ON COLLISIONS OF BROWNIAN PARTICLES

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ABSTRACT. We examine the behavior of n Brownian particles diffusing on the real line, with bounded, measurable drift and bounded, piecewise continuous diffusion coëfficients that depend on the current configuration of particles. Sufficient conditions are established for the absence of triple collisions, as well as for the presence of (infinitely-many) triple collisions among the particles. As an application to the Atlas model of equity markets, we study a special construction of such systems of diffusing particles using Brownian motions with reflection on polyhedral domains.

1. Introduction

It is well known that, with probability one, the n-dimensional Brownian motion started away from the origin will hit the origin infinitely often for $n = 1$, while it will never hit the origin for $n \geq 2$. This is also true for $n-$dimensional Brownian motion with constant drift and diffusion coëfficients, by Girsanov's theorem and re-orientation of coördinates. The next step of generalization is the case of bounded drift and diffusion coëfficients. The existence of weak solutions for the stochastic equations that describe such processes was discussed by Stroock & Varadhan (18) and by Krylov (12) through the study of appropriate martingale problems.

Now let us suppose that $\mathbb{R}^n$ is partitioned as a finite union of disjoint polyhedra. Bass & Pardoux (3) established the existence and uniqueness of a weak solution to the stochastic integral equation

$$X(t) = x_0 + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW(s)\,, \qquad 0 \leq t < \infty \tag{1.1}$$

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with initial condition $x_0 \in \mathbb{R}^n$, where the measurable functions $\mu : \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are bounded and, moreover, σ is everywhere non-singular and piecewise constant (that is, constant on each polyhedron). The continuous process $\{W(t), 0 \le t < \infty\}$ is $n-$dimensional Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Here uniqueness is understood in the sense of the probability distribution.

Bass and Pardoux also discovered an interesting phenomenon, namely, that the weak solution to (1.1) may satisfy

$$\mathbb{P}_{x_0}\big(X(t) = 0, \quad \text{i.o.}\big) = 1\,; \quad x_0 \in \mathbb{R}^n, \tag{1.2}$$

for a diffusion matrix $\sigma(\cdot)$ with special structure. Here $\mathbb{P}_{x_0}$ is the solution to the martingale problem corresponding to (1.1). In the Bass & Pardoux (3) example the whole space $\mathbb{R}^n$ is partitioned into a finite number of polyhedral domains with common vertex at the origin, carefully chosen *small* apertures, and $\sigma(\cdot)$ constant in each domain. We review this example in Section 2.3.1.

In the present paper we find conditions sufficient for ruling (1.2) out. More specifically, we are interested in the case of a bounded, measurable drift vector $\mu(\cdot)$ and of a bounded, piecewise continuous diffusion matrix of the form

$$\sigma(x) = \sum_{\nu=1}^{m} \sigma_\nu(x) \mathbf{1}_{\mathcal{R}_\nu} \equiv \sigma_{\mathfrak{p}(x)}(x)\,; \quad x \in \mathbb{R}^n\,, \tag{1.3}$$

under the assumption of *well-posedness* (existence and uniqueness of solution) when $n \ge 3$. Here $\mathbf{1}_{\{.\}}$ is the indicator function; the sets $\big\{\mathcal{R}_\nu\big\}_{\nu=1}^{m}$ form a partition of $\mathbb{R}^n$ for some $m \in \mathbb{N}$, namely, $\mathcal{R}_\nu \cap \mathcal{R}_\kappa = \emptyset$ for $\nu \ne \kappa$ and $\cup_{\nu=1}^m \mathcal{R}_\nu = \mathbb{R}^n$; and the mapping $\mathfrak{p}(x) : \mathbb{R}^n \to \{1, \ldots, m\}$ satisfies $x \in \mathcal{R}_{\mathfrak{p}(x)}$ for every $x \in \mathbb{R}^n$. Throughout this paper we shall assume that each $\mathcal{R}_\nu$ is an n-dimensional polyhedron for $\nu = 1, \ldots, m$, and that the $(n \times n)$ matrix-valued functions $\big\{\sigma_\nu(\cdot)\big\}_{\nu=1}^{m}$ are positive-definite everywhere.

We shall also assume throughout that there exists a unique weak solution for the equation (1.1). Existence is guaranteed by the measurability and boundedness of the functions $\mu(\cdot)$ and $\sigma(\cdot)\sigma'(\cdot)$, as well as the uniform strong non-degeneracy of $\sigma(\cdot)\sigma'(\cdot)$ (eg. Krylov (13), Remark 2.1) where the superscript $'$ represents the transposition. Uniqueness holds when $n = 1$ or $n = 2$; for $n \ge 3$, the argument of Chapter 7 of Stroock & Varadhan (18) implies uniqueness if the function $\sigma(\cdot)$ in (1.3) is continuous in $\mathbb{R}^n$ (Theorem 7.2.1 of (18)) or close to constant (Corollary 7.1.6 of (18)), namely, if there exists a constant $(n \times n)$ matrix α and a sufficiently small $\delta > 0$ depending on the dimension n and the bounds of eigenvalues of $\sigma(\cdot)$ such that

$$\sup_x \|\sigma(x)\sigma'(x) - \alpha\| \le \delta$$

where $\|\cdot\|$ is the matrix norm. Bass & Pardoux (3) showed uniqueness for piecewise-constant coëfficients, i.e., $\sigma_\nu(\cdot) \equiv \sigma_\nu$, $\nu = 1, \ldots, m$. For further discussion on uniqueness and non-uniqueness, see the paper by Krylov (13) and the references in it. The structural assumption (1.3) may be weakened to more general bounded cases, under modified conditions.

Our main concern is to obtain sufficient conditions on $\mu(\cdot)$, and on $\sigma(\cdot)$ of the form (1.3), so that with $n \geq 3$ we have

$$\begin{aligned} &\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t) \quad \text{for some } t > 0\big) = 0 \quad \text{or} \\ &\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t) \quad \text{i.o.}\big) = 1\,; \quad x_0 \in \mathbb{R}^n \end{aligned} \tag{1.4}$$

for some $1 \leq i < j < k \leq n$. Put differently, we study conditions on their drift and diffusion coëfficients, under which three Brownian particles moving on the real line can collide at the same time, and conditions under which such "*triple collisions*" never occur. Propositions 1 and 2 provide partial answers to these questions, in Section 2.

Since the properties of (1.4) concern the paths of weak solutions to the stochastic differential equation (1.1), we may construct weak solutions to (1.1) which have different path properties. Proposition 3 prescribes a possible construction in Section 3.

The results have consequences in the computations of the local time for the differences $\{X_i(t) - X_j(t), X_j(t) - X_k(t)\}$. We discuss such local times in Section 4. Proofs of selected results are presented in Section 5.

Recent work related to this problem was done by Cépa & Lépingle (4). These authors consider a system of mutually repelling Brownian particles and show the absence of triple collisions. The electrostatic repulsion they consider comes from drift coëfficients that are unbounded; in our model all drifts are bounded.

2. A First Approach

2.1. **The Setting.** Consider the stochastic integral equation (1.1) with coëfficients $\mu(\cdot)$ and $\sigma(\cdot)$ as in (1.3), and assume that the matrix-valued functions $\sigma_\nu(\cdot)$, $\nu = 1, \ldots, m$ are uniformly positive-definite. Then, the inverse $\sigma^{-1}(\cdot)$ of the diffusion coëfficient $\sigma(\cdot)$ exists in the sense $\sigma^{-1}(\cdot) = \sum_{\nu=1}^m \sigma_\nu^{-1}(\cdot)\mathbf{1}_{\mathcal{R}_\nu}$. As usual, a *weak solution* of this equation consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$; a filtration $\{\mathcal{F}_t\,, 0 \leq t < \infty\}$ of sub-σ-fields of $\mathcal{F}$ which satisfies the *usual conditions* of right-continuity and augmentation by the $\mathbb{P}-$negligible sets in $\mathcal{F}$; and two adapted, n-dimensional processes on this space $X(\cdot)$, $W(\cdot)$ on this space, such that $W(\cdot)$ is Brownian motion and (1.1) is satisfied $\mathbb{P}-$almost surely. The concept of uniqueness associated with this notion of solvability, is that of *uniqueness in distribution* for the process $X(\cdot)$.

2.1.1. *Removal of Drift.* We start by observing that the piece-wise constant drift part has no effect on the probabilities (1.4). In fact, define an $n-$dimensional process $\xi(\cdot)$ by

$$\xi(t) := \sigma^{-1}(X(t))\mu(X(t))\,, \qquad 0 \leq t < \infty\,.$$

By the nature of the functions $\mu(\cdot)$ and $\sigma(\cdot)$ in (1.3), the mapping $t \mapsto \xi(t)$ is right-continuous or left-continuous on each boundary $\partial \mathcal{R}_{\mathfrak{p}(X(t))}$ at time t, deterministically, according to the position $\mathcal{R}_{\mathfrak{p}(X(t-))}$ of $X(t-)$. Then, although the sample path of n-dimensional process $\xi(\cdot)$ is not entirely right-continuous or left-continuous, it *is* progressively measurable. Moreover, $\xi(\cdot)$ is bounded, so the exponential process

$$\eta(t) = \exp\left[\sum_{i=1}^{n} \int_0^t \langle \xi_i(u), d\,W_i(u)\rangle - \frac{1}{2}\int_0^t \|\xi(s)\|^2 d\,u\right]; \quad 0 \le t < \infty$$

is a continuous martingale, where $\|x\|^2 := \sum_{j=1}^n x_j^2$, $x \in \mathbb{R}^n$ stands for n-dimensional Euclidean norm and the bracket $\langle x, y\rangle := \sum_{j=1}^n x_j y_j$ is the inner product of two vectors $x, y \in \mathbb{R}^n$. By Girsanov's theorem

$$\widetilde{W}(t) := W(t) + \int_0^t \sigma^{-1}(X(u))\mu(X(u))d\,u\,,\ \mathcal{F}_t\,; \quad 0 \le t < \infty$$

is an n-dimensional standard Brownian motion under the new probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, that satisfies by $\mathbb{Q}(C) = \mathbb{E}^{\mathbb{P}}\big(\eta(T)1_C\big)$ for $C \in \mathcal{F}_T$, $0 \le T < \infty$.

Thus, it suffices to consider the case of $\mu(\cdot) \equiv 0$ in (1.1), namely

$$X(t) \,=\, x_0 + \int_0^t \sigma\big(X(s)\big)\, dW(s)\,, \qquad 0 \le t < \infty\,. \tag{2.1}$$

The infinitesimal generator $\mathcal{A}$ of this process, defined on the space $C^2(\mathbb{R}^n;\mathbb{R})$ of twice continuously differentiable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, is given as

$$\mathcal{A}\varphi(x) := \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(x)\,\frac{\partial^2}{\partial x_i \partial x_k}\,[\varphi(x)]\,; \quad \varphi \in C^2(\mathbb{R}^n;\mathbb{R}), \tag{2.2}$$

where

$$a_{ik}(x) := \sum_{j=1}^{n} \sigma_{ij}(x)\,\sigma_{kj}(x)\,, \quad A(x) := \big\{a_{ij}(x)\big\}_{1\le i,j\le n}\,; \quad x \in \mathbb{R}^n. \tag{2.3}$$

Here $\sigma_{ij}(\cdot)$ is the (i,j)-th element of the matrix-valued function $\sigma(\cdot)$ for $1 \le i, j \le n$. By the assumption of uniform positive-definiteness on the matrices $\{\sigma_\nu\}(\cdot)$, $\nu = 1, \cdots, m$ in (1.3), the operator $\mathcal{A}$ is uniformly elliptic. As is well known, existence (respectively, uniqueness) of a weak solution to the stochastic integral equation (2.1), is equivalent to the solvability (respectively, well-posedness) of the martingale problem associated with the operator $\mathcal{A}$.

2.2. **Equivalent Problems.** Without loss of generality we start from the case $i = 1, j = 2, k = 3$ in (1.4). Let us define $(n \times 1)$ vectors d_1, d_2, d_3 to extract the *information* of the diffusion matrix $\sigma(\cdot)$ on (X_1, X_2, X_3), namely

$$d_1 := (1, -1, 0, \cdots, 0)',\ d_2 := (0, 1, -1, 0, \cdots, 0)',\ d_3 := (-1, 0, 1, 0, \cdots, 0)',$$

where the superscript $'$ stands for transposition. Define the $(n \times 3)$-matrix $D = (d_1,\, d_2,\, d_3)$ for notational simplicity. The cases we consider in (1.4) for $i = 1,\ j = 2,\ k = 3$ are equivalent to

$$\begin{aligned} &\mathbb{P}_{x_0}\Big(s^2(X(t)) = 0\,,\ \text{for some } t > 0\Big) = 0 \quad \text{or} \\ &\mathbb{P}_{x_0}\Big(s^2(X(t)) = 0, \quad \text{for infinitely many } t \geq 0\Big) = 1\,; \quad x_0 \in \mathbb{R}^n, \end{aligned} \tag{2.4}$$

where the continuous function $s^2 : \mathbb{R}^n \to \mathbb{R}_+$ is defined as

$$\begin{aligned} s^2(x) &:= (x_1 - x_3)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2 \\ &= d_1' x x' d_1 + d_2' x x' d_2 + d_3' x x' d_3 = x' D D' x\,; \quad x \in \mathbb{R}^n\,. \end{aligned} \tag{2.5}$$

This function measures the sum of squared distances for the three particles we are interested in. Thus, it suffices to study the behavior of the continuous, non-negative process $\{s^2(X(t)); 0 \leq t < \infty\}$ around its zero set

$$\mathcal{Z} := \{x \in \mathbb{R}^n : s(x) = 0\}. \tag{2.6}$$

We shall set up related stopping times and derive equivalent conditions (2.9) and (2.16) to (2.4) in the rest of this subsection.

2.2.1. *Some Useful Stopping Times.* Suppose that $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$ is fixed, take an integer $k_0 > 1$ such that $k^{-k} < s(X(0)) = s(x_0) < k$ holds for every $k \geq k_0$, and define the stopping times

$$\begin{aligned} S_k &:= \inf\{t > 0 : s(X(t)) = k\}, \\ T_k &:= \inf\{t > 0 : s(X(t)) = k^{-k}\}; \quad k \geq k_0, \\ T &:= \inf\{t > 0 : s(X(t)) = 0\} = \lim_{k \to \infty} T_k. \end{aligned} \tag{2.7}$$

It can be verified by the time-change for Brownian motion, that

$$\mathbb{P}_{x_0}(S_k < \infty) = 1 = \mathbb{P}_{x_0}\left(\lim_{\ell \to \infty} S_\ell = \infty\right); \quad k \geq k_0\,. \tag{2.8}$$

By the Strong Markov property, the cases considered in (2.4) can be cast as

$$\mathbb{P}_{x_0}(T < \infty) = 0 \quad \text{or} \quad \mathbb{P}_{x_0}(T < \infty) = 1; \quad x_0 \in \mathbb{R}^n \setminus \mathcal{Z}. \tag{2.9}$$

2.2.2. *An Integral Form.* We define the test function $f : \mathbb{R}^n \setminus \mathcal{Z} \to \mathbb{R}$ by

$$\begin{aligned} f(x) &:= \frac{\sum_{i=1}^3 d_i' \sigma(x) \sigma'(x) d_i}{\sum_{i=1}^3 d_i' x x' d_i} - \frac{2\|\sum_{i=1}^3 d_i' x \sigma'(x) d_i\|^2}{(\sum_{i=1}^3 d_i' x x' d_i)^2} \\ &= \frac{\operatorname{trace}(D' A(x) D)}{x' D D' x} - \frac{2 x' D D' A(x) D D' x}{(x' D D' x)^2}\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}. \end{aligned} \tag{2.10}$$

It follows from (2.1) that

$$d\left(X_i(t) - X_j(t)\right) = \sum_{\ell=1}^n \left(\,\sigma_{i\ell}(X(t)) - \sigma_{j\ell}(X(t))\,\right) dW_\ell(t)\,; \quad 1 \leq i, j \leq 3\,,$$

and an application of Itô's formula to $\log\big(s^2(X(\cdot))\big)$ gives

$$\log\big(s^2(X(t))\big) = \log s^2(x_0)+\int_0^t f(X(s))ds+M(t)\,, \qquad 0\le t<T \tag{2.11}$$

where $f(\cdot)$ is the test function in (2.10) and $M(\cdot)$ is the local martingale

$$M(t) := \int_0^t \Big\langle \Big(\frac{2}{s^2(x)}\sum_{i=1}^3 \sigma'(x)\,d_i d_i' x\Big)\Big|_{x=X(u)}\,,\ dW(u)\Big\rangle\,. \tag{2.12}$$

Define a sequence of bounded stopping times $V_k^{(N)} := S_k\wedge T_k\wedge N$ for $k\ge k_0$ and a fixed $N\in\mathbb{N}$. From the observation $\mathbb{E}_{x_0}\big[M\big(V_k^{(N)}\big)\big]=0$, we obtain for every $k\ge k_0$, $N\in\mathbb{N}$:

$$\mathbb{E}_{x_0}\Big[\log\Big(s\big(X(V_k^{(N)})\big)\Big)\Big] = \log s(x_0)+\frac{1}{2}\,\mathbb{E}_{x_0}\Big[\int_0^{V_k^{(N)}} f(X(t))dt\Big]\,. \tag{2.13}$$

Indeed, as long as the process $X(\cdot)$ remains inside the tubular domain

$$\mathcal{D}_k := \{x\in\mathbb{R}^n : k^{-k}<s(x)<k\}\,;\quad k\ge k_0\ , \tag{2.14}$$

we have $t<S_k\wedge T_k$ and all the elements of the integrand on the right-hand side of (2.12) are bounded by

$$\frac{\|2\sum_{i=1}^3\sigma'(x)\,d_id_i'x\|}{s^2(x)} = \frac{2x'D(D'A(x)D)D'x}{x'DD'x}\le 2\max_{\substack{1\le i\le 3\\1\le\nu\le m}}\Big(\max_{x\in\mathcal{R}_\nu}\lambda_{i\nu}^D(x)\Big)<\infty\,. \tag{2.15}$$

Here $\{\lambda_{i\nu}^D(x)\,,\ i=1,2,3\}$ are the eigenvalues of the (3×3) matrix $D'\sigma_\nu(x)\sigma_\nu'(x)D$, $x\in\mathcal{R}_\nu$, with $\sigma_\nu(x)$ defined in (1.3) for $\nu=1,\dots,m$. Applying the optional sampling theorem, and using the boundedness of the matrix-valued function $\sigma(\cdot)$, we obtain $\mathbb{E}_{x_0}\big[M\big(V_k^{(N)}\big)\big]=0$ for every $k\ge k_0$, $N\in\mathbb{N}$.

