i$ to the original terms $b_i(\cdot)$, for every $i = 1, \ldots, d$. In the context of this new model $\mathcal{M}_\nu$, *we can formulate and solve the analogue*

$$V_\nu(x) := \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_\nu \left(\frac{C - X^{x,\pi}(T)}{S_0(T)}\right)^+ , \quad A(0) \le x < \infty \tag{5.5}$$

of the stochastic control problem (2.15), where $\mathbf{E}_\nu$ denotes expectation with respect to the measure $\mathbf{P}_\nu$ of (5.2). Indeed, consider the analogues

$$G_\nu(\zeta) := \mathbf{E}_\nu\left[\frac{C - A}{S_0(T)}\, 1_{\{\zeta Z_\nu(T) \ge 1\}}\right] , \quad 0 < \zeta \le \infty \tag{5.6}$$

$$H_\nu(\zeta) := \mathbf{E}\left[\frac{C-A}{S_0(T)}\, 1_{\{\zeta Z_\nu(T)\geq 1\}}\right] \ , \quad 0<\zeta\leq\infty \tag{5.7}$$

of the functions in (3.5), (3.6). We have denoted by

$$Z_\nu(t) := \exp\left[-\int_0^t \theta_\nu'(s)dW_\nu(s) - \frac{1}{2}\int_0^t ||\theta_\nu(s)||^2 ds\right], \quad 0\leq t\leq T \tag{5.8}$$

the likelihood ratio $(d\mathbf{P}/d\mathbf{P}_\nu)|_{\mathcal{F}(t)}$, where

$$\theta_\nu(t) := \sigma^{-1}(t)[b(t)+\nu(t)-r(t)\tilde{\mathbf{1}}] \ , \qquad W(t) = W_\nu(t) + \int_0^t \theta_\nu(s)ds. \tag{5.9}$$

Then just as in Theorem 3.1, we have

$$V_\nu(x) = \left\{ \begin{array}{ccc} G_\nu(0)=0 & ; & x\geq C(0) \\ G_\nu(\infty) = \mathbf{E}_\nu\left[\frac{C-A}{S_0(T)}\right] & ; & x = A(0) \\ G_\nu(\mathcal{Y}_\nu(x)) & ; & A(0)<x<C(0) \end{array} \right\}, \tag{5.10}$$

where we have denoted by $\mathcal{Y}_\nu(x)$ the smallest number $\zeta\in(0,\infty)$ that satisfies $H_\nu(\zeta)\geq C(0)-x$.

Remark 5.1 Suppose that

$$|b_i(t)-r(t)|\leq N_i \ , \quad \forall\ 0\leq t\leq T \tag{5.11}$$

holds a.s. for every $i=1,\ldots,d$; then with $\hat{\nu}(\cdot) := r(\cdot)\tilde{\mathbf{1}} - b(\cdot)$, we have

$$V_{\hat{\nu}}(x) = (C(0)-x)^+. \tag{5.12}$$

This is because, under the condition (5.11), the process $\hat{\nu}(\cdot)$ belongs to the space $\mathcal{D}$ and has $\theta_{\hat{\nu}}(\cdot)\equiv 0$, $Z_{\hat{\nu}}(\cdot)\equiv 1$ in (5.9), (5.8); then (5.12) follows from Proposition 3.2.

□

The least expected discounted net loss $V_\nu(x)$ of (5.5), (5.10) provides a measure of the risk involved when an agent tries to hedge the liability C in the market-model $\mathcal{M}_\nu$ of (5.4), starting with initial capital $x\geq A(0)$ and using portfolios $\pi(\cdot)$ that satisfy the margin requirement (2.7). Then the "max-min" quantity

$$\underline{V}(x) := \sup_{\nu(\cdot)\in\mathcal{D}} \ \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_\nu\left(\frac{C-X^{x,\pi}(T)}{S_0(T)}\right)^+ \tag{5.13}$$

is the *maximal risk that can be incurred, over all possible random perturbations* $\nu(\cdot)\in\mathcal{D}$ *of the stock appreciation rates.* It is dominated by its "min-max" counterpart

$$\overline{V}(x) := \inf_{\pi(\cdot)\in\mathcal{A}(x)} \ \sup_{\nu(\cdot)\in\mathcal{D}} \mathbf{E}_\nu\left(\frac{C-X^{x,\pi}(T)}{S_0(T)}\right)^+ , \tag{5.14}$$

the upper-value of a fictitious *stochastic game* between an agent (who tries to choose $\pi(\cdot) \in \mathcal{A}(x)$ so as to minimize his risk) and "the market" (whose "goal" is to perturb the stock appreciation rates to the agent's utmost detriment). A question of immediate interest, is to settle whether the "upper-value" (5.14) and the "lower-value" (5.13) of this game coincide and, if they do, to compute this common value. We shall answer this question only in the relatively straightforward context of (5.11); the general case will be considered elsewhere (see, however, Example 5.3 below).

Theorem 5.1 *Under the assumption* (5.11) *and with the notation* $\hat{\nu}(\cdot) := r(\cdot)\tilde{\mathbf{1}} - b(\cdot) \in \mathcal{D}$, *we have*

$$(C(0) - x)^+ = \inf_{\hat{\pi}(\cdot) \in \mathcal{A}(x)} \mathbf{E}_{\hat{\nu}} \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ \leq \underline{V}(x). \tag{5.15}$$

If, in addition,

$$A \leq (x - C(0))S_0(T) + C =: \hat{C} \tag{5.16}$$

holds almost surely, then

$$\overline{V}(x) = \underline{V}(x) = V_{\hat{\nu}}(x) = (C(0) - x)^+. \tag{5.17}$$

In particular, there exists then a portfolio $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ *such that the pair* $(\hat{\nu}(\cdot), \hat{\pi}(\cdot)) \in \mathcal{D} \times \mathcal{A}(x)$ *is a saddle-point*

$$\mathbf{E}_{\nu} \left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right)^+ \leq (C(0) - x)^+ \leq \mathbf{E}_{\hat{\nu}} \left(\frac{C - X^{x,\pi}(T)}{S_0(T)} \right)^+ ; \ \forall \ \nu(\cdot) \in \mathcal{D}, \ \pi(\cdot) \in \mathcal{A}(x) \tag{5.18}$$

of the stochastic game with value (5.17).

Proof: We have argued the validity of all (5.15), (5.17) and (5.18) for $x \geq C(0)$, so let us concentrate on $A(0) \leq x < C(0)$. The conditions of (5.15) and the second inequality of (5.18) follow from (5.12). Next, if (5.16) holds, then starting with initial capital x, we can find a portfolio $\hat{\pi}(\cdot) \in \mathcal{A}(x)$ that replicates the contingent claim $\hat{C}$, i.e., with $X^{x,\hat{\pi}}(T) = \hat{C}$, a.s. (note that $\hat{C}(0) := \mathbf{E}\left[\frac{\hat{C}}{S_0(T)}\right] = x$). Indeed, as we mentioned in the discussion following the proof of Proposition 3.2, we can borrow the amount $C(0) - x$ from the bank at $t = 0$, and from then on invest in the stock according to the portfolio $\pi_C(\cdot)$. In particular, this gives

$$\left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right)^+ = C(0) - x \ , \ a.s.$$

which leads to the first inequality of (5.18) - in fact, valid as equality. Taking the supremum over $\nu(\cdot) \in \mathcal{D}$ in (5.18), we deduce

$$\sup_{\nu(\cdot) \in \mathcal{D}} \mathbf{E}_{\nu} \left(\frac{C - X^{x,\hat{\pi}}(T)}{S_0(T)} \right)^+ \leq C(0) - x \ ,$$

thus also $\overline{V}(x) \leq C(0) - x$. But we have $C(0) - x \leq \underline{V}(x) \leq \overline{V}(x)$ from (5.12), so (5.17) follows.

□

Example 5.5 Consider the setting of Example 4.2 with $d = 1$, constant r, b, $\sigma > 0$ and $C = c > 0$, $A = 0$. Denote by $\mathcal{D}_c$ the class of *constant perturbations* $\nu(\cdot) \equiv \nu \in [-N, N]$. Then $\theta_\nu = (b - r + \nu)/\sigma$ of (5.9) is also constant and, by analogy with (4.20), the value function of (5.5) is given by

$$V_\nu(x) = ce^{-rT}\left[1 - \Phi\left(\Phi^{-1}\left(\frac{xe^{rT}}{c}\right) + |\theta_\nu|\sqrt{T}\right)\right], \quad |\nu| \leq N \tag{5.19}$$

for $0 < x < ce^{-rT}$. If $|b - r| \leq N$, then $\hat{\nu} := r - b \in [-N, N]$ and we have