The left-hand side in (2.13) is

$$-k(\log k)\,\mathbb{P}_{x_0}(T_k\le S_k\wedge N)+(\log k)\,\mathbb{P}_{x_0}(S_k\le T_k\wedge N) + \mathbb{E}_{x_0}\Big(\log\big(X(S_k\wedge T_k\wedge N)\big)\,\mathbf{1}_{\{N<T_k\wedge S_k\}}\Big)\,.$$

Substituting this expression into (2.13) and sending N to infinity, we use (2.8), the boundedness of $f(\cdot)$ and $\mathbb{E}[S_k]$ (see Remark 2.2 below), and dominated convergence, to obtain

$$\mathbb{P}_{x_0}(T_k\le S_k) = \frac{1}{k}\mathbb{P}_{x_0}(S_k\le T_k)-\frac{\log s(x_0)}{k\log k}-\frac{\mathbb{E}_{x_0}\Big[\int_0^{T_k\wedge S_k} f(X(t))dt\Big]}{2k\log k}\,.$$

The first and second terms on the right-hand side of this identity tend to zero as $k\to\infty$, so it follows from (2.7) and (2.8) that

$$\mathbb{P}_{x_0}(T<\infty)=\lim_{k\to\infty}\mathbb{P}_{x_0}(T_k<S_k) = -\lim_{k\to\infty}\frac{\mathbb{E}_{x_0}\Big[\int_0^{T_k\wedge S_k} f(X(t))dt\Big]}{2k\log k}.$$

Therefore, the cases considered in (2.4), (2.9) amount to

$$\lim_{k\to\infty} \frac{\mathbb{E}_{x_0}\left[\int_0^{T_k\wedge S_k} f(X(t))dt\right]}{2k\log k} = 0 \;\text{ or }\; -1\,, \tag{2.16}$$

and we need to study the behavior as $k \to \infty$ of the expectation of the integral functional $\int_0^{T_k\wedge S_k} f(X(t))dt$ for the diffusion process $X(\cdot)$, where $f(\cdot)$ is defined by (2.10).

As a summary of this subsection, we pose the following problem:

Problem 1: *Under which conditions on $\sigma(\cdot)$ of the form (1.3), do the equivalent conditions (2.4), (2.9) and (2.16) hold?*

Remark 2.1. For the standard, $n-$dimensional Brownian motion, i.e., $\sigma(\cdot) \equiv I_n$, $n \geq 3$, we compute $f(\cdot) \equiv 0$ in (2.10); the limit in (2.16) is trivially equal to zero in this case, so we recover the well-known fact that there are no triple-collisions along the standard Brownian path: We have $\mathbb{P}_{x_0}(T < \infty) = 0$.

More generally, suppose that the variance covariance rate $A(\cdot)$ is

$$A(x) := \sum_{\nu=1}^{m} \big(\alpha_\nu I_n + \beta_\nu DD' + \mathbb{I}\mathbb{I}' \operatorname{diag}(\gamma_\nu)\big)\cdot \mathbf{1}_{\mathcal{R}_\nu}(x)\;; \quad x\in\mathbb{R}^n\,,$$

for some scalar constants α_ν, β_ν and $(n\times 1)$ constant vectors γ_ν, $\nu = 1,\ldots,m$. Here $\operatorname{diag}(x)$ is the $(n\times n)$ diagonal matrix whose diagonal entries are the elements of $x\in\mathbb{R}^n$, and $\mathbb{I}$ is the $(n\times 1)$ vector with all entries equal to one. Then $f(\cdot)\equiv 0$, because $\mathbb{I}'D = (0,0,0)\in\mathbb{R}^{1\times 3}$ and

$$DD' = \frac{1}{3}DD'DD' = \begin{pmatrix} 2 & -1 & -1 & \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & \\ & 0 & & 0 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Hence, if the coëfficients α_ν, β_ν and γ_ν, $\nu = 1,\ldots,m$ are chosen above so that $A(\cdot)$ is positive-definite, then we have $\mathbb{P}_{x_0}(T<\infty) = 0$. $\square$

Other cases are not so trivial; the form of the function $f(\cdot)$ does not give any immediate answer to the problem. For instance, with a diagonal matrix-valued function $\sigma(x) = \operatorname{diag}\,(\sigma_{11}(x),\cdots,\sigma_{nn}(x))$, we get

$$f(x) = \frac{2}{s^4(x)}\left(\begin{array}{c} \sigma_{11}^2(x)\big((x_2-x_3)^2 + 2(x_1-x_2)(x_3-x_1)\big) \\ +\sigma_{22}^2(x)\big((x_3-x_1)^2 + 2(x_2-x_3)(x_1-x_2)\big) \\ +\,\sigma_{33}^2(x)\big((x_1-x_2)^2 + 2(x_3-x_1)(x_2-x_3)\big) \end{array}\right).$$

Remark 2.2. The function $f(\cdot)$ of (2.10) can be bounded from above on the tube $\mathcal{D}_k$ by

$$|f(x)| \leq \frac{1}{s^2(x)}\left|\operatorname{trace}\,\big(D'A(x)D\big) - \frac{2x'D\big(D'A(x)D\big)D'x}{x'DD'x}\right|; \quad x\in\mathcal{D}_k\,.$$

The matrix-valued function $A(\cdot)$ is bounded, and so is the term inside $|\cdot|$ as in (2.15); however, the $s^{-2}(x)$ term explodes at the order of $O(k^{2k})$ for $x \in \mathcal{D}_k$ in (2.14), as $k \to \infty$.

An elementary estimate (Lemma 5.1 of (3)) gives $\mathbb{E}_{x_0}[S_k] \le ck^2$ for some constant c. These bounds do not work well in (2.16), as they lead to estimates that grow much faster than $k \log k$ in (2.16). $\Box$

It becomes clear that we need some more elaborate tools for deciding when the conditions of (2.16) will hold. We shall develop such tools in the next subsection, using partial differential equations and stochastic calculus.

2.3. **Partial Differential Equation.** Consider a strictly elliptic partial differential operator $\widetilde{\mathcal{A}}$ of the second order, defined by

$$\widetilde{\mathcal{A}}\varphi(x) := \frac{1}{2}\sum_{i,j=1}^{n} \tilde{a}_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}[\varphi(x)]\,, \quad x \in \mathbb{R}^n\,, \tag{2.17}$$

where $\tilde{a}_{ij}(\cdot)$ is bounded and belongs to the class $C^\alpha(\mathbb{R}^n)$ of $\alpha-$Hölder continuous functions with $0 < \alpha < 1$. Define the $(n \times n)$ matrix-valued function

$$\widetilde{A}(x) := \Big\{\tilde{a}_{ij}(x)\Big\}_{1 \le i,j \le n}\,; \quad x \in \mathbb{R}^n\,, \tag{2.18}$$

as well as a $C^\alpha-$function $g : \mathbb{R}^n \setminus \mathcal{Z} \to \mathbb{R}$ by

$$g(x) := \frac{\operatorname{trace}(D'\widetilde{A}(x)D)}{x'DD'x} - \frac{2x'DD'\widetilde{A}(x)DD'x}{(x'DD'x)^2}\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,, \tag{2.19}$$

and its truncated version $g_r(\cdot)$ for $r \ge 2$ by $g_r(\cdot) := g(\cdot)\zeta_r(\cdot)$. Here $\zeta_r(\cdot)$ is a $C^\infty-$function from $\mathbb{R}^n$ to $\mathbb{R}$ which takes the value 1 for $||x|| \le r-1$, the value 0 for $||x|| \ge r$, and values in $[0,1]$ for $r-1 \le ||x|| \le r$ for $r \ge 2$.

Note that $g(\cdot)$ is a "smooth counterpart" of the test-function $f(\cdot)$ in (2.10), that $g_r(\cdot)$ is a "smooth and truncated" counterpart of $f(\cdot)$, and that

$$g(x) = \lim_{r \to \infty} g_r(x)\,, \quad x \in \mathbb{R}^n.$$

Consider a diffusion process $\widetilde{X} = \{\widetilde{X}(t);\, 0 \le t < \infty\}$ with infinitesimal generator $\widetilde{\mathcal{A}}$, starting at $\widetilde{X}(0) = x_0$. Existence and uniqueness for the associated martingale problem are established in (18). In this subsection we shall denote by $\widetilde{\mathbb{P}}_{x_0}$ the induced probability distribution of solution to the corresponding martingale problem.

Our motivation for employing the theory of partial differential equations is to obtain the estimate (2.25) below, under the following assumption.

Assumption. *For $k \ge k_0$, $r \ge 2$, assume* temporarily *that the Dirichlet problem*

$$\widetilde{\mathcal{A}}u_{k,r}(\cdot) + g_r(\cdot) = 0 \ \text{ in } \mathcal{D}_k\,, \qquad u_{k,r}(\cdot) = 0 \text{ on } \partial\mathcal{D}_k\,, \qquad \lim_{||x|| \to \infty} u_{k,r}(x) = 0 \tag{2.20}$$

has a unique solution $u_{k,r}(\cdot)$ *in* $C^{2,\alpha}(\overline{D}_k)$ *that satisfies* $\sup_{x\in\mathcal{D}_k}|u_{k,r}(x)| < \infty$.

Moreover, assume that the values $u_{k,r}(x_0)$ *of this solution at* $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$, *for* $k \geq k_0$ *and* $r \geq 2$, *behave asymptotically in the order*

$$\sup_{r\geq 2} \big|u_{k,r}(x_0)\big| \;=\; o(k\log k)\,, \quad \text{as } k \to \infty\,. \tag{2.21}$$

We shall validate this assumption under some sufficient conditions (2.35) explored in the rest of this section.

Under the above Assumption, we apply Itô's formula to $u_{k,r}(\widetilde{X}(\cdot))$ and integrate over the interval $\big[0, \tilde{V}_k^{(N)}\big]$, where $\tilde{V}_k^{(N)} := \widetilde{T}_k \wedge \widetilde{S}_k \wedge N$ and

$$\widetilde{S}_k := \inf\big\{t>0 : s(\widetilde{X}(t)) = k\big\}\,, \qquad \widetilde{T}_k := \inf\big\{t>0 : s(\widetilde{X}(t)) = k^{-k}\big\}$$

are stopping times for $\widetilde{X}(\cdot)$. Taking expectations with respect to $\widetilde{\mathbb{P}}_{x_0}$, we obtain then from (2.20):

$$\begin{aligned}
&\widetilde{\mathbb{E}}_{x_0}\Big[u_{k,r}\big(\widetilde{X}\big(\widetilde{V}_k^{(N)}\big)\big)\Big] - u_{k,r}(x_0) \\
&= \widetilde{\mathbb{E}}_{x_0}\left[\int_0^{\widetilde{V}_k^{(N)}} \big\langle \nabla u_{k,r}(\widetilde{X}(t)), d\,\widetilde{X}(t)\big\rangle + \int_0^{\widetilde{V}_k^{(N)}} \widetilde{\mathcal{A}} u_{k,r}(\widetilde{X}(t))d\,t\right] \\
&= -\widetilde{\mathbb{E}}_{x_0}\left[\int_0^{\widetilde{V}_k^{(N)}} g_r(\widetilde{X}(t))d\,t\right]\,,
\end{aligned} \tag{2.22}$$

because the expectation of the local martingale part is zero (reasoning as in (2.13)). We note that $\widetilde{\mathbb{P}}_{x_0}(\widetilde{S}_k < \infty) = 1$, so the boundary condition $u_{k,r}(\cdot) = 0$ on $\partial\mathcal{D}_k$ implies $u_{k,r}(\widetilde{X}(\widetilde{S}_k \wedge \widetilde{T}_k)) = 0$, a.s. From this, and $\sup_{x\in\mathcal{D}_k}|u_{k,r}(x)| < \infty$, we obtain the estimate

$$\begin{aligned}
&\Big|\widetilde{\mathbb{E}}_{x_0}\Big[u_{k,r}\big(\widetilde{X}\big(\widetilde{V}_k^{(N)}\big)\big)\Big]\Big| \;=\; \Big|\widetilde{\mathbb{E}}_{x_0}\Big[u_{k,r}\big(\widetilde{X}(N)\big)\,\mathbf{1}_{\{N\leq \widetilde{S}_k\wedge\widetilde{T}_k\}}\Big]\Big| \\
&\leq \sup_{x\in\mathcal{D}_k}|u_{k,r}(x)|\cdot\widetilde{\mathbb{P}}_{x_0}\big(N < \widetilde{S}_k\wedge\widetilde{T}_k\big) \xrightarrow[N\to\infty]{} 0\,; \quad k\geq k_0\,,\;\; r\geq 2\,.
\end{aligned}$$

Thus, letting $N \to \infty$ in (2.22) and invoking the dominated convergence theorem (evaluating $g_r(\cdot)$ and $\widetilde{\mathbb{E}}\big(\widetilde{S}_k\big)$ as in Remark 2.2), we obtain

$$\widetilde{\mathbb{E}}_{x_0}\left[\int_0^{\widetilde{S}_k\wedge\widetilde{T}_k} g_r(\widetilde{X}(t))d\,t\right] = u_{k,r}(x_0)\,; \quad k\geq k_0,\, r\geq 2\,. \tag{2.23}$$

Moreover, since $\widetilde{\mathbb{E}}\big(\widetilde{S}_k\big) \le ck^2 < \infty$ for some constant c (Remark 2.2 again), we get

$$\begin{aligned}
&\left| \widetilde{\mathbb{E}}_{x_0}\left[\int_0^{\widetilde{S}_k \wedge \widetilde{T}_k} g_r(\widetilde{X}(t))\,d\,t - \int_0^{\widetilde{S}_k \wedge \widetilde{T}_k} g(\widetilde{X}(t))\,d\,t \right] \right| \\
&\le \widetilde{\mathbb{E}}_{x_0}\left[\int_0^{\widetilde{S}_k \wedge \widetilde{T}_k} |g_r(\widetilde{X}(t)) - g(\widetilde{X}(t))|\,d\,t \right] \qquad (2.24)\\
&\le \sup_{x \in \mathcal{D}_k} |g(x)| \cdot \widetilde{\mathbb{E}}_{x_0}\left[\widetilde{S}_k \cdot \mathbf{1}\Big\{ \sup_{t \le \widetilde{S}_k} |\widetilde{X}(t)| \ge r \Big\} \right] \xrightarrow[r \to \infty]{} 0\,.
\end{aligned}$$

Therefore, under the assumption (2.21) on the asymptotic behavior of the Dirichlet problem (2.20), and using (2.23), (2.24) we obtain

$$\lim_{k\to\infty} \left| \frac{\widetilde{\mathbb{E}}_{x_0}\left[\int_0^{\widetilde{S}_k \wedge \widetilde{T}_k} g(\widetilde{X}(t))\,d\,t \right]}{k \log k} \right| \le \lim_{k \to \infty} \left(\sup_{r \ge 2} \frac{|u_{k,r}(x_0)|}{k \log k} \right) = 0\,. \qquad (2.25)$$

This is the special case of (2.16) when the function $f(\cdot) \equiv g(\cdot)$ and the coëfficients of the differential operator $\mathcal{A} \equiv \widetilde{\mathcal{A}}$ are bounded and belong to $C^\alpha(\mathbb{R}^n)$, and when the above Assumption holds. Since the first case of (2.16) is equivalent back then to that of (2.9), (2.4) and (1.4), as discussed in Section 2.2, this implies that there is *no triple-collision* of Brownian particles. Thus, it is worth considering the following problem.