$$\overline{V}(x) = \underline{V}(x) = \sup_{|\nu| \leq N} V_\nu(x) = V_{\hat{\nu}}(x) = ce^{-rT} - x$$

from (5.17), as well as from maximizing directly the expression of (5.19) over $\nu \in [-N, N]$. We can do this, however, even if $|b - r| > N$; indeed, the expression of (5.19) is maximized by

$$\hat{\nu} := \left\{ \begin{array}{ccc} N & ; & \text{if } r - b > N \\ -N & ; & \text{if } r - b < -N \\ r - b & ; & \text{if } |r - b| \leq N \end{array} \right\}, \tag{5.20}$$

so that

$$\sup_{|\nu| \leq N} V_\nu(x) = V_{\hat{\nu}}(x) = ce^{-rT}\left[1 - \Phi\left(\Phi^{-1}\left(\frac{xe^{rT}}{c}\right) + \frac{d}{\sigma}\sqrt{T}\right)\right], \tag{5.21}$$

where $d := \mathrm{dist}(r - b, [-N, N])$.

□

Remark 5.2 In the setting of Example 4.4 with $d = 1$, constant r, b, $\sigma > 0$ and $C - A = kS_0(T)$ for some $k > 0$, and considering the class $\mathcal{D}_c$ of constant perturbations, we can show similarly that $\hat{\nu}$ of (5.20) is again the perturbation that maximizes $V_\nu(x)$.

6 A Bayesian measure of risk

In Section 5 we discussed a min-max method for measuring risk in the presence of uncertainty about the appreciation rates of stocks. Another way to incorporate such uncertainty into our model, is to adopt the *Bayesian approach*, which assumes that these rates are unobservable random variables $B_1, \cdots, B_d$ independent of the driving Brownian motion $W_0(\cdot)$, and with some known *prior joint-distribution* μ. As observations about stock-prices keep coming in, the agent has to update this distribution constantly, while at the same time trying to hedge the liability C subject to a margin requirement of the type (2.7).

Such a Bayesian approach necessitates a few changes in the model of Section 2, which we now carry out. In order to help concentrate on the novel aspects of this approach while keeping notation reasonable, we shall take

$$r(\cdot) \equiv 0, \qquad \sigma(\cdot) \equiv I_d \tag{6.1}$$

throughout this section.

Let us start then with a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, which carries an $\mathbb{R}^d-$valued Brownian motion $W(t) = (W_1(t), \cdots, W_d(t))'$, $0 \le t \le T$ as well as an *independent* random vector $B = (B_1, \cdots, B_d)'$ with known distribution $\mu(E) = \mathbf{P}[B \in E]$, $E \in \mathcal{B}(\mathbb{R}^d)$ that satisfies

$$\mu(\{\tilde{\mathbf{0}}\}) < 1, \quad \int_{\mathbb{R}^d} ||b|| \mu(db) < \infty. \tag{6.2}$$

We shall denote by $\mathbf{F} = \{\mathcal{F}(t)\}_{0 \le t \le T}$ the $\mathbf{P}-$augmentation of the filtration

$$\mathcal{F}^W(t) = \sigma(W(s); 0 \le s \le t), \quad 0 \le t \le T$$

generated by $W(\cdot)$, and by $\mathbf{G} = \{\mathcal{G}(t)\}_{0 \le t \le T}$ the $\mathbf{P}-$augmentation of the *enlarged* filtration

$$\mathcal{F}^{B,W}(t) = \sigma(B, W(s); 0 \le s \le t), \quad 0 \le t \le T$$

generated by *both* $W(\cdot)$ *and* B. Then the process

$$M(t) := exp\Big(B'W(t) - ||B||^2 t/2\Big), \quad 0 \le t \le T \tag{6.3}$$

is a $(\mathbf{G}, \mathbf{P})-$martingale, and the measure

$$\mathbf{P}_0(\Lambda) := \mathbf{E}[M(T) \cdot 1_\Lambda] \tag{6.4}$$

is a probability on $\mathcal{G}(T)$. Under this new probability measure $\mathbf{P}_0$, the process

$$W_0(t) := W(t) - Bt, \ \mathcal{G}(t); \quad 0 \le t \le T \tag{6.5}$$

is Brownian motion, *independent of the random vector* B, and $\mathbf{P}_0[B \in E] = \mathbf{P}[B \in E] = \mu(E), \;\; \forall\, E \in \mathcal{B}(\mathbb{R}^d)$.

6.1 The Model: With these ingredients, and with the simplifications of (6.1), the model $\mathcal{M}$ for the financial market takes the new form

$$S_0(\cdot) \equiv 1$$

$$dS_i(t) = S_i(t)[B_i dt + dW_0^i(t)] = S_i(t)dW^i(t), \quad S_i(0) = s_i > 0 \;\; \text{for } i = 1, \cdots, d \qquad (6.6)$$

on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}_0), \mathbf{F}$. Notice that $\mathbf{F}$ coincides with the augmentation of the filtration

$$\mathcal{F}^S(t) = \sigma(S(u); 0 \le u \le t), \quad 0 \le t \le T$$

generated by the vector $S(\cdot) = (S_1(\cdot), \cdots, S_d(\cdot))'$ of stock-prices, since $S_i(t) = s_i \cdot \exp(W^i(t) - t^2/2), \;\; 0 \le t \le T$ from (6.6). Since these prices are directly observable, we call $\mathbf{F}$ the *observations filtration* of the model.

With this interpretation of the filtration $\mathbf{F}$, the rest of the model of Section 2 stays the same as before, starting with the equation (2.6) for the wealth-process $X(\cdot) \equiv X^{x,\pi}(\cdot)$ now in the simpler form

$$dX(t) = \pi'(t)dW(t), \quad X(0) = x. \qquad (6.7)$$

As in Definition 2.1, *the portfolio processes* $\pi(\cdot) \in \mathcal{A}(x)$ *are adapted to the observations-filtration* $\mathbf{F}$ (which contains information about $W(\cdot)$, or equivalently about the stock-prices $S(\cdot)$), *not to the enlarged filtration* $\mathbf{G}$ (which contains information also about the unobservable stock-appreciation-rates $(B_1, \ldots, B_d)'$). Our effort will focus again on computing the least expected net loss

$$V_0(x) \equiv V_0(x; C) := \inf_{\pi(\cdot) \in \mathcal{A}(x)} \mathbf{E}_0 \Big(C - X^{x,\pi}(T) \Big)^+ , \qquad (6.8)$$

as a measure of the risk associated with hedging the liability C using ($\mathbf{F}-$adapted) portfolios $\pi(\cdot) \in \mathcal{A}(x)$.

6.2 Results: In order to translate the results of Section 3 to our new setting, we have to compute the $(\mathbf{F}, \mathbf{P}_0)-$martingale

$$Z_0(t) = \left.\frac{d\mathbf{P}}{d\mathbf{P}_0}\right|_{\mathcal{F}(t)} = \frac{1}{\hat{M}(t)} , \quad \hat{M}(t) := \mathbf{E}\left[\frac{d\mathbf{P}_0}{d\mathbf{P}} \;\Big|\; \mathcal{F}(t) \right] \qquad (6.9)$$

as in (2.3). But the $(\mathbf{F}, \mathbf{P})-$martingale $\hat{M}(\cdot)$ in (6.9) is easy to compute, once we recall the $(\mathbf{G}, \mathbf{P})-$martingale property of $M(\cdot)$ in (6.3) and the independence of B, $W(\cdot)$ under $\mathbf{P}$, namely

$$\begin{aligned}\hat{M}(t) &= \mathbf{E}[M(T)|\mathcal{F}(t)] = \mathbf{E}[\mathbf{E}(M(T)|\mathcal{G}(t))|\mathcal{F}(t)] = \mathbf{E}[M(t)|\mathcal{F}(t)] \\ &= \mathbf{E}\left[e^{B'W(t)-\frac{1}{2}\|B\|^2 t} \,\Big|\, \mathcal{F}(t)\right] \\ &= \left\{\begin{array}{ccc} 1 & ; & t=0 \\ F(t,W(t)) & ; & 0<t\le T \end{array}\right\}. \end{aligned} \tag{6.10}$$

Therefore

$$Z_0(t) = \left\{\begin{array}{ccc} 1 & ; & t=0 \\ \frac{1}{F(t,W(t))} & ; & 0<t\le T \end{array}\right\}, \tag{6.11}$$

where we have set

$$F(s,y) := \int_{\mathbb{R}^d} e^{b'y-\frac{1}{2}\|b\|^2 s}\mu(db)\ ; \quad s>0,\ y\in\mathbb{R}^d. \tag{6.12}$$