Problem 2:

(i) *Is it possible, under appropriate conditions, to find a sequence of solutions $\{u_{k,r}(\cdot);\, k \ge k_0,\, r \ge 2\}$ to the sequence of Dirichlet problems (2.20) that satisfy (2.21), so we obtain (2.25) and give a partial answer to Problem 1?*

(ii) *If so, under which conditions does (2.25) still hold, to wit, we get "*no triple collision*", even when $\widetilde{\mathcal{A}}$ and $g(\cdot)$ are replaced by $\mathcal{A}$ and $f(\cdot)$ as in (2.2) and (2.10), respectively, which allow discontinuities?*

In the next subsections we shall introduce the *effective dimension* for the process $X(\cdot)$, by analogy with the theory of the exterior Dirichlet problem (recalled in the next subsection 2.3.1); we shall verify then the existence of a solution for Problem 2(i) in subsection 2.3.2, and control the growth of this solution using a barrier function in subsection 2.3.3. We shall discuss an approximation procedure for Problem 2 (ii) in subsection 2.3.4, and formulate the results in subsection 2.3.5.

2.3.1. *Effective Dimension.* In the example of (3), mentioned briefly in Section 1, the diffusion matrix $\sigma(\cdot) = \sum_{\nu=1}^m \sigma_\nu(\cdot) 1_{\mathbb{R}_\nu}$ in (1.3) has a special characteristic in the allocation of its eigenvalues: *All eigenvalues but the largest are small;*, namely, they are of the form $(1, \varepsilon, \cdots, \varepsilon)$, where $0 < \varepsilon < 1/2$

satisfies, for some $0 < \delta < 1/2$:

$$\left| \frac{x'\sigma(x)\sigma(x)'x}{\|x\|^2} - 1 \right| \le \delta \qquad \text{for } x \in \mathbb{R}^n\,, \quad \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1\,. \tag{2.26}$$

This is the case when the diffusion matrix $\sigma(\cdot)$ can be written as $\sum_{\nu=1}^{m} \sigma_\nu 1_{\mathcal{R}_\nu}(\cdot)$ where the constant $(n \times n)$ matrices $\{\sigma_\nu,\, \nu = 1 \ldots m\}$ have the decomposition

$$\sigma_\nu \sigma_\nu' := (y_\nu\,,\, B_\nu)\ \text{diag}\ (1, \varepsilon^2, \ldots, \varepsilon^2) \begin{pmatrix} y_\nu' \\ B_\nu' \end{pmatrix} ,$$

where the fixed $(n \times 1)$ vector $y_\nu \in \mathbb{R}_\nu$ satisfyies

$$\|y_\nu\| = 1\,, \quad \frac{|\langle\, x\,,\, y_\nu \,\rangle|^2}{\|x\|^2} \ge 1 - \varepsilon\,; \quad x \in \mathbb{R}_\nu\,, \tag{2.27}$$

and the $(n\times(n-1))$ matrix B_ν consists of $(n-1)$ orthonormal $n-$dimensional vectors orthogonal to each other and orthogonal to y_ν, for $\nu = 1, \ldots m$. Then, for $x \in \mathbb{R}^n$,

$$\frac{\|x\|^2 \text{trace}\ (\sigma(x)\sigma(x)')}{x'\sigma(x)\sigma(x)'x} - 1 \le \frac{(n-1)\varepsilon^2 + \delta}{1-\delta} < 1.$$

This is sufficient for the process X to hit the origin in a finite time.

To exclude this situation, we introduce the *effective dimension* $\text{ED}_{\mathcal{A}}(\cdot)$ of the elliptic second-order operator $\mathcal{A}$ defined in (2.2), namely

$$\text{ED}_{\mathcal{A}}(x) := \frac{\|x\|^2 \text{trace}\ (\sigma(x)\sigma(x)')}{x'\sigma(x)\sigma(x)'x} = \frac{\|x\|^2 \text{trace}\ (A(x))}{x'A(x)x} \tag{2.28}$$

for $x \in \mathbb{R}^n \setminus \{0\}$. This function comes from the theory of the so-called *exterior Dirichlet problem* for second-order elliptic partial differential equations, pioneered by Meyers & Serrin (14). In the next subsections we will see that

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \text{ED}_{\widetilde{\mathcal{A}}}(\cdot) > 2 \tag{2.29}$$

is sufficient for the existence of solution to the Dirichlet problem (2.20).

With $\sigma(\cdot)$ defined in (1.3), the effective dimension $\text{ED}_{\mathcal{A}}(\cdot)$ satisfies

$$\text{ED}_{\mathcal{A}}(x) \ge \min_{\nu=1,\cdots,m} \left(\frac{|x|^2 \text{trace}\ (\sigma_\nu(\cdot)\sigma_\nu'(\cdot))}{x'\sigma_\nu(\cdot)\sigma_\nu'(\cdot)x} \right) \ge \min_{\nu=1,\cdots,m} \left(\frac{\sum_{i=1}^n \lambda_{i\nu}(x)}{\max_{i=1,\cdots,n} \lambda_{i\nu}(x)} \right)$$

for $x \in \mathbb{R}^n \setminus \{0\}$, where $\{\lambda_{i\nu}(\cdot),\, i = 1, \cdots, n\}$ are the eigenvalues of the matrix-valued functions $\sigma_\nu(\cdot)\sigma_\nu'(\cdot)$, for $\nu = 1, \cdots, m$ in (1.3). Thus, $\text{ED}_{\mathcal{A}}(\cdot) > 2$ if

$$\inf_{x \in \mathbb{R}^n \setminus \{0\}} \min_{\nu=1,\cdots,m} \left(\frac{\sum_{i=1}^n \lambda_{i\nu}(x)}{\max_{i=1,\cdots,n} \lambda_{i\nu}(x)} \right) > 2\,. \tag{2.30}$$

This condition can be interpreted as mandating that the relative size of the maximum eigenvalue is not too large, when compared to the other eigenvalues.

Remark 2.3. In the above example of (2.26), which leads to attainability (1.2) of the origin, we have $\mathrm{ED}_{\mathcal{A}}(\cdot) \le 2 - \eta$ for some $\eta > 0$. On the other hand, by adding another condition to (2.29) we settle the issue of "*no triple collision*", as explained in Proposition 1 of Section 2.3.5. For the discrepancy between these conditions, see Remarks 2.10. and 2.12 following Propositions 1 and 2. For the $n-$dimensional Laplacian, the effective dimension is n. □

2.3.2. *Dirichlet Problems for the Smooth Case.* In this subsection we discuss the rôle of the effective dimension. For $k \ge k_0$ the domain $\mathcal{D}_k$ is an infinite tube containing x_0: it is translation-invariant in the direction of $x \to (x_1 + \xi, x_2 + \xi, x_3 + \xi, x_4, \ldots, x_n)$ for any point $x \in \mathcal{D}_k$ and $\xi \in \mathbb{R}$.

Since $\mathcal{D}_k$ is unbounded, we use results of Meyers & Serrin (14) on the *exterior Dirichlet problem,* whose domain is generated by removing a smooth bounded domain from $\mathbb{R}^n$. These authors develop the so-called ϕ-sequence of solutions, where ϕ is the boundary value. Each element of the ϕ-sequence is a solution with the common boundary condition ϕ for a sub-domain which is parametrized by the distance from the origin. By obtaining sufficient conditions for the existence of barriers at infinity, they discuss the well-posedness of the problem.

For our problem (2.20) we also consider ϕ-sequences and apply the sufficient conditions in (14). In order to explain the construction of solutions, define an increasing sequence $\{\mathcal{E}_{k,p}; p \ge k+1, k \ge k_0\}$ of *smooth, bounded* sub-domains of $\mathcal{D}_k$ by

$$\mathcal{E}_{k,p} := \mathcal{S}(\mathcal{D}_k \cap B_p(0)); \quad p \ge k+1\,,\ k \ge k_0\,.$$

Here $B_p(x)$ is the n-dimensional ball with center $x \in \mathbb{R}^n$ and radius $p > 0$; whereas the operator $\mathcal{S}$ acts on a subset $A \subset \mathbb{R}^n$ of the form $D_k \cap B_p(0)$, whose boundary ∂A is not of class $C^{2,\alpha}$, in such a way that the image of the boundary ∂A *becomes* of class $C^{2,\alpha}$ and we also have

$$\mathcal{E}_{k,p} = \mathcal{S}\left(D_k \cap B_p(0)\right) \subset D_k \cap B_p(0) \subset \mathcal{E}_{k,p+1}\,; \quad p \ge k+1\,,\ k \ge k_0$$

as well as $\mathcal{D}_k = \cup_{p=k+1}^{\infty} \mathcal{E}_{k,p}$.

For example, consider the case $n = 3$. For $p \ge k+1, k \ge k_0$ the set $\mathcal{D}_k \cap B_p(0)$ is a disjoint union of a tube with finite height $2\sqrt{p^2 - k^2}$ and two identical oppositely-directed spherical segments. The finite tube is placed between the spherical segments. Each spherical segment has a hole hollowed in its center with radius k^{-k} and depth $p - \sqrt{p^2 - k^2}$ in $\mathbb{R}^3$. The surface of the hole is non-smooth at a circle $\partial B_p(0) \cap \{x \in \mathbb{R}^3 : s(x) = k^{-k}\}$. Moreover, the conjunctions between the finite tube and the spherical segments are non-smooth at the circles with centers of coordinates $\sqrt{(p^2 - k^2)/3}\,(1,1,1)'$ and $-\sqrt{(p^2 - k^2)/3}\,(1,1,1)'$ with the same radius k. We make the boundary smooth by acting on it with $\mathcal{S}$. In higher dimensions $n \ge 4$ we can carry out a similar construction, since the restriction $\mathcal{D}_k$ concerns the first three coördinates of $\mathbb{R}^n$.

Let us start with a sequence of Dirichlet problems for the bounded $C^{2,\alpha}$-domains $\mathcal{E}_{k,p}$: for $k \geq k_0$, $p \geq k+1$, $r \geq 2$,

$$\widetilde{\mathcal{A}} u_{k,p,r}(\cdot) + g_r(\cdot) = 0 \quad \text{in } \mathcal{E}_{k,p}\,, \qquad u_{k,p,r}(\cdot) = 0 \ \text{ on } \partial\mathcal{E}_{k,p}\,. \tag{2.31}$$

Now we build up the solutions to the Dirichlet problems. First, from the Perron method we obtain the following result.

Lemma 2.1 ((8) Theorem 6.14.)**.** *The Dirichlet problem* (2.31) *has a unique solution* $u_{k,p,r}(\cdot)$ *in* $C^{2,\alpha}(\bar{\mathcal{E}}_{p,k})$*, for every* $k \geq k_0$*,* $p \geq k+1$*,* $r \geq 2$*.*

Then, with the help of the existence of a barrier at infinity (Lemma 1, Theorem 1, Lemma 3 and Theorem 10 of (14)) and Schauder's interior estimate, we construct the solution of the Dirichlet problem (2.20) as the limit of $u_{k,p,r}(\cdot)$ as $p \to \infty$.

Lemma 2.2. *If* (2.29) *holds, the Dirichlet problem* (2.20) *has a unique solution* $u_{k,r}(\cdot)$ *in the space* $C^{2,\alpha}(\bar{\mathcal{D}}_k)$ *for every* $k \geq k_0$*,* $r \geq 2$*.*

Note that the Dirichlet problem is well-posed because of the truncation, namely, because $g_r(\cdot)$ is zero outside of the ball $B_r(0)$.

Remark 2.4. This condition on the effective dimension can be weakened slightly. In fact, in Theorem 2 of (14) it is shown that if $\mathrm{ED}_{\widetilde{\mathcal{A}}}(\cdot) \geq 2 + \varepsilon(\cdot)$ in some neighborhood of infinity, where $\varepsilon(\cdot)$ can be written as a function $\varepsilon(r)$ of the distance $r = \|x\|$, $x \in \mathbb{R}^n$ from the origin such that

$$v(t) := \int_t^\infty \xi^{-1} \exp\Big(- \int_a^\xi r^{-1} \varepsilon(r) dr \Big) d\xi < \infty\,; \quad \forall\, t > 0\,,$$

then the Dirichlet problem is well-posed. The function $v(\cdot)$ defined above satisfies $Lv(\|x\|) \leq 0$ for $x \in \mathbb{R}^n$ and $v'(t) < 0$ for $t \in \mathbb{R}_+$. It follows that $v(\cdot)$ serves as a barrier at infinity. □

2.3.3. *Barrier function.* We shall control the behavior of the solution $u_{k,r}(\cdot)$ to the Dirichlet problem (2.20) by reducing the n-dimensional problem to a *one*-dimensional problem and finding an appropriate barrier function.

We shall use the following Sturm-type lemma.

Lemma 2.3 ((14) Lemma 2)**.** *Suppose that the* C^2-*function* $v : \mathbb{R}_+ \to \mathbb{R}$ *satisfies the second-order ordinary differential equation*

$$v''(t) + B(t) v'(t) = F(t); \quad 0 \leq a \leq t \leq b\,, \tag{2.32}$$

where $B(\cdot)$ *and* $F(\cdot)$ *are continuous functions satisfying* $F(\cdot) \geq 0$*. If* $v(a) = v'(a) = 0$ *, then we have* $v(\cdot) \geq 0, v'(\cdot) \geq 0$ *in the interval* $[a, b]$*.*

From here onward we shall denote by $v(\cdot)$ the particular solution of (2.32) such that $v(\cdot) \geq 0$ with its derivative $v'(\cdot) \geq 0$ in the interval $[a, b]$ and the boundary conditions $v(a) = v'(a) = 0$ with $a = k^{-k}$, $b = k$ for some functions $B(\cdot)$ and $F(\cdot)$ to be specified in (2.36). Simple calculation gives

$$\widetilde{\mathcal{A}} v(s(\cdot)) = \frac{1}{2} \widetilde{Q}(\cdot) \left(v''(s(\cdot)) + \frac{v'(s(\cdot))}{s(\cdot)} (\widetilde{R}(\cdot) - 1) \right) \ \text{ in } \mathbb{R}^n \setminus \mathcal{Z}\,, \tag{2.33}$$

where we have set

$$\widetilde{Q}(x) := \frac{x'DD'\widetilde{A}(x)DD'x}{x'DD'x}\,, \quad \widetilde{R}(x) := \frac{\mathrm{trace}(D'\widetilde{A}(x)D)x'DD'x}{x'DD'\widetilde{A}(x)DD'x}\,. \tag{2.34}$$

Recall the definition (2.28) of the effective dimension, assume

$$\inf_{x\in\mathbb{R}^n\setminus\{0\}} \mathrm{ED}_{\widetilde{\mathcal{A}}}(x) > 2\,, \qquad \inf_{x\in\mathbb{R}\setminus\mathcal{Z}} \widetilde{R}(x) > 2\,, \tag{2.35}$$

and choose $B(\cdot)$ and $F(\cdot)$ in Lemma 2.3 such that

$$B(s(\cdot)) := \frac{c_1}{s(\cdot)} \le \frac{\widetilde{R}(\cdot)-1}{s(\cdot)}\,, \quad \frac{2|g_r(\cdot)|}{\widetilde{Q}(\cdot)} \le F(s(\cdot)) := \frac{c_2}{s^2(\cdot)} \ \text{ in } \mathbb{R}^n\setminus\mathcal{Z}\,, \tag{2.36}$$

where

$$c_1 := \inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}} \widetilde{R}(x)-1 > 1\,,\ \ c_2 := 2\sup_{x\in\mathbb{R}^n\setminus\mathcal{Z}} |\widetilde{R}(x)-2| > 0\,.$$

Such a real number c_2 exists, since $\widetilde{R}(\cdot)$ is bounded in $\mathbb{R}^n\setminus\mathcal{Z}$; see the following Remarks 2.6 and 2.7.