The functions of (3.5), (3.6) can thus be written in the form

$$G_0(\zeta) := \mathbf{E}\left[F(T,W(T))(C-A)\ 1_{\{F(T,W(T))\le\zeta\}}\right]\ ,\ 0<\zeta\le\infty \tag{6.13}$$

$$H_0(\zeta) := \mathbf{E}\left[(C-A)\ 1_{\{F(T,W(T))\le\zeta\}}\right]\ ,\ 0<\zeta\le\infty. \tag{6.14}$$

With $x\in[A(0),C(0))$ fixed, the value function of (6.8) is given by $V_0(x) = G_0(\mathcal{Y}_0(x))$ according to Theorem 3.1, where $\mathcal{Y}_0(x)$ is the smallest $\zeta\in[0,\infty)$ that satisfies $H_0(\hat{\zeta})\ge C(0)-x$. Furthermore, there exists then a portfolio $\hat{\pi}(\cdot)\in\mathcal{A}(x)$ that is optimal (i.e., attains the infimum) in (6.8); and the wealth-process $\hat{X}(\cdot)\equiv X^{x,\hat{\pi}}(\cdot)$ corresponding to this optimal portfolio is

$$\begin{aligned}\hat{X}(t) &= x+\int_0^t \hat{\pi}'(s)dW(s) && (6.15)\\ &= \mathbf{E}\left[C\ 1_{\{F(T,W(T))>\mathcal{Y}_0(x)\}} + A\ 1_{\{F(T,W(T))\le\mathcal{Y}_0(x)\}}\,\Big|\,\mathcal{F}(t)\right]\ ,\quad 0\le t\le T. \end{aligned}$$

Example 6.3 Consider $A=0$, and $C=g(S(T))$ for some continuous $g:(0,\infty)\to[0,\infty)$. Then, from (6.6) we have also $C=u(W(T))$ for a suitable continuous function $u:\mathbb{R}^d\to[0,\infty)$, and the expressions of (3.5)-(6.15) become

$$H_0(\zeta) = \int_{\{z\in\mathbb{R}^d;F(T,z)\le\zeta\}} u(z)\varphi_T(z)dz \tag{6.16}$$

$$G_0(\zeta) = \int_{\{z\in\mathbb{R}^d;F(T,z)\le\zeta\}} F(T,z)u(z)\varphi_T(z)dz\ ,\quad 0<\zeta\le\infty \tag{6.17}$$

$$\hat{X}(t) = \left\{\begin{array}{ccc} \mathcal{X}(T-t,W(t)) & ; & 0\le t<T \\ u(W(T))\ 1_{\{F(T,W(T))>\mathcal{Y}_0(x)\}} & ; & t=T \end{array}\right\}. \tag{6.18}$$

We have set

$$\mathcal{X}(s,y) := \int_{\{z\in\mathbb{R}^d; F(T,z)>\mathcal{Y}_0(x)\}} u(z)\varphi_s(y-z)dz \ , \qquad \varphi_s(y) := \frac{e^{-\frac{\|y\|^2}{2s}}}{(2\pi s)^{\frac{d}{2}}} \tag{6.19}$$

for $s>0$, $y\in\mathbb{R}^d$. It is checked easily that the functions F, $\mathcal{X}$ of (6.12), (6.19) satisfy the heat equation $Q_s+\frac{1}{2}\Delta Q=0$ on $(0,\infty)\times\mathbb{R}^d$; and in conjunction with (6.18), this leads to the expression

$$\hat{\pi}(t) = \nabla\mathcal{X}(T-t,W(t)) = \int_{\{z\in\mathbb{R}^d; F(T,z)>\mathcal{Y}_0(x)\}} u(z)\left(\frac{z-y}{s}\right)\varphi_s(y-z)dz\,\Bigg|_{\substack{y=W(t)\\ s=T-t}} \ , \quad 0\le t<T \tag{6.20}$$

for the optimal portfolio $\hat{\pi}(\cdot)$ of (6.15). Here $\nabla\mathcal{X}(s,\cdot)$ denotes the gradient of the function $\mathcal{X}(s,\cdot)$.

Example 6.4 *Maximizing the probability of perfect hedge.* With $A=0$, $C=k>0$ the quantities of (6.16), (6.17) become

$$H_0(\zeta) = k\int_{\{z;F(T,z)\le\zeta\}} \varphi_T(z)dz = k\left[1-\int_{\{z;F(T,z)>\zeta\}} \varphi_T(z)dz\right]$$

$$G_0(\zeta) = k\int_{\{z;F(T,z)\le\zeta\}} F(T,z)\varphi_T(z)dz = k\left[1-\int_{\{z;F(T,z)>\zeta\}} F(T,z)\varphi_T(z)dz\right]$$

since $\int_{\mathbb{R}^d} F(T,z)\varphi_T(z)dz = \mathbf{E}F(T,W(T)) = \mathbf{E}\hat{M}(T) = \hat{M}(0) = 1$. As in Karatzas (1997), it can be shown that, for every $0<x<k$, there exists a unique $\hat{\zeta}=\mathcal{Y}_0(x)>0$ which satisfies

$$\int_{\{z;F(T,z)>\mathcal{Y}_0(x)\}} \varphi_T(z)dz = \frac{x}{k} \ , \ \text{ or equivalently } \ H_0(\mathcal{Y}_0(x)) = k-x. \tag{6.21}$$

The value function

$$\begin{aligned} V_0(x) \ &:= \inf_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{E}_0\left(k-X^{x,\pi}(T)\right)^+ = G_0(\mathcal{Y}_0(x)) \\ &= k\left[1-\int_{\{z;F(T,z)>\mathcal{Y}_0(x)\}} F(T,z)\varphi_T(z)dz\right] \end{aligned} \tag{6.22}$$

is thus related to the maximal probability of perfect hedge

$$\sup_{\pi(\cdot)\in\mathcal{A}(x)} \mathbf{P}_0[X^{x,\pi}(T)\ge k] = 1-\frac{V_0(x)}{k} = \int_{\{z;F(T,z)>\mathcal{Y}_0(x)\}} F(T,z)\varphi_T(z)dz.$$

Furthermore, the optimal portfolio- and wealth-processes

$$\hat{\pi}(t) = \nabla\mathcal{X}(T-t,W(t)) = k\int_{\{z;F(T,z)>\mathcal{Y}_0(x)\}} \left(\frac{z-y}{s}\right)\varphi_s(y-z)dz\,\Bigg|_{\substack{y=W(t)\\ s=T-t}} \ , \quad 0\le t<T$$

$$\hat{X}(t) = X^{x,\hat{\pi}}(t) = \mathcal{X}(T-t, W(t)) = k \int_{\{z; F(T,z) > \mathcal{Y}_0(x)\}} \varphi_s(y-z)dz \,\Big|_{\substack{y=W(t)\\ s=T-t}} \quad , \quad 0 \le t < T$$

are the same for *both problems* (ibid. p. 333, where the case $k = 1$, $d = 1$ is treated in detail).

Example 6.5 *Margin Requirement, Arbitrary Contingent Claim.* With arbitrary contingent claim C, and $A := C - k$ for some $k > 0$ as in (2.19), we fix $x \in (C(0) - k, C(0))$ and denote $\xi := x + k - C(0) \in (0, k)$. Reasoning as in Example 4.4, we conclude that the value of (6.8) is given as

$$V_0(x) = k \int_{\{z; F(T,z) \le \mathcal{Y}_0(\xi)\}} F(T,z)\varphi_T(z)dz$$

by analogy with (6.22), where $\mathcal{Y}_0(\xi) > 0$ is defined as in (6.21), that is, via

$$\int_{\{z; F(T,z) > \mathcal{Y}_0(\xi)\}} \varphi_T(z)dz = \frac{\xi}{k}.$$

Reasoning again as in Example 4.4, we see that the optimal wealth- and portfolio processes take now the form

$$\hat{X}(t) = X^{x,\hat{\pi}}(t) = C(t) - k \int_{\{z; F(T,z) \le \mathcal{Y}_0(\xi)\}} \varphi_s(y-z)dz \,\Big|_{\substack{y=W(t)\\ s=T-t}} \quad , \quad 0 \le t < T$$

$$\hat{\pi}(t) = \pi_C(t) + k \int_{\{z; F(T,z) \le \mathcal{Y}_0(\xi)\}} \left(\frac{y-z}{s}\right) \varphi_s(y-z)dz \,\Big|_{\substack{y=W(t)\\ s=T-t}} \quad , \quad 0 \le t < T$$

respectively.

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