Remark 2.5. Comparing $\widetilde{R}(\cdot)$ with $\mathrm{ED}_{\widetilde{\mathcal{A}}}(\cdot)$, we may view $\widetilde{R}(\cdot)$ as a *local* effective dimension which characterizes the behavior of the first three coördinates of n-dimensional process, while $\mathrm{ED}_{\widetilde{\mathcal{A}}}(\cdot)$ is the *global* one. In fact, replacing the matrix D in the definition (2.34) of $\widetilde{R}(\cdot)$ by the identity matrix I, we obtain $\mathrm{ED}_{\widetilde{\mathcal{A}}}(\cdot)$. Moreover, replacing the matrix D in the definition $s^2(x) = x'DD'x$ by I, we obtain $\|x\|^2$. With these correspondence between $(I, \mathrm{ED}_{\widetilde{\mathcal{A}}}(\cdot), \|\cdot\|^2)$ and $(D, R(\cdot), s^2(\cdot))$ we take the global effective dimension $\mathrm{ED}_{\widetilde{\mathcal{A}}}(\cdot)$ for the behavior of n-coördinates and take the local effective dimension $\mathrm{R}(\cdot)$ for the first three coördinates. Thus, in this sense, the matrix D defined in the beginning of Section 2.2 extracts the *information* on the first three coördinates from the diffusion matrix $\sigma(\cdot)$. $\Box$

Remark 2.6. Observe that $g(\cdot)$ of (2.19) can be written as

$$\frac{g(\cdot)}{\widetilde{Q}(\cdot)} = \frac{\widetilde{R}(\cdot)-2}{s^2(\cdot)} \quad \text{in } \mathbb{R}^n\setminus\mathcal{Z}\,.$$

Thus, $\widetilde{R}(\cdot)\ge 2$ is equivalent to $g(\cdot)\ge 0$. $\Box$

Remark 2.7. Since $\widetilde{A}(\cdot)$ is positive-definite and $\mathrm{rank}(D) = 2$, the matrix $D'\widetilde{A}(\cdot)D$ is non-negative-definite and the number of its non-zero eigenvalues is equal to $\mathrm{rank}(D'\widetilde{A}(\cdot)D) = 2$. This implies

$$\widetilde{R}(x) \ge \frac{\sum_{i=1}^3 \widetilde{\lambda}_i^D(x)}{\max_{1\le i\le 3}\widetilde{\lambda}_i^D(x)} > 1; \quad x\in\mathbb{R}^n\setminus\mathcal{Z}\,, \tag{2.37}$$

where $\{\widetilde{\lambda}_i^D(\cdot), i = 1, 2, 3\}$ are the eigenvalues of the (3×3) matrix $D'\widetilde{A}(\cdot)D$. On the other hand, another upper bound for $\widetilde{R}(\cdot)$ is given by

$$\widetilde{R}(x) \le \frac{\text{trace } (D'\widetilde{A}(x)D)}{3 \min_{1 \le i \le n} \widetilde{\lambda}_i(x)}; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,, \tag{2.38}$$

where $\{\widetilde{\lambda}_i(\cdot), 1 \le i \le n\}$ are the eigenvalues of $\widetilde{A}(\cdot)$. In fact, we can verify $DD'DD' = 3DD'$, $\{x \in \mathbb{R}^n : DD'x = 0\} = \mathcal{Z}$, and so if $DD'x \ne 0 \in \mathbb{R}^n$,

$$\min_{1 \le i \le n} \widetilde{\lambda}_i(x) \le \frac{x'DD'\widetilde{A}(x)DD'x}{x'DD'DD'x} = \frac{\widetilde{Q}(x)}{3} = \frac{\text{trace } (D'\widetilde{A}(x)D)}{3\widetilde{R}(x)}\,;$$

this gives the upper bound (2.38) for $\widetilde{R}(\cdot)$ above. □

By choosing $B(\cdot)$ and $F(\cdot)$ as in (2.36), we obtain from (2.31)- (2.33) that

$$\begin{aligned} \widetilde{\mathcal{A}}v(s(\cdot)) &\ge \frac{1}{2}\widetilde{Q}(\cdot)[v''(s(\cdot)) + B(s(\cdot))v'(s(\cdot))] = \frac{1}{2}\widetilde{Q}(\cdot)F(s(\cdot)) \\ &\ge |g_r(\cdot)| = |\widetilde{\mathcal{A}}u_{k,p,r}(\cdot)| \quad \text{ in } \mathcal{D}_k\,. \end{aligned} \tag{2.39}$$

Applying the weak maximum principle in the bounded domain $\mathcal{E}_{k,p}$ to the inequality (2.39), i.e., $-\widetilde{\mathcal{A}}v(s(\cdot)) \le \widetilde{\mathcal{A}}u(\cdot) \le \widetilde{\mathcal{A}}v(s(\cdot))$ for the ϕ-sequence $\{\, u_{k,p,r}(\cdot)\,\}$ with the boundary condition $u_{k,p,r}|_{\partial\mathcal{E}_{k,p}} = 0$, we obtain

$$\begin{aligned} v(s(\cdot)) - u_{k,p,r}(\cdot) &\le \max_{y \in \partial\mathcal{E}_{k,p}} [v(s(y)) - u_{k,p,r}(y)] = \max_{y \in \partial\mathcal{E}_{k,p}} v(s(y))\,, \\ v(s(\cdot)) + u_{k,p,r}(\cdot) &\le \max_{y \in \partial\mathcal{E}_{k,p}} [v(s(y)) + u_{k,p,r}(y)] = \max_{y \in \partial\mathcal{E}_{k,p}} v(s(y))\,, \end{aligned}$$

thus $|u_{k,p,r}(x)| \le \max_{y \in \partial\mathcal{E}_{k,p}} v(s(y)) - v(s(x))\,,\ x \in \mathcal{E}_{k,p}\,.$

It follows from Lemma 2.3 that $v'(s) \ge 0$, $v(a) = 0$ for $k^{-k} = a \le s \le b = k$ and hence the maximum of $v(\cdot)$ on the interval $[a, b]$ is attained at $b = k$. Therefore, we obtain for every $x \in \mathcal{E}_{k,p}\,,\ p \ge k+1\,,\ k \ge k_0$ the bound

$$|u_{k,p,r}(x)| \le \max_{y \in \partial\mathcal{E}_{k,p}} v(s(y)) - v(s(x)) = \int_{s(x)}^{k} v'(t)d\,t\,. \tag{2.40}$$

Solving the second-order ordinary differential equation (2.32) with the specification of (2.36) by the substitution method, we obtain the same bound on $u_{k,p,r}(\cdot)$ *irrespectively* of the truncation parameters (p, r). Since the solution $u_{k,r}(\cdot)$ is the limit of the ϕ-sequence $u_{k,p,r}(\cdot)$ as $p \to \infty$, the same bound works for $u_{k,r}(\cdot)$ as well.

We are now in a position to verify the asymptotic property (2.21); the proof of the following result is in subsection 5.1 of the Appendix.

Lemma 2.4. *Suppose that the conditions* (2.35) *on both the global and the local effective dimensions are satisfied. With the constants* $c_1 > 1$, $c_2 > 0$ *chosen as in* (2.36), *the solution* $u_{k,r}(\cdot)$ *to the Dirichlet problem* (2.20) *satisfies*

$$\sup_{r \ge 2} |u_{k,r}(x_0)| < \frac{c_2}{c_1 - 1} \log\left(\frac{k}{s(x_0)}\right) + \frac{c_2\, k^{-(c_1-1)k}}{(c_1-1)^2}\left(\frac{1}{k^{(c_1-1)}} - \frac{1}{(s(x_0))^{(c_1-1)}}\right); \quad k \ge k_0\,. \tag{2.41}$$

Remark 2.8. We may replace the condition (2.35) on the local effective dimension $\widetilde{R}(\cdot)$ by one of the following weaker conditions only on a neighborhood of the zero-set $\mathcal{Z}$ of $s(\cdot)$: namely, that there exist some constant $\delta_0 > 0$ such that

$$\inf_{0 < s(x) \le \delta_0} \widetilde{R}(x) \; > \; 2 \tag{2.42}$$

or

$$\widetilde{R}(x) \; = \; 2 \quad \text{whenever } 0 < s(x) \le \delta_0\,. \tag{2.43}$$

If (2.42) holds, then we may modify the continuous function $B(\cdot)$ thanks to (2.37) in such a way that

$$B(s) = \begin{cases} c_1/s; \quad 0 < s \le \delta_1\,, \\ \text{linear interpolation in } \delta_1 \le s \le \delta_0\,, \\ c_3/s; \quad \delta_0 \le s < \infty\,, \end{cases}$$

for some constants

$$c_1 = \inf_{0 < s(x) \le \delta_0} \widetilde{R}(x) - 1 > 1\,, \quad c_3 = \left(\inf_{s(x) \ge \delta_0} \widetilde{R}(x) - 1\right) \vee c_1 > 0\,,$$

with $0 < \delta_1 < \delta_0$, and obtain a similar inequality for $u_{k,r}(\cdot)$.

If (2.43) holds, then we may choose $B(\cdot)$ and $F(\cdot)$ with

$$B(s) := \begin{cases} 0; \quad 0 < s \le \delta_1\,, \\ \text{linear interpolation in } \delta_1 \le s \le \delta_0\,, \\ c_3/s; \quad \delta_0 \le s < \infty\,, \end{cases}$$

and

$$F(s) := \begin{cases} 0; \quad 0 < s \le \delta_1\,, \\ \text{linear interpolation in } \delta_1 \le s \le \delta_0\,, \\ c_2/s^2\,; \quad \delta_0 \le s < \infty\,. \end{cases}$$

□

2.3.4. *Approximation.* In this subsection we discuss how to approximate the non-smooth $\mathcal{A}$, $A(\cdot)$ and $f(\cdot)$ of (2.2), (2.3) and (2.10), respectively, by the smooth $\widetilde{\mathcal{A}}$, $\widetilde{A}(\cdot)$ and $g(\cdot)$ as in (2.17), (2.18) and (2.19), and apply the results from the previous subsection. Recall the definition of the effective dimension in (2.28) and define the local effective dimension $R(\cdot)$ for the diffusion $X(\cdot)$ by

$$R(x) := \frac{\text{trace}(D'A(x)D) \cdot x'DD'x}{x'DD'A(x)DD'x}\,; \quad x \in \mathbb{R}^n \tag{2.44}$$

by analogy with (2.34). Throughout this subsection we shall assume that the effective dimension $\mathrm{ED}_{\mathcal{A}}(\cdot)$ and the local effective dimension $R(\cdot)$ satisfy

$$\inf_{x\in\mathbb{R}^n\setminus\{0\}} \mathrm{ED}_{\mathcal{A}}(x) > 2\,, \qquad \inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R(x) > 2\,. \tag{2.45}$$

Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be the n-dimensional standard Gaussian density function, with $\int_{\mathbb{R}^n}\rho(x)dx = 1$ and $|\rho(x)| \le C(1+\|x\|)^{-n-c}$, $\forall\, x\in\mathbb{R}^n$ for some constants $C>0$ and $c>0$. Define $\rho_\ell(x) = \ell^n\rho(\ell x)$ for $\ell \ge 1$ and $x\in\mathbb{R}^n$. We obtain the approximating sequences by the convolution:

$$a_{ij}^{(\ell)}(x) := \rho_\ell * a_{ij}(x) := \int_{\mathbb{R}^n}\rho_\ell(x-z)a_{ij}(z)d\,z\,; \quad x\in\mathbb{R}^n\,. \tag{2.46}$$

By analogy with (2.17), (2.18), we define the sequence of matrix-valued functions $A^{(\ell)}(\cdot) := \{a_{ij}^{(\ell)}(\cdot)\}_{1\le i,j\le n}$ for $\ell\ge 1$, and from this the corresponding sequence of infinitesimal generators $\mathcal{A}^{(\ell)}$; the effective dimension $\mathrm{ED}_{\mathcal{A}^{(\ell)}}(\cdot)$ on $\mathbb{R}^n\setminus\{0\}$ as in (2.28); the test-function $f^{(\ell)}(\cdot)$, along with its truncation $f_r^{(\ell)}(\cdot) = f^{(\ell)}(\cdot)\zeta_r(\cdot)$ for $r\ge 2$; and finally the local effective dimension $R^{(\ell)}(\cdot)$ in $\mathbb{R}^n\setminus\mathcal{Z}$, for $l\ge 1$ as in (2.34). The proof of the following result is in the Appendix.

Lemma 2.5. *Under the the assumption (2.45) and the positive-definiteness of $\sigma_\nu(\cdot)$ defined in (1.3) for $1\le\nu\le m$, the approximating sequences satisfy*

$$y'A^{(\ell)}(x)y \;\ge\; \sup_{z\in\mathbb{R}^n}\left(\min_{\substack{1\le i\le n\\ 1\le\nu\le m}}\lambda_{i\nu}(z)\right)\|y\|^2\,; \quad x,y\in\mathbb{R}^n\,,$$

and

$$\inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R^{(\ell)}(x) \;\ge\; \inf_{x\in\mathbb{R}^n\setminus\mathcal{Z}} R(x) > 2\,,$$
$$\inf_{x\in\mathbb{R}^n\setminus\{0\}} ED_{\mathcal{A}^{(\ell)}}(x) \ge \inf_{x\in\mathbb{R}^n\setminus\{0\}} ED_{\mathcal{A}}(x) > 2\,; \quad \ell\ge 1\,,$$

where $\{\lambda_{i\nu}(\cdot)\,,1\le i\le n\}$ are the eigenvalues of $\sigma_\nu(\cdot)\sigma'_\nu(\cdot)$ for $1\le\nu\le m$.

For $k\ge k_0$, $\ell\ge 1$, and $r\ge 2$, consider the sequence of Dirichlet problems

$$\begin{aligned}\mathcal{A}^{(\ell)}u_{k,\ell,r}(\cdot) + f_r^{(\ell)}(\cdot) = 0 \;\text{ in }\; \mathcal{D}_k\,,\; u_{k,\ell,r}(\cdot) = 0 \;\text{ on }\; \partial\mathcal{D}_k\,,\\ \lim_{|x|\to\infty} u_{k,\ell,r}(x) = 0\,.\end{aligned} \tag{2.47}$$

It follows from Lemma 2.5 that we can choose $c_1 > 1$ and c_2 independent of ℓ as in (2.36). From the previous subsections we know that the solution $u_{k,\ell,r}(\cdot)$ exists and behaves asymptotically as in (2.41).

Lemma 2.6. *Under the assumption (2.45) and the positive-definiteness of $\sigma_\nu(\cdot)$ defined in (1.3) for $1\le\nu\le m$, there exist constants $c_1>1, c_2>0$ such that the solution $u_{k,\ell,r}(\cdot)$ to the Dirichlet problem (2.47) exists and satisfies*

$$\mathbb{E}_{x_0}\left[\int_0^{S_k\wedge T_k} f_r^{(\ell)}(X(t))d\,t\right] = u_{k,\ell,r}(x_0)\,; \quad k\ge k_0,\ \ell\ge 1,\ r\ge 2\,,$$

as well as

$$\sup_{r \ge 2} |u_{k,\ell,r}(x_0)| < \frac{c_2}{c_1 - 1} \log\left(\frac{k}{s(x_0)}\right) + \frac{k^{-(c_1-1)k} c_2}{(c_1-1)^2} \left(\frac{1}{k^{(c_1-1)}} - \frac{1}{(s(x_0))^{(c_1-1)}}\right); \quad k \ge k_0,\, \ell \ge 1\,.$$

Consider now the approximating sequence (X, W), $(\Omega, \mathcal{F}, \mathbb{P}^{(\ell)})$, $\{\mathcal{F}_t\}$ $(\ell \in \mathbb{N})$ of the weak solution (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}$. Let $\mathbb{P}^{(\ell)}_{x_0}$ be the probability measure induced by the martingale problem corresponding to the approximating elliptic operator $\mathcal{A}^{(\ell)}$ with initial condition $X(0) = x_0$. We have the following Alexandroff-type estimate.

Lemma 2.7 (Exercise 7.3.2 of (18))**.** *If the support of the $C_0(\mathbb{R}^n)-$function $\mathfrak{g} : \mathbb{R}^n \to \mathbb{R}$ is bounded, then for all $p > n$, $K > 0$ we have*

$$\left| \mathbb{E}_{x_0} \int_0^K \mathfrak{g}(X(t))dt \right| \le C \,||\, \mathfrak{g} \,||_{\mathbb{L}^p(\mathbb{R}^n)}$$

for some constant C that depends only on p, K and $\inf_x \min_{1 \le i \le n, 1 \le \nu \le m} \lambda_{i\nu}(x)$, a lower bound on the eigenvalues of $\sigma_\nu(\cdot)$, $\nu = 1, \dots, m$.

Using this result we obtain the tightness of the sequence of probability measures $\{\mathbb{P}^{(\ell)}_{x_0}\}_{\ell \ge 1}$, and its vague convergence to $\mathbb{P}_{x_0}$. Applying the continuous mapping theorem to the integral, and deriving estimates similar to those in the previous section, we obtain the following.

Lemma 2.8.

$$\lim_{\ell \to \infty} \mathbb{E}^{(\ell)}_{x_0}\left[\int_0^{T_k \wedge S_k} f^{(\ell)}_r (X(t))dt\right] = \mathbb{E}_{x_0}\left[\int_0^{T_k \wedge S_k} f_r(X(t))dt\right], \tag{2.48}$$

and hence

$$\begin{aligned} \mathbb{E}_{x_0}\left[\int_0^{T_k \wedge S_k} f(X(t))dt\right] &= \lim_{r \to \infty} \lim_{\ell \to \infty} \mathbb{E}^{(\ell)}_{x_0}\left[\int_0^{T_k \wedge S_k} f^{(\ell)}_r(X(t))dt\right] \\ &= \lim_{r \to \infty} \lim_{\ell \to \infty} u_{k,\ell,r}(x_0)\,. \end{aligned} \tag{2.49}$$

The proof is in subsection 5.3 of the Appendix. It follows from Lemmata 2.6 and 2.8 that

$$\lim_{k \to \infty} \frac{\mathbb{E}_{x_0}\left[\int_0^{T_k \wedge S_k} f(X(t))dt\right]}{k \log k} = \lim_{k \to \infty} \lim_{r \to \infty} \lim_{\ell \to \infty} \left(\frac{u_{k,\ell,r}(x_0)}{k \log k}\right) = 0\,, \tag{2.50}$$

which answers the question raised in Problem 2(ii) of Section 2.3.

2.3.5. *Results.* With this estimate at hand, we are ready to state the first main result, on the absence of triple collisions:

Proposition 1. *Suppose that the matrices $\sigma_\nu(\cdot)$, $\nu = 1, \cdots, m$ in (1.3) are uniformly bounded and positive-definite, and satisfy the conditions of (2.45).*

Then for the weak solution $X(\cdot)$ to (2.1) starting at any $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$, we have

$$\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t) \ \text{ for some } t > 0\big) = 0\,. \tag{2.51}$$

The proof is in Section 5.4, and a class of examples satisfying (2.45) is given in Remark 2.9 and Section 5.6 below.

On the other hand, regarding the presence of triple collisions we have the following.

Proposition 2. *Suppose that the matrices $\sigma_\nu(\cdot)$, $\nu = 1, \cdots, m$ in (1.3) are uniformly bounded and positive-definite, and that $R(\cdot) \le 2 - \eta$ holds in $\mathbb{R}^n \setminus \mathcal{Z}$ for some $\eta \in (0,2)$. Then the weak solution $X(\cdot)$ to (2.1) starting at any $x_0 \in \mathbb{R}^n$ satisfies*

$$\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = X_3(t), \ \text{ for infinitely many } t > 0\big) \;=\; 1\,. \tag{2.52}$$

Remark 2.9. Proposition 1 holds under several circumstances. For example, take $n = 3$ and fix the elements $a_{11}(\cdot) = a_{22}(\cdot) = a_{33}(\cdot) \equiv 1$ of the symmetric matrix $A(\cdot) = \sigma\sigma'(\cdot)$ in (2.3) and choose the other parameters by

$$\begin{aligned} a_{12}(x) = a_{21}(x) &:= \alpha_{1+}\mathbf{1}_{\mathcal{R}_{1+}}(x) + \alpha_{1-}\mathbf{1}_{\mathcal{R}_{1-}}(x)\,, \\ a_{23}(x) = a_{32}(x) &:= \alpha_{2+}\mathbf{1}_{\mathcal{R}_{2+}}(x) + \alpha_{2-}\mathbf{1}_{\mathcal{R}_{2-}}(x)\,, \\ a_{31}(x) = a_{13}(x) &:= \alpha_{3+}\mathbf{1}_{\mathcal{R}_{3+}}(x) + \alpha_{3-}\mathbf{1}_{\mathcal{R}_{3-}}(x)\,; \quad x \in \mathbb{R}^3\,, \end{aligned} \tag{2.53}$$

where $\mathcal{R}_{i\pm}$, $i = 1, 2, 3$ are subsets of $\mathbb{R}^3$ defined by

$$\begin{aligned} &\mathcal{R}_{1+} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) > 0\,\}\,, \quad \mathcal{R}_{2+} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = 0\,, \mathfrak{f}_2(x) > 0\,\}\,, \\ &\mathcal{R}_{1-} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) < 0\,\}\,, \quad \mathcal{R}_{2-} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = 0\,, \mathfrak{f}_2(x) < 0\,\}\,, \\ &\mathcal{R}_{3+} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = 0\,, \mathfrak{f}_3(x) > 0\,\}\,, \\ &\mathcal{R}_{3-} := \{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = 0\,, \mathfrak{f}_3(x) < 0\,\}\,, \\ &\mathfrak{f}_1(x) := [x_3 - x_1 - (-2 + \sqrt{3})(x_2 - x_3)] \cdot [x_3 - x_1 - (-2 - \sqrt{3})(x_2 - x_3)]\,, \\ &\mathfrak{f}_2(x) := [x_2 - x_3 - (-2 + \sqrt{3})(x_1 - x_2)] \cdot [x_2 - x_3 - (-2 - \sqrt{3})(x_1 - x_2)]\,, \\ &\mathfrak{f}_3(x) := [x_1 - x_2 - (-2 + \sqrt{3})(x_3 - x_1)] \cdot [x_1 - x_2 - (-2 - \sqrt{3})(x_3 - x_1)]\,, \end{aligned}$$

for $x \in \mathbb{R}^3$ with the six constants $\alpha_{i\pm}$ satisfying $0 < \alpha_{i+} \le 1/2$, $-1/2 \le \alpha_{i-} < 0$ for $i = 1, 2, 3$. Note that the zero set $\mathcal{Z}$ defined in (2.6) is $\{\, x \in \mathbb{R}^3 \,:\, \mathfrak{f}_1(x) = \mathfrak{f}_2(x) = \mathfrak{f}_3(x) = 0\,\}$. Thus, we split the region $\mathbb{R}^3 \setminus \mathcal{Z}$ into six disjoint polyhedral regions $\mathcal{R}_{i\pm}$, $i = 1, 2, 3$. See Figure 2. In Section 5.6 we show that this is an example. □

Remark 2.10. Note that in the example of Bass & Pardoux (3), *all n Brownian particles collide at the origin at the same time, infinitely often:* for $x_0 \in \mathbb{R}^n$ we have

$$\mathbb{P}_{x_0}\big(X_1(t) = X_2(t) = \cdots = X_n(t) = 0\,, \ \text{ for infinitely many } t > 0\big) \;=\; 1\,.$$

This is a special case of Proposition 2. Under the setting (2.26) we may show that $R(\cdot) \le 2-\eta$ for some $\eta > 0$. In fact, it follows from (2.27) that there exists a constant $\xi \in (0,1/2)$ such that

$$\frac{|\langle\, D'x\,,D'y_\nu\,\rangle|^2}{\|D'x\|^2\cdot\|D'y_\nu\|^2} \ge 1-\xi\,; \quad x\,,y_\nu \in \mathbb{R}^n\setminus\mathcal{Z}\,, \tag{2.54}$$

and hence, we obtain

$$R(x) = \sum_{\nu=1}^{m} \frac{\|D'y_\nu\|^2 + 6\varepsilon^2}{\frac{|\langle\, D'x,D'y_\nu\,\rangle|^2}{\|D'x\|^2} + 3\varepsilon^2} 1_{\mathcal{R}_\nu}(x) \le \frac{1}{1-\xi} < 2\,; \quad x\in\mathbb{R}^n\setminus\mathcal{Z}\,.$$

□

Remark 2.11. Friedman (7) established theorems on the non-attainability of lower dimensional manifolds by non-degenerate diffusions. Let $\mathcal{M}$ be a closed $k-$dimensional C^2-manifold in $\mathbb{R}^n$, with $k \le n-1$. At each point $x\in\mathcal{M}$, let $N_{k+i}(x)$ form a set of linearly independent vectors in $\mathbb{R}^n$ which are *normal to* $\mathcal{M}$ and x. Consider the $(n-k)\times(n-k)$ matrix $\alpha(x) := (\alpha_{ij}(x))$ where

$$\alpha_{ij}(x) = \langle A(x)N_{k+i}(x), N_{k+j}(x)\rangle\,; \quad 1\le i,j\le n-k\,,\ x\in\mathcal{M}\,.$$

Roughly speaking, the strong solution of (1.1) under linear growth condition and Lipschitz condition on the coëfficients cannot attain $\mathcal{M}$, if rank $(\alpha(x)) \ge 2$ holds for all $x\in\mathcal{M}$. The rank indicates how wide the orthogonal complement of $\mathcal{M}$ is. If the rank is large, the manifold $\mathcal{M}$ is too *thin* to be attained. The fundamental lemma there is based on the solution $u(\cdot)$ of partial differential inequality $\mathcal{A}u(\cdot) \le \mu u(\cdot)$ for some $\mu \ge 0$, outside but near $\mathcal{M}$ with $\lim_{\mathrm{dist}(x,\mathcal{M})\to\infty} u(x) = \infty$, which is different from our treatment in the previous sections.

Ramasubramanian (16) (17) examined the recurrence and transience of projections of weak solution to (1.1) for *continuous* diffusion coëfficient $\sigma(\cdot)$, showing that any $(n-2)-$dimensional C^2-manifold is not hit. The integral test developed there has the integrand similar to the effective dimension studied in (14), as pointed out by M. Cranston in MathSciNet Mathematical Reviews on the Web.

The above Propositions 1 and 2 are complementary with those previous general results, since the coëfficients here are allowed to be *piece-wise continuous*, however, they depend on the typical geometric characteristic on the manifold $\mathcal{Z}$ we are interested in. Since the manifold of interest in this work is the zero set $\mathcal{Z}$ of the function $s(\cdot)$, the projection $s(X(\cdot))$ of the process and the corresponding effective dimensions $\mathrm{ED}_{\mathcal{A}}(\cdot)$, $R(\cdot)$ are studied. □

Remark 2.12. As V. Papathanakos first pointed out, the conditions in Proposition 1 and Proposition 2 are disjoint, and there is a "gray" zone of sets of coëfficients which satisfy neither of the conditions. This is because we chose a particular barrier function $v(\cdot)$, when replacing n-dimensional problem by the solvable *one*-dimensional problem. In order to look at a finer structure, we discuss a special case in the next section by reducing it to a *two*-dimensional problem. This follows a suggestion of A. Banner. □

3. A Second Approach

In this section we discuss a consequence of uniqueness in the sense of probability distribution. The definition of weak uniqueness allows the existence of two processes X and $\widetilde{X}$ which have different path properties. We shall construct a weak solution which has "*no triple collision*".

3.1. **Ranked process.** Given a vector process $X(\cdot) := \{(X_1(t), \ldots, X_n(t)\,; 0 \le t < \infty\}$, we define the vector $X_{(\cdot)} := \{(X_{(1)}(t), \ldots, X_{(n)}(t))\,; 0 \le t < \infty\}$ of *ranked processes,* ordered from largest to smallest, by

(3.1)
$$X_{(k)}(t) := \max_{1 \le i_1 < \ldots < i_k \le n} \Big(\min \big(X_{i_1}(t), \ldots, X_{i_k}(t)\big)\Big)\,; \quad 0 \le t < \infty\,,\ k = 1, \cdots, n\,.$$

If, for every $j = 1, \ldots, n-2$, the two-dimensional process

$$\big(Y_j(\cdot),\, Y_{j+1}(\cdot)\big)' := \big(X_{(j)}(\cdot) - X_{(j+1)}(\cdot),\, X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot)\big)' \tag{3.2}$$

obtained by looking at the "gaps" among the three adjacent ranked processes

$$X_{(j)}(\cdot),\ X_{(j+1)}(\cdot),\ X_{(j+2)}(\cdot)\,,$$

never reaches the corner $(0,0)'$ of $\mathbb{R}^2$ almost surely, then the process $X(\cdot)$ satisfies

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t)\,,\ \text{for some } (i,j,k)\,,\ t > 0\big) = 0 \tag{3.3}$$

for $x_0 \in \mathbb{R}^n \setminus \mathcal{Z}$. On the other hand, if for some $j = 1, \ldots, n-2$ the vector of gaps $(\,X_{(j)}(\cdot) - X_{(j+1)}(\cdot),\, X_{(j+1)}(\cdot) - X_{(j+2)}(\cdot)\,)'$ does reach the corner $(0,0)'$ of $\mathbb{R}^2$ almost surely, then we have

$$\mathbb{P}_{x_0}\big(X_i(t) = X_j(t) = X_k(t),\ \text{for some } (i,j,k),\ t > 0\big) \ = \ 1\,; \quad x_0 \in \mathbb{R}^n.$$

Thus, we study the ranked process $X_{(\cdot)}$ and its adjacent differences. In the following we use the parametric result of Varadhan & Williams (19) on Brownian motion in a two-dimensional wedge with oblique reflection at the boundary, and the result of Williams (20) on Brownian motion with reflection along the faces of a polyhedral domain.

3.2. **Reflected Brownian motion.** Let $(e_1, \ldots, e_{n-1})$ be the orthonormal basis of $\mathbb{R}^{n-1}$, where the e_k is $(n-1)-$dimensional vector whose k-th component is equal to one and all other components are equal to zero, for $k = 1, \ldots n-1$ and $n \ge 3$. We shall define Brownian motion with reflection on the faces of the non-negative orthant

$$\mathfrak{S} := \mathbb{R}^{n-1}_+ = \left\{ \sum_{k=1}^{n-1} x_k e_k \,:\, x_1 \ge 0, \ldots,\, x_{n-1} \ge 0 \right\}\,,$$

whose $(n-2)-$dimensional faces $\mathfrak{F}_1, \ldots, \mathfrak{F}_{n-1}$ are given as

$$\mathfrak{F}_i := \left\{ \sum_{k=1}^{n-1} x_k e_k : x_k \ge 0 \text{ for } k = 1, \ldots, n-1,\, x_i = 0 \right\}; \quad 1 \le i \le n-1 \, .$$

Let us denote the $(n-3)-$dimensional faces of intersection by $\mathfrak{F}^o_{ij} := \mathfrak{F}_i \cap \mathfrak{F}_j$ for $1 \le i < j \le n-1$ and their union by $\mathfrak{F}^o := \cup_{1 \le i < j \le n-1} \mathfrak{F}^o_{ij}$.

For $n \ge 3$, we shall define the $(n-1)-$dimensional reflected Brownian motion $Y(\cdot) := \{(Y_1(t), \ldots Y_{n-1}(t))\, ;\ t \ge 0\}$ on the orthant $\mathbb{R}^{n-1}_+$ with zero drift, constant $((n-1) \times (n-1))$ constant variance/covariance matrix $\mathfrak{A} := \Sigma\Sigma'$, and reflection along the faces of the boundary along constant directions, by

$$\begin{aligned} Y(t) &= Y(0) + \Sigma B(t) + \mathfrak{R} L(t)\, ; \quad 0 \le t < \infty \, , \\ Y(0) &\in \mathbb{R}^{n-1}_+ \setminus \mathfrak{F}^o \, . \end{aligned} \tag{3.4}$$

Here, $\{B(t); 0 \le t < \infty\}$ is $(n-1)-$dimensional standard Brownian motion starting at the origin of $\mathbb{R}^{n-1}$. The $((n-1) \times (n-1))$ reflection matrix $\mathfrak{R}$ has all its diagonal elements equal to one, and spectral radius strictly smaller than one. Finally the components of the $(n-1)-$dimensional process $L(t) := (L_1(t), \ldots, L_{n-1}(t));\ 0 \le t < \infty$ are adapted, non-decreasing, continuous and satisfy $\int_0^\infty Y_i(t)\, dL_i(t) = 0$ (that is, $L_i(\cdot)$ is flat off the set $\{t \ge 0 : Y_i(t) = 0\}$) almost surely, for each $i = 1, \ldots n-1$. Note that, if $Y(t)$ lies on $\mathfrak{F}^o_{ij} = \mathfrak{F}_i \cap \mathfrak{F}_j$, then $Y_i(t) = Y_j(t) = 0$ for $1 \le i \ne j \le n-1$.

See Harrison & Reiman (9) for the pathwise construction through of this multi-dimensional Skorohod reflection problem.

3.2.1. *Rotation and Rescaling.* Assume that the constant matrix $\mathfrak{A}$ is positive-definite. Let U be the unitary matrix whose columns are the orthonormal eigenvectors of the covariance matrix $\mathfrak{A} = \Sigma\Sigma'$, and let $\mathfrak{L}$ be the corresponding diagonal matrix of eigenvalues such that $\mathfrak{L} = U' \mathfrak{A}\, U$. Note that all the eigenvalues of $\mathfrak{A}$ are positive. Define $\widetilde{Y}(\cdot) := \mathfrak{L}^{-1/2} U Y(\cdot)$. By this rotation and rescaling, we obtain

$$\widetilde{Y}(t) = \widetilde{Y}(0) + \widetilde{B}(t) + \mathfrak{L}^{-1/2} U \mathfrak{R} L(t)\, ; \quad 0 \le t < \infty \tag{3.5}$$

from (3.4), where $\widetilde{B}(\cdot) := \mathfrak{L}^{-1/2} U \Sigma B(\cdot)$ is another standard $(n-1)-$dimensional Brownian motion. We may regard $\widetilde{Y}(\cdot)$ as reflected Brownian motion in a new state-space $\widetilde{\mathfrak{S}} := \mathfrak{L}^{-1/2} U \mathbb{R}^{n-1}_+$. The transformed reflection matrix $\widetilde{\mathfrak{R}} := \mathfrak{L}^{-1/2} U \mathfrak{R}$ can be written as

$$\widetilde{\mathfrak{R}} \,=\, \mathfrak{L}^{-1/2} U \mathfrak{R} \,=\, (\widetilde{\mathfrak{N}} + \widetilde{\mathfrak{Q}}) \mathfrak{C} \,=\, (\widetilde{\mathfrak{r}}_1, \ldots, \widetilde{\mathfrak{r}}_{n-1})\, , \tag{3.6}$$

where

$$\begin{aligned} &\mathfrak{C} := \mathfrak{D}^{-1/2}\, , \quad \mathfrak{D} := diag(\mathfrak{A})\, , \quad \widetilde{\mathfrak{N}} := \mathfrak{L}^{1/2} U \mathfrak{C} \,\equiv\, (\widetilde{\mathfrak{n}}_1, \ldots, \widetilde{\mathfrak{n}}_{n-1})\, , \\ &\widetilde{\mathfrak{Q}} := \mathfrak{L}^{-1/2} U \mathfrak{R} \mathfrak{C}^{-1} - \widetilde{\mathfrak{N}} \,\equiv\, (\widetilde{\mathfrak{q}}_1, \ldots, \widetilde{\mathfrak{q}}_{n-1})\, . \end{aligned} \tag{3.7}$$

Here $\mathfrak{D} = diag(\mathfrak{A})$ is the $((n-1) \times (n-1))$ diagonal matrix with the same diagonal elements as those of $\mathfrak{A} = \Sigma\Sigma'$ (the variances). The constant vectors $\widetilde{\mathfrak{r}}_i, \widetilde{\mathfrak{q}}_i, \widetilde{\mathfrak{n}}_i$, $i = 1, \ldots, n-1$ are $((n-1) \times 1)$ column vectors.

Since U is an orthonormal matrix which rotates the state space $\mathfrak{S} = \mathbb{R}_+^{n-1}$, and $\mathfrak{L}^{1/2}$ is a diagonal matrix which changes the scale in the positive direction, the new state-space $\widetilde{\mathfrak{S}}$ is an $(n-1)-$dimensional polyhedron whose i-th face $\widetilde{\mathfrak{F}}_i := \mathfrak{L}^{-1/2} U \mathfrak{F}_i$ has dimension $(n-2)$, for $i = 1, \ldots, n-1$.

Note that $diag(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}}) = 0$ and $diag(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{N}}) = I$, that is, $\widetilde{\mathfrak{n}}_i$ and $\widetilde{\mathfrak{q}}_i$ are orthogonal and $\widetilde{\mathfrak{n}}_i$ is a unit vector, i.e., $\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{q}}_i = 0$ and $\widetilde{\mathfrak{n}}_i'\widetilde{\mathfrak{n}}_i = 1$ for $i = 1, \ldots, n-1$. Also note that $\widetilde{\mathfrak{n}}_i$ is the inward unit normal to the new i-th face $\widetilde{\mathfrak{F}}_i$ on which the continuous, non-decreasing process $L_i(\cdot)$ actually increases, for $i = 1, \ldots, n-1$. The i-th face $\widetilde{\mathfrak{F}}_i$ can be written as $\{x \in \widetilde{\mathfrak{S}} : \widetilde{\mathfrak{n}}_i' x = \mathfrak{b}_i\}$ for some $\mathfrak{b}_i \in \mathbb{R}$, for $i = 1, \ldots, n-1$.

Moreover, the i-th column $\widetilde{\mathfrak{r}}_i$ of the new reflection matrix $\widetilde{\mathfrak{R}}$ is decomposed into components that are normal and tangential to $\widetilde{\mathfrak{F}}_i$, i.e., $\widetilde{\mathfrak{r}}_i = \mathfrak{C}_{ii}(\widetilde{\mathfrak{n}}_i + \widetilde{\mathfrak{q}}_i)$ for $i = 1, \ldots n-1$, where $\mathfrak{C}_{ii}$ is the (i,i)-element of the diagonal matrix $\mathfrak{C}$. Note that, since the matrix $\mathfrak{L}^{-1/2}U$ of the transformation is invertible, we obtain

$$\widetilde{Y}(\cdot) \in \widetilde{\mathfrak{F}}_{ij}^{o} := \widetilde{\mathfrak{F}}_i \cap \widetilde{\mathfrak{F}}_j \iff Y(\cdot) \in \mathfrak{F}_{ij}^{o}\,; \quad 1 \le i < j \le n-1\,. \tag{3.8}$$

Thus, it suffices to work on the transformed process $\widetilde{Y}(\cdot)$ to obtain (3.3) for $Y(\cdot)$ in (3.2).

3.3. **Attainability.** With (3.8) we consider, for $n = 3$ and $n > 3$ separately, the hitting times

$$\begin{aligned} \tau_{ij} &:= \inf\{t > 0 : Y(t) \in \mathfrak{F}_{ij}^{o}\} \\ &= \inf\{t > 0 : \widetilde{Y}(t) \in \widetilde{\mathfrak{F}}_{ij}^{o}\}\,; \quad 1 \le i \ne j \le n-1\,. \end{aligned} \tag{3.9}$$

First we look at the case $n = 3$, i.e., two-dimensional reflected Brownian motion and the hitting time τ_{12}. The directions of reflection $\widetilde{\mathfrak{r}}_1$ and $\widetilde{\mathfrak{r}}_2$ can be written in terms of angles. Note that the angle ξ of the two-dimensional wedge $\widetilde{\mathfrak{S}}$ is positive and smaller than π, since all the eigenvalues of $\mathfrak{A}$ are positive. Let θ_1 and θ_2 with $-\pi/2 < \theta_1, \theta_2 < \pi/2$ be the angles between $\widetilde{\mathfrak{n}}_1$ and $\widetilde{\mathfrak{r}}_2$ and between $\widetilde{\mathfrak{n}}_1$ and $\widetilde{\mathfrak{r}}_2$, respectively, measured in such a way that θ_1 is positive if and only if $\widetilde{\mathfrak{r}}_1$ points towards the corner with local coördinate $(0,0)'$ and similar for θ_2. See Figure 1 in the end of this paper.

Paraphrasing the result of Varadhan & Williams (19) for Brownian motion reflected on the *two*-dimensional wedge, we obtain the following dichotomous result on the relationship between the stopping time and the sum $\theta_i + \theta_j$ of angles of reflection directions, when $n-1 = 2$. We shall denote $\widetilde{\mathfrak{F}}^{o} := \mathfrak{L}^{-1/2}U\mathfrak{F}^{o} = \bigcup_{1 \le i < j \le n-1} \widetilde{\mathfrak{F}}_{ij}^{o}$.

Lemma 3.1. [Theorem 2.2 of (19)] *Suppose that* $\widetilde{Y}(0) = \tilde{y}_0 \in \widetilde{\mathfrak{S}} \setminus \widetilde{\mathfrak{F}}^o$. *If* $\beta := (\theta_1 + \theta_2)/\xi > 0$, *then we have* $\mathbb{P}(\tau_{12} < \infty) = 1$; *if, on the other hand,* $\beta \le 0$, *then we have* $\mathbb{P}(\tau_{12} < \infty) = 0$.

In terms of the reflection vectors $\widetilde{\mathfrak{n}}_1, \widetilde{\mathfrak{r}}_1$ and $\widetilde{\mathfrak{n}}_2, \widetilde{\mathfrak{r}}_2$, and with the aid of (3.8) we can cast this result as follows:

Lemma 3.2. *Suppose that* $Y(0) = y_0 \in \mathbb{R}^2 \setminus \mathfrak{F}^o$. *If* $\widetilde{\mathfrak{n}}_1' \widetilde{\mathfrak{q}}_2 + \widetilde{\mathfrak{n}}_2' \widetilde{\mathfrak{q}}_1 > 0$, *then we have* $\mathbb{P}(\tau_{12} < \infty) = 1$. *If, on the other hand,* $\widetilde{\mathfrak{n}}_1' \widetilde{\mathfrak{q}}_2 + \widetilde{\mathfrak{n}}_2' \widetilde{\mathfrak{q}}_1 \le 0$, *then we have* $\mathbb{P}(\tau_{12} < \infty) = 0$.

The proof is in Section 5.7.1.

We consider the general case $n > 3$ next. With (3.8) and Theorem 1.1 of Williams (20) we obtain the following Lemma, valid for $n \ge 3$.

Lemma 3.3. *Suppose that* $Y(0) = y \in \mathbb{R}_+^{n-1} \setminus \mathfrak{F}^o$ *and* $n \ge 3$, *and that the so-called* skew-symmetry condition

$$\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j' \widetilde{\mathfrak{q}}_i = 0\,; \quad 1 \le i < j \le n-1 \tag{3.10}$$

holds. Then we have

$$\mathbb{P}_y(\tau < \infty) = 0\,, \qquad \textit{where} \quad \tau := \inf\{t > 0 : Y(t) \in \mathfrak{F}^o\}\,.$$

Moreover, the components of adapted non-decreasing continuous process $L(\cdot)$ *defined in* (3.4) *are identified as the local times of one-dimensional processes at level zero:*

$$2L_i(t) = Y_i(t) - Y_i(0) - \int_0^t \mathit{sgn}(Y_i(s))\, dY_i(s)\,; \quad 0 \le t < \infty\,, \quad i = 1, \dots, n\,.$$

Remark 3.1. In the planar (two-dimensional) setting of Lemma 3.2, the skew-symmetry condition (3.10) takes a weaker form, that of an inequality. □

Using Lemmata 3.2 and 3.3, we obtain the following result on the absence of triple-collisions for a system of n one-dimensional Brownian particles interacting through their ranks. For this purpose, let us introduce a collection $\{Q_k^{(i)}\}_{1 \le i,k \le n}$ of polyhedral domains in $\mathbb{R}^n$, such that $\{Q_k^{(i)}\}_{1 \le i \le n}$ is partition $\mathbb{R}^n$ for each fixed k, and $\{Q_k^{(i)}\}_{1 \le k \le n}$ is partition $\mathbb{R}^n$ for each fixed i. The interpretation is as follows:

$$y = (y_1, \cdots, y_n)' \in Q_k^{(i)} \quad \text{means that } y_i \text{ is ranked } k\text{th among } y_1, \cdots, y_n\,,$$

with ties resolved by resorting to the smallest index for the highest rank, by analogy with (3.1).

Proposition 3. *For* $n \ge 3$, *consider the weak solution of the equation* (2.1) *with diffusion coëfficient* (1.3), *where* $\sigma(\cdot)$ *is the diagonal matrix*

$$\sigma(x) := \mathit{diag}\left(\sum_{k=1}^n \tilde{\sigma}_k 1_{Q_k^{(1)}}(x), \dots, \sum_{k=1}^n \tilde{\sigma}_k 1_{Q_k^{(n)}}(x) \right)\,; \quad x \in \mathbb{R}^n\,. \tag{3.11}$$

If the positive constants $\{\, \widetilde{\sigma}_k \,; 1 \le k \le n \,\}$ *satisfy the linear growth condition*

$$\widetilde{\sigma}_2^2 - \widetilde{\sigma}_1^2 \,=\, \widetilde{\sigma}_3^2 - \widetilde{\sigma}_2^2 \,=\, \cdots \,=\, \widetilde{\sigma}_n^2 - \widetilde{\sigma}_{n-1}^2 \,, \tag{3.12}$$

then (3.3) holds, i.e., there are no triple-collisions among the n *one-dimensional particles.*

If $n = 3$*, the weaker condition* $\widetilde{\sigma}_2^2 - \widetilde{\sigma}_1^2 \ge \widetilde{\sigma}_3^2 - \widetilde{\sigma}_2^2$ *is sufficient for the absence of triple collisions.*

Remark 3.2. This special structure (3.11) has been studied in the context of mathematical finance. Recent work on interacting particle systems by Pal & Pitman (15) clarifies the long-range behavior of the spacings between the arranged Brownian particles under the equal variance condition: $\widetilde{\sigma}_1 = \cdots = \widetilde{\sigma}_n$. The setting of systems with countably many particle is also studied there, and related work from the Physics literature on competing tagged particle systems is surveyed. □

In the next section we shall discuss some details of the resulting model, as an application of Lemma 3.3.

4. Application

4.1. **Atlas model for an Equity Market.** Let us recall the Atlas model

$$\begin{aligned} d\,X_i(t) &= \left(\sum_{k=1}^{n} g_k 1_{Q_k^{(i)}}(X(t)) + \gamma \right) d\,t + \sum_{k=1}^{n} \tilde{\sigma}_k 1_{Q_k^{(i)}}(X(t)) d\,W_i(t); \\ &\text{for } 1 \le i \le n,\ 0 \le t < \infty, \qquad (X_1(0), \ldots, X_n(0))' = x_0 \in \mathbb{R}^n \,, \end{aligned} \tag{4.1}$$

introduced by Fernholz (5) and studied by Banner, Fernholz & Karatzas (1). Here $X(\cdot) = (X_1(\cdot), \cdots, X_n(\cdot))'$ represents the vector of asset capitalizations in an equity market, and we are using the notation of Proposition 3. We assume that $\tilde{\sigma}_j > 0$ and g_j, $j = 1, \cdots, n$ are constants satisfying the conditions

$$g_1 < 0\,,\ g_1 + g_2 < 0\,, \cdots,\ g_1 + \cdots + g_{n-1} < 0\,,\ g_1 + \cdots + g_n = 0\,, \tag{4.2}$$

imposed to ensure that the resulting diffusion $X(\cdot)$ is ergodic.

The dynamics of (4.1) induce corresponding dynamics for the ranked processes $X_{(1)}(\cdot) \ge X_{(2)}(\cdot) \ge \cdots \ge X_{(n)}(\cdot)$ of (3.1). These involve the local times $\Lambda^{k,\ell}(\cdot) \equiv L^{X_{(k)} - X_{(\ell)}}(\cdot)$ for $1 \le k < \ell \le n$, where the notation $L^Y(\cdot)$ is used to signify the local time at the origin of a continuous semi-martingale $Y(\cdot)$. An increase in $\Lambda^{k,\ell}(\cdot)$ is due to, and signifies, a collision of $\ell - k + 1$ particles in the ranks k through ℓ. In general, when multiple collisions can occur, there are $(n-1)\,n/2$ such possible local times; all of them appear then in the dynamics of the ranked processes, in the manner of Banner & Ghomrasni (2).

Let $S_k(t) := \{ i \,:\, X_i(t) = X_{(k)}(t) \}$ be the set of indexes of processes which are kth ranked, and denote its cardinality by $N_k(t) := |S_k(t)|$ for $0 \le t < \infty$. Banner & Ghomrasni show in Theorem 2.3 of (2) that for any

$n-$dimensional continuous semi-martingale process $X(\cdot) = (X_1(\cdot), \ldots, X_n(\cdot))$, its ranked process $X_{(\cdot)}(\cdot)$ with components $X_{(k)}(t) = X_{p_t(k)}(t)$, $k = 1, \ldots, n$ can be written as

$$d\,X_{(k)}(t) = (N_k(t))^{-1}\Big[\sum_{i=1}^{n} \mathbf{1}_{\{X_{(k)}(t)=X_i(t)\}}\,d\,X_i(t) + \sum_{j=k+1}^{n} d\,\Lambda^{k,j}(t) - \sum_{j=1}^{k-1} d\,\Lambda^{j,k}(t)\Big]\,, \tag{4.3}$$

for $0 \le t < \infty$. Here $p_t := \{\,(p_t(1), \ldots, p_t(n))\,\}$ is the random permutation of $\{1, \ldots, n\}$ which describes the relation between the indexes of $X(t)$ and the ranks of $X_{(\cdot)}(t)$ such that $p_t(k) < p_t(k+1)$ if $X_{(k)}(t) = X_{(k+1)}(t)$ for $0 \le t < \infty$.

Let Π be the symmetric group of permutations of $\{\,1, \ldots, n\,\}$. The map $p_t : \Omega \times [0, \infty) \mapsto \Pi$ is measurable with respect to σ-field generated by the adapted continuous process $\{X(s)\,, 0 \le s \le t\}$ and hence is predictable. Since Π is bijective, let us define the inverse map $p_t^{-1} := (p_t^{-1}(1), \ldots, p_t^{-1}(n))$ such that

$$X_{(p_t^{-1}(i))}(t) = X_i(t)\,; \quad i = 1, \ldots, n\,, \quad 0 \le t < \infty\,. \tag{4.4}$$

That is, $p_t^{-1}(i)$ indicates the rank of $X_i(t)$ in the $n-$dimensional process $X(t)$. The map $p_t^{-1} : \Omega \times [0, \infty) \mapsto \Pi$ is also predictable.

Under the assumption of "no triple collisions" (that is, when the only non-zero change-of-rank local times are those of the form $\Lambda^{k,k+1}(\cdot)$, $1 \le k \le n-1$), Fernholz (5) considered the stochastic differential equation of the vector of ranked process $X_{(\cdot)}$ in a general framework and Banner, Fernholz & Karatzas (1) obtained a rather complete analysis of the Atlas model (4.1).

In this section we apply the main results obtained in the previous sections to the Atlas model. Note that there are some cases of piece-wise constant diffusion coëfficients, which satisfy the conditions in Proposition 1 or 3. Obviously, if $\{\tilde{\sigma}_k^2\}$ are the same, we are in the case of standard Brownian motion. A bit more interestingly, if $\{\tilde{\sigma}_k^2\}$ are linearly growing in the sense of (3.12), we can construct a weak solution to (4.1) with no collision of three or more particles.

Remark 4.1. On page 2305, the paper (1) contains the erroneous statement that "the uniform non-degeneracy of the variance structure and boundedness of the drift coëfficients" are enough to preclude triple collisions. Part of our motivation in undertaking the present work was a desire to correct this error. □

4.2. **Construction of weak solution.**

4.2.1. *Reflected Brownian motion.* Let us start by observing that the dynamics of the sum (total capitalization) $\mathfrak{X}(t) := X_1(\cdot) + \cdots + X_n(\cdot)$ can be written as

$$
\begin{aligned}
d\,\mathfrak{X}(t) \;&=\; n\gamma d\,t + \sum_{i=1}^{n}\sum_{k=1}^{n} \tilde{\sigma}_k 1_{Q_k^{(i)}}(X(t))d\,W_i(t) \\
&=\; n\gamma d\,t + \sum_{k=1}^{n} \widetilde{\sigma}_k \, d\,B_k(t); \quad 0 \le t < \infty,
\end{aligned} \tag{4.5}
$$

where $B(\cdot) := \{(B_1(t), \cdots, B_n(t))', 0 \le t < \infty\}$ is an $n-$dimensional Brownian motion starting at the origin, by Dambis-Dubins-Schwarz theorem, with components $B_k(t) := \sum_{i=1}^{n} \int_0^t 1_{Q_k^{(i)}}(X(s))d\,W_i(s)$ for $1 \le k \le n$, $0 \le t < \infty$.

Next, let h and $\widetilde{\Sigma}$ be the $(n-1)\times 1$ vector and the $(n-1)\times n$ triangular matrix with entries

$$
h := (g_1 - g_2, \cdots, g_{n-1} - g_n)', \quad \widetilde{\Sigma} := \begin{pmatrix} \tilde{\sigma}_1 & -\tilde{\sigma}_2 & & & \\ & \tilde{\sigma}_2 & -\tilde{\sigma}_3 & & \\ & & \ddots & \ddots & \\ & & & \tilde{\sigma}_{n-1} & \tilde{\sigma}_n \end{pmatrix},
$$

where the elements in the lower-triangular part and the upper-triangular part, except the first diagonal above the main diagonal, are zeros. Then the process $\{ht + \widetilde{\Sigma}B(t), 0 \le t < \infty\}$ is an $(n-1)-$dimensional Brownian motion starting at the origin of $\mathbb{R}^{n-1}$, with constant drift h and variance/covariance matrix

$$
\mathfrak{A} := \widetilde{\Sigma}\widetilde{\Sigma}' := \begin{pmatrix} \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 & -\tilde{\sigma}_2^2 & & \\ -\tilde{\sigma}_2^2 & \tilde{\sigma}_2^2 + \tilde{\sigma}_3^2 & \ddots & \\ & \ddots & \ddots & -\sigma_{n-1}^2 \\ & & -\tilde{\sigma}_{n-1}^2 & \tilde{\sigma}_{n-1}^2 + \tilde{\sigma}_n^2 \end{pmatrix}. \tag{4.6}
$$

Now we construct as in Section 3.2 an $(n-1)-$dimensional process $Z(\cdot) := \{(Z_1(t), \cdots, Z_{n-1}(t))', 0 \le t < \infty\}$ on $\mathbb{R}_+^{n-1}$ by

$$
\begin{aligned}
Z_k(t) \;:=&\; (g_k - g_{k+1})t + \tilde{\sigma}_k B_k(t) - \tilde{\sigma}_{k+1} B_{k+1}(t) \\
&+ \Lambda^{k,k+1}(t) - \frac{1}{2}\big(\Lambda^{k-1,k}(t) + \Lambda^{k+1,k+2}(t)\big); \quad 0 \le t < \infty
\end{aligned} \tag{4.7}
$$

for $k = 1, \cdots, n-1$. Here $\Lambda^{k,k+1}(\cdot)$ is a continuous, adapted and non-decreasing process with $\Lambda^{k,k+1}(0) = 0$ and $\int_0^\infty Z_k(t)\, d\Lambda^{k,k+1}(t) = 0$ almost surely. Setting $\Lambda^{0,1}(\cdot) \equiv \Lambda^{n,n+1}(\cdot) \equiv 0$ for notational convenience, we write in matrix form:

$$
Z(t) \;=\; h\,t + \widetilde{\Sigma}\,B(t) + \mathfrak{R}\,\Lambda(t); \quad 0 \le t < \infty,
$$

where $\Lambda(\cdot) = (\Lambda^{1,2}(\cdot), \cdots, \Lambda^{k-1,k}(\cdot))'$ and the reflection matrix $\mathfrak{R} = I - \mathfrak{Q}$ is given by

$$\mathfrak{R} = I - \mathfrak{Q} := \begin{pmatrix} 1 & -1/2 & & & \\ -1/2 & 1 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -1/2 \\ & & & -1/2 & 1 \end{pmatrix} . \tag{4.8}$$

Recall that *if the process* $X(\cdot)$ *has no "triple collisions"*, then it follows from (4.3) that

$$d\,X_{(k)}(t) = \sum_{i=1}^{n} \mathbf{1}_{\{X_i(t) = X_{(k)}(t)\}} d\,X_i(t) + \frac{1}{2}\Big(d\Lambda^{k,k+1}(t) - d\Lambda^{k-1,k}(t) \Big) ,$$

for $0 \le t < \infty$. Hence by substituting (4.1) into this equation and subtracting, we obtain that

$$X_{(k)}(t) - X_{(k+1)}(t) = Z_k(t); \quad 1 \le k \le n-1,\ 0 \le t < \infty , \tag{4.9}$$

and that $\Lambda^{k,k+1}(\cdot)$ is the local time of the one-dimensional process $Z_k(t)$ at level zero for $k = 1, \dots n-1$. In general, the process $X(\cdot)$ may have triple or more collisions so that we have additional terms in (4.9):

$$X_{(k)}(t) - X_{(k+1)}(t) = Z_k(t) + \zeta_k(t) , \quad 1 \le k \le n-1,\ 0 \le t < \infty , \tag{4.10}$$

where the contribution $\zeta(\cdot) := (\zeta_1(\cdot), \dots, \zeta_{n-1}(\cdot))$ from the triple or more collisions can be written for $1 \le k \le n-1$, $0 \le t < \infty$ as

$$\zeta_k(t) = \sum_{\ell=3}^{n} \ell^{-1} \Big[\sum_{j=k+2}^{n} \Lambda^{k,j}(t) - \sum_{j=1}^{k-2} \Lambda^{j,k}(t) \Big] \mathbf{1}_{\{N_k(t)=\ell\}}$$
$$- \sum_{\ell=3}^{n} \ell^{-1} \Big[\sum_{j=k+3}^{n} \Lambda^{k+1,j}(t) - \sum_{j=1}^{k-1} \Lambda^{j,k+1}(t) \Big] \mathbf{1}_{\{N_{k+1}(t)=\ell\}} .$$

Remark 4.2. Note that $\zeta(\cdot)$ consists of the (random) linear combination of the local times from collisions of three or more particles, and hence it is *flat*, unless there are triple collisions, i.e., $\int_0^t \mathbf{1}_{\mathfrak{S}}\, d\,\zeta(s) = 0$ for $0 \le t < \infty$, where the set $\mathfrak{S}$ is defined as $\{ s \ge 0 : X_i(s) = X_j(s) = X_k(s)$ for some $1 \le i < j < k \le n \}$. We use this fact with Lemma 4.1 in the next subsection. □

4.2.2. *Application of Lemma 3.3.* Under the assumption of Proposition 3 we can apply Lemma 3.3 to obtain

$$\mathbb{P}(Z_i(t) = Z_j(t) = 0\,, \exists t > 0\,, \exists (i,j)\,,\ 1 \le i \ne j \le n) = 0\,. \tag{4.11}$$

See (5.13) in Section 5.7. Thus, $Z(\cdot)$ is a special case of reflected Brownian motion whose each $\Lambda^{k,k+1}(\cdot)$ of non-decreasing finite variation part is exactly the local time of $Z_k(\cdot)$ at level zero.

Now let us state the following lemma to examine the local times from collisions of three or more particles. Its proof is in Section 5.7.3.

Lemma 4.1. *Let $\alpha(\cdot) = \{\alpha(t)\,;\, 0 \le t < \infty\}$ be a non-negative continuous function with decomposition $\alpha(t) = \beta(t) + \gamma(t)$, where $\beta(\cdot)$ is a strictly positive continuous function and $\gamma(\cdot)$ is a continuous function that can be written as a difference of two non-decreasing functions which are flat off $\{t \ge 0 \,:\, \alpha(t) = 0\}$, i.e., $\int_0^t \mathbf{1}_{\{\alpha(s)>0\}} d\gamma(s) = 0$ for $0 \le t < \infty$. Assume that $\gamma(0) = 0$ and $\alpha(0) = \beta(0) > 0$. Then, $\gamma(t) = 0$ and $\alpha(t) = \beta(t)$ for $0 \le t < \infty$.*

Under the assumption of Proposition 3, applying the above Lemma 4.1 with (4.10), (4.11) and $\alpha(\cdot) = X_{(k)}(\cdot,\omega) - X_{(k+2)}(\cdot,\omega)$, $\beta(\cdot) = Z_k(\cdot,\omega) + Z_{k+1}(\cdot,\omega)$ and $\gamma(\cdot) = \zeta_k(\cdot,\omega) + \zeta_{k+1}(\cdot,\omega)$ for $\omega \in \Omega$, we obtain $\alpha(\cdot) = \beta(\cdot)$, i.e.,

$$X_{(k)}(\cdot) - X_{(k+2)}(\cdot) = Z_k(\cdot) + Z_{k+1}(\cdot)\,, \quad k = 1, \ldots, n-2\,. \tag{4.12}$$

In fact, the decomposition $\alpha(\cdot) = X_{(k)}(\cdot) - X_{(k+1)}(\cdot) + X_{(k+1)}(\cdot) - X_{(k+2)}(\cdot) = \beta(\cdot) + \gamma(\cdot)$ with (4.10) validates the assumption of Lemma 4.1. See Remark 4.2. Combining (4.12) with (4.11), we obtain $X_{(k)}(\cdot) - X_{(k+2)}(\cdot) > 0$ or

$$\mathbb{P}(X_{(k)}(t) = X_{(k+1)}(t) = X_{(k+2)}(t)\,, \exists t > 0\,, \exists k\,,\ 1 \le k \le n-2) = 0\,. \tag{4.13}$$

Therefore, there are "*no triple collisions*" under the assumption of Proposition 3. This concludes the proof of Proposition 3.

In summary, we *recover* the n-dimensional ranked process $X_{(\cdot)}$ of X by considering the linear transformation. Specifically, construct n-dimensional ranked process

$$\Psi_{(\cdot)}(t) := (\Psi_{(1)}(t), \cdots, \Psi_{(n)}(t))\,;\ 0 \le t < \infty$$

from the sum $\mathfrak{X}(t)$, $0 \le t < \infty$ defined in (4.5) and the reflected Brownian motion $Z(\cdot)$, such that the differences satisfy

$$\Psi_{(k)}(t) - \Psi_{(k+1)}(t) = Z_k(t)\,, \quad k = 1, \ldots n-1\,, \tag{4.14}$$

and the sum satisfies

$$\sum_{k=1}^{n} \Psi_{(k)}(t) = \mathfrak{X}(t)\,; \quad 0 \le t < \infty\,. \tag{4.15}$$

Each element is uniquely determined by

$$\begin{pmatrix} \Psi_{(1)}(\cdot) \\ \Psi_{(2)}(\cdot) \\ \vdots \\ \Psi_{(n)}(\cdot) \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \mathfrak{X}(\cdot) + Z_{n-1}(\cdot) + (n-2)Z_{n-2}(\cdot) + \cdots + (n-1)Z_1(\cdot) \\ \mathfrak{X}(\cdot) + Z_{n-1}(\cdot) + (n-2)Z_{n-2}(\cdot) + \cdots - Z_1(\cdot) \\ \vdots \\ \mathfrak{X}(\cdot) - (n-1)Z_{n-1}(\cdot) - (n-2)Z_{n-2}(\cdot) - \cdots - Z_1(\cdot) \end{pmatrix}$$

for $0 \le t < \infty$. Under the assumption of Proposition 3, we obtain (4.11) and hence with (4.14) we arrive at

$$\mathbb{P}(\Psi_{(k)}(t) = \Psi_{(k+1)}(t) = \Psi_{(k+2)}(t),\ \exists t > 0,\ 1 \le \exists k \le n-2) = 0\,, \tag{4.16}$$

in the same way as discussed in (3.3).

Thus, the ranked process $\{X_{(\cdot)}(t), 0 \le t < \infty\}$ of the original process $X(\cdot)$ without collision of three or more particles, and the ranked process $\Psi_{(\cdot)}(\cdot)$ defined in the above, are equivalent, since both of them have the same sum (4.15) and the same non-negative difference processes $Z(\cdot)$ identified in (4.9) and (4.14). Then, we may view $\Psi_{(\cdot)}(\cdot)$ as the weak solution to the SDE for the ranked process $X_{(\cdot)}(\cdot)$. Finally, we define $\Psi(\cdot) := (\Psi_1(\cdot), \dots, \Psi_n(\cdot))$ where $\Psi_i(\cdot) = \Psi_{(p_t^{-1}(i))}(\cdot)$ for $i = 1, \dots, n$, and $p_t^{-1}(i)$ is defined in (4.4). Then, $\Psi(\cdot)$ is the weak solution of SDE (4.1). This construction of solution leads us to the invariant properties of the Atlas model given in (1) and (11).

5. Appendix

5.1. **Proof of Lemma 2.4.** With $B(\cdot)$ and $F(\cdot)$ as in (2.36), the second-order ordinary differential equation (2.32) is solved as

$$
\begin{aligned}
H(t) &:= \exp\left[\int_a^t B(w)d\,w\right] = \exp\left[\int_a^t \frac{c_1}{w}d\,w\right] = \left(\frac{t}{a}\right)^{c_1}, \\
v'(t) &= \frac{1}{H(t)}\int_a^t H(w)F(w)d\,w = \frac{c_2}{c_1-1}\left[\frac{1}{t} - \left(\frac{a^{c_1-1}}{t^{c_1}}\right)\right]
\end{aligned}
\tag{5.1}
$$

for $k^{-k} = a \le t \le b = k$. Thus, from (2.40) we obtain the bound

$$
\begin{aligned}
&u_{k,p,r}(x) \le \max_{y\in\partial\mathcal{E}_{k,p}} v\big(s(y)\big) - v\big(s(x)\big) = \int_{s(x)}^k v'(t)d\,t \\
&= \frac{c_2}{c_1-1}\log\left(\frac{k}{s(x)}\right) + \frac{k^{-(c_1-1)k}c_2}{(c_1-1)^2}\left(\frac{1}{k^{(c_1-1)}} - \frac{1}{(s(x))^{(c_1-1)}}\right); \quad x \in \mathcal{E}_{k,p}
\end{aligned}
$$

which does not depend on p, r, as well as the bound for $\sup_{r\ge 2} u_{k,r}(\cdot)$:

$$
\begin{aligned}
&\sup_{r\ge 2} u_{k,r}(x) = \lim_{p\to\infty} u_{k,p,r}(x) \\
&\le \frac{c_2}{c_1-1}\log\left(\frac{k}{s(x)}\right) + \frac{k^{-(c_1-1)k}c_2}{(c_1-1)^2}\left(\frac{1}{k^{(c_1-1)}} - \frac{1}{(s(x))^{(c_1-1)}}\right); \quad x \in \mathcal{E}_{k,p}.
\end{aligned}
$$

5.2. **Proof of Lemma 2.5.** For the effective dimension, we observe that under the assumptions of Lemma 2.5

$$
\begin{aligned}
\frac{x' A^{(\ell)}(x) x}{|x|^2} &= \sum_{j=1}^m \int_{\mathcal{R}_j} \rho_\ell(x-z)\cdot\frac{x'\sigma_j(x)\sigma_j'(x)x}{|x|^2}d\,z \\
&\le \sum_{j=1}^m \int_{\mathcal{R}_j} \rho_\ell(x-z)\frac{\text{trace }(\sigma_j(x)\sigma_j'(x))}{\inf_{y\in\mathbb{R}^n\setminus\{0\}} \mathrm{ED}_{\mathcal{A}}(y)}d\,z = \frac{\text{trace }\big(A^{(\ell)}(x)\big)}{\inf_{y\in\mathbb{R}^n\setminus\{0\}} \mathrm{ED}_{\mathcal{A}}(y)}
\end{aligned}
\tag{5.2}
$$

holds for $x \in \mathbb{R}^n \setminus \{0\}$, hence

$$
\inf_{x\in\mathbb{R}^n\setminus\{0\}} \mathrm{ED}_{\mathcal{A}^{(\ell)}}(x) \ge \inf_{x\in\mathbb{R}^n\setminus\{0\}} \mathrm{ED}_{\mathcal{A}}(x) > 2.
$$

under the assumptions; the other quantities are treated similarly.

5.3. **Proof of Lemma 2.8.** First we observe

$$\lim_{\widetilde{\ell}\to\infty}\left|f_r^{(\widetilde{\ell})}(X(t))-f_r(X(t))\right|=0,\qquad 0\le t\le S_k\wedge T_k\,;\quad \mathbb{P}_{x_0}^{(\ell)}-\text{a.s.},\ \ell\ge 1.$$

In fact, since $a_{ij}^{(\widetilde{\ell})}(x)$ converges to $a_{ij}(x)$ as $\widetilde{\ell}$ tends to infinity, for every x at which $a_{ij}(x)$ is continuous (see e.g. Theorem 8.15 of Folland (6)), for every $1\le i,j\le n$, the matrix norm $\left\|A^{(\widetilde{\ell})}(\cdot)-A(\cdot)\right\|:=\max_{1\le i,j\le n}\left|a_{ij}^{(\widetilde{\ell})}(\cdot)-a_{ij}(\cdot)\right|$ converges to 0 except for the union $\cup_{j=1}^m\partial R_j$ of possible discontinuity points. The discontinuity set is of zero Lebesgue measure. Then, since the process X under $\mathbb{P}_{x_0}^{(\ell)}$ is non-degenerate, we have for $t\in[0,S_k\wedge T_k]$:

$$\begin{aligned}\left|f_r^{(\widetilde{\ell})}(X(t))-f_r(X(t))\right| &\le \left|\frac{\text{trace }\left(D'\left(A^{(\widetilde{\ell})}(x)-A(x)\right)D\right)}{x'DD'x}\right|\zeta_r(x)\Bigg|_{x=X(t)}\\ &\quad+\left|\frac{2x'DD'\left(A^{(\widetilde{\ell})}(x)-A(x)\right)DD'x}{(x'DD'x)^2}\right|\zeta_r(x)\Bigg|_{x=X(t)}\\ &\xrightarrow[l'\to\infty]{\mathbb{P}_{x_0}^{(\ell)}\ a.s.}0,\quad \ell\ge 1.\end{aligned}$$

Moreover, as in Remark 2.2, the random variables $f_r(X(t))$ and $f_r^{(\widetilde{\ell})}(X(t))$ are bounded by some constant times k^k when $t\in[0,S_k\wedge T_k]$ i.e.,

$$|f_r(X(t))|\vee\left|f_r^{(\widetilde{\ell})}(X(t))\right|\le C\cdot k^k;\quad 0\le t\le S_k\wedge T_k \tag{5.3}$$

for some positive constant C which is independent of $(k,\widetilde{\ell},r,t)$, and so is the difference $\left|f_r^{(\ell)}(X(t))-f_r(X(t))\right|$, when $t\in[0,S_k\wedge T_k]$. Therefore, by the bounded convergence theorem,

$$\lim_{\widetilde{\ell}\to\infty}\int_0^{S_k\wedge T_k}\left(f_r^{(\widetilde{\ell})}(X(t))-f_r(X(t))\right)dt=0,\quad \mathbb{P}_{x_0}^{(\ell)}-\text{a.s.};\quad \ell\ge 1\,. \tag{5.4}$$

Next, since $\int_0^{S_k\wedge T_k}f_r(X(t))\,dt$ is a bounded continuous functional of the process $X(\cdot)$, we obtain by the weak convergence of $\mathbb{P}_{x_0}^{(\ell)}$ to $\mathbb{P}_{x_0}$ and by the continuous mapping theorem,

$$\lim_{\ell\to\infty}\left|\mathbb{E}_{x_0}^{(\ell)}\left[\int_0^{S_k\wedge T_k}f_r(X(t))dt\right]-\mathbb{E}_{x_0}\left[\int_0^{S_k\wedge T_k}f_r(X(t))dt\right]\right|=0\,. \tag{5.5}$$

Furthermore, it follows from (5.3) that

$$\left|\int_0^{S_k\wedge T_k}\left(f_r^{(\widetilde{\ell})}(X(t))-f_r(X(t))\right)dt\right|\le 2C\cdot k^kS_k<\infty.$$

Then, by using (5.4) and the estimate $\mathbb{E}_{x_0}^{(\ell)}[S_k]\le C'k^2<\infty$ obtained as in Remark 2.2 for some constant C', we get by the dominated convergence theorem:

$$\lim_{\widetilde{\ell}\to\infty} \left| \mathbb{E}^{(\ell)} \left[\int_0^{S_k \wedge T_k} \left(f_r^{(\widetilde{\ell})}(X(t)) - f_r(X(t)) \right) d\,t \right] \right| = 0\,; \quad \ell \geq 1\,. \tag{5.6}$$

Finally, let $\varepsilon > 0$ be given. Use (5.5) to choose ℓ_0 so that

$$\left| \mathbb{E}^{(\ell_0)}_{x_0} \left[\int_0^{S_k \wedge T_k} f_r(X(t)) d\,t \right] - \mathbb{E}_{x_0} \left[\int_0^{S_k \wedge T_k} f_r(X(t)) d\,t \right] \right| < \varepsilon.$$

Then, use (5.4) and (5.6) to choose $\widetilde{\ell}$ so that

$$\left| \int_0^{S_k \wedge T_k} \left(f_r^{(\ell_0)}(X(t)) - f_r^{(\widetilde{\ell})}(X(t)) \right) d\,t \right| < \varepsilon, \quad \mathbb{P}^{(\ell_0)}_{x_0} - \text{a.s.}$$

and

$$\left| \mathbb{E}^{(\ell_0)} \left[\int_0^{S_k \wedge T_k} \left(f_r^{(\widetilde{\ell})}(X(t)) - f_r(X(t)) \right) d\,t \right] \right| < \varepsilon.$$

Then, by triangle inequality we obtain

$$\begin{aligned}
&\left| \mathbb{E}^{(\ell_0)} \left[\int_0^{S_k \wedge T_k} f_r^{(\ell_0)}(X(t)) d\,t \right] - \mathbb{E} \left[\int_0^{S_k \wedge T_k} f_r(X(t)) d\,t \right] \right| \\
&\leq \left| \mathbb{E}^{(\ell_0)}_{x_0} \left[\int_0^{S_k \wedge T_k} \left(f_r^{(\ell_0)}(X(t)) - f_r^{(\widetilde{\ell})}(X(t)) \right) d\,t \right] \right| \\
&+ \left| \mathbb{E}^{(\ell_0)}_{x_0} \left[\int_0^{S_k \wedge T_k} f_r(X(t)) d\,t \right] - \mathbb{E}_{x_0} \left[\int_0^{S_k \wedge T_k} f_r(X(t)) d\,t \right] \right| \\
&+ \left| \mathbb{E}^{(\ell_0)} \left[\int_0^{S_k \wedge T_k} \left(f_r^{(\widetilde{\ell})}(X(t)) - f_r(X(t)) \right) d\,t \right] \right| < 3\,\varepsilon\,,
\end{aligned}$$

the desired result (2.48). For (2.49) we use the same argument as in (2.24).

5.4. **Proof of Proposition 1.** In subsection 2.2 we noted the equivalence of the conditions in (1.4) with those of (2.4), (2.9) and (2.16), for $i = 1$, $j = 2$, $k = 3$. In subsection 2.3.4 we derived Lemmata 2.6 and 2.8 to obtain (2.50) that is, one of the cases in (2.16); the claim (2.51) follows.

5.5. **Proof of Proposition 2.** The semimartingale decomposition of the process $s(X(\cdot))$ is given by $d\,s(X(t)) = h(X(t))d\,t + d\,\Theta(t)$, where

$$\begin{aligned}
h(x) &:= \frac{1}{2\,s^3(x)} \left(s^2(x) \sum_{i=1}^3 d_i' \sigma(x) \sigma(x)' d_i - \Big|\Big| \sum_{i=1}^3 d_i' x \sigma(x)' d_i \Big|\Big|^2 \right) \\
&= \frac{x' D D' x \cdot \; \text{trace} \; (D' A(x) D) - x' D D' A(x) D D' x}{2 (x' D D' x)^{3/2}} \\
&= \frac{(R(x) - 1) Q(x)}{2 s(x)}\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}
\end{aligned} \tag{5.7}$$

and the continuous local martingale $\Theta(\cdot)$, along with its quadratic variation process $\langle\Theta\rangle(\cdot)$, are given by

$$\Theta(t) := \int_0^t \left(\sum_{i=1}^{3} \frac{d_i' X(\tau) \sigma'(X(\tau)) d_i}{s(X(\tau))} \right) d\,W(\tau)\,,$$
$$\langle\Theta\rangle(t) = \int_0^t \frac{x' D D' A(x) D D' x}{x' D D' x}\bigg|_{x=X(\tau)} d\,\tau = \int_0^t Q(X(\tau)) d\,\tau\,; \quad 0 \le t < \infty\,,$$

respectively. Note the following relations among the functions $f(\cdot), h(\cdot)$, $s(\cdot), Q(\cdot), R(\cdot)$ and between the local martingales $M(\cdot)$ in (2.12) and $\Theta(\cdot)$:

$$s^2(x)\, f(x) = (R(x) - 2\,) Q(x) = 2\, s(x)\, h(x) - Q(x)\,; \quad x \in \mathbb{R}^n \setminus \mathcal{Z}\,,$$
$$s(X(t))\, d\, M(t) = d\,\Theta(t)\,; \quad 0 \le t < \infty\,.$$

Applying Itô's formula to $\log s(X(\cdot))$ with the above semimartingale decomposition and these relations, we can replicate the stochastic differential equation (2.11) of $\log s(X(\cdot))$.

Now define the stopping time $\Lambda_u := \inf\{t \ge 0 : \langle\Theta\rangle\,(t) \ge u\}$. Then, by the Dambis-Dubins-Schwartz theorem of time-change for martingales, we have

$$s(X(\Lambda_u)) - s(x_0) = \int_0^{\Lambda_u} h(X(t)) d\,t + B(u)\,; \quad 0 \le u < \infty\,,$$

where $B(u) := \Theta(\Lambda_u)\,,\, 0 \le u < \infty$ is standard Brownian motion. With $\mathfrak{s}(\cdot) := s(X(\Lambda_\cdot))$ we can write

$$d\,\mathfrak{s}(u) = h\big(X(\Lambda_u)\big)\,\Lambda_u'\, d\,u + d\,B(u)\,; \quad 0 \le u < \infty\,, \tag{5.8}$$

where

$$h\big(X(\Lambda_u)\big)\,\Lambda_u' = \frac{[R(X(\Lambda_u)) - 1]\, Q(X(\Lambda_u))}{2\, s(X(\Lambda_u))} \cdot \frac{1}{Q(X(\Lambda_u))} = \frac{\mathfrak{d}(u) - 1}{2\,\mathfrak{s}(u)}$$

with $\mathfrak{d}(u) := R(X(\Lambda_u))$. The dynamics of the process $\mathfrak{s}(\cdot)$ are comparable to those for $d-$dimensional Bessel process, namely

$$d\,\mathfrak{r}(u) = \frac{d-1}{2\,\mathfrak{r}(u)}\, d\,u + d\,B(u)\,; \quad 0 \le u < \infty$$

since, under the conditions of Proposition 2, we have $\mathfrak{d}(\cdot) \le 2 - \eta =: d$. By the comparison theorem for one dimensional diffusions, the process $\mathfrak{s}(\cdot)$ is dominated by the Bessel diffusion $\mathfrak{r}(\cdot)$ with $d < 2$; the claim (2.52) follows from this, and from the strong Markov property.

5.6. **Example in Remark 2.9.** With some computations we obtain the following simplification

$$\mathrm{ED}_{\mathcal{A}}(x) = 2 + \frac{\begin{bmatrix} \|x\|^2 - 4a_{12}(x) \cdot x_1 x_2 \mathbf{1}_{\mathcal{R}_{1+} \cup \mathcal{R}_{1-}} \\ -4a_{23}(x) \cdot x_2 x_3 \mathbf{1}_{\mathcal{R}_{2+} \cup \mathcal{R}_{2-}} \\ -4a_{31}(x) \cdot x_3 x_1 \mathbf{1}_{\mathcal{R}_{3+} \cup \mathcal{R}_{3-}} \end{bmatrix}}{x' A(x) x}\,, \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}$$

and

$$R(x) = 2 + \frac{\begin{pmatrix} 4a_{12}(x) \cdot [(x_1 - x_2)^2 + 2(x_2 - x_3)(x_3 - x_1)]\mathbf{1}_{\mathcal{R}_{1+} \cup \mathcal{R}_{1-}} \\ + 4a_{23}(x) \cdot [(x_2 - x_3)^2 + 2(x_3 - x_1)(x_1 - x_2)]\mathbf{1}_{\mathcal{R}_{2+} \cup \mathcal{R}_{2-}} \\ + 4a_{31}(x) \cdot [(x_3 - x_1)^2 + 2(x_2 - x_3)(x_1 - x_2)]\mathbf{1}_{\mathcal{R}_{3+} \cup \mathcal{R}_{3-}} \end{pmatrix}}{x' DD' A(x) DD' x}$$

for $x \in \mathbb{R}^3 \setminus \mathcal{Z}$ Under the specification (2.53), we verify $\mathrm{ED}(\cdot) > 2$ and $R(\cdot) > 2$, since the denominators of the fractions on the right hands are positive quadratic forms and their numerators can be written as

$$\|x\|^2 - 4a_{12}(x) x_1 x_2 = (1 - 4a_{12}^2) x_2^2 + x_3^2 + (x_1 - 2a_{12} x_2)^2 > 0 \, ; \quad x \in \mathcal{R}_{1+} \cup \mathcal{R}_{1-}$$

$$4a_{12}(x)[(x_1 - x_2)^2 + 2(x_2 - x_3)(x_3 - x_1)] = 4a_{12}(x)\mathfrak{f}_1(x) > 0 \, ; \quad x \in \mathcal{R}_{1+} \cup \mathcal{R}_{1-}$$

with similar formulas for $x \in \mathcal{R}_{i+} \cup \mathcal{R}_{i-}$, $i = 2, 3$.

5.7. **Proof of Proposition 3.**

5.7.1. *Proof of Lemma 3.2.* Now recall the special geometric structure of orthogonality $\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_i = 0$ and $\|\widetilde{\mathfrak{n}}_i\| = 1$ and observe that

$$\left(\widetilde{\mathfrak{N}}' \widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}' \widetilde{\mathfrak{N}}\right)_{ij} \; \substack{\geq \\ <} \; 0 \iff \widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j' \widetilde{\mathfrak{q}}_i \; \substack{\geq \\ <} \; 0 \; ; \; \forall \, (i, j). \tag{5.9}$$

Note that if $n = 3$, i.e. $n - 1 = 2$, then $\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j = \|\widehat{\mathfrak{q}}_j\| \, sgn(-\theta_j) \sin(\xi)$ for $1 \le i \ne j \le 2$, where $sgn(x) := 1_{\{x > 0\}} - 1_{\{x < 0\}}$. The length $\|\widetilde{\mathfrak{q}}_2\|$ of $\widetilde{\mathfrak{q}}_2$ determines the angle θ_2 and vice versa, i.e.,

$$\|\widetilde{\mathfrak{q}}_i\| \; \substack{\geq \\ <} \; \|\widetilde{\mathfrak{q}}_j\| \iff |\theta_i| \; \substack{\geq \\ <} \; |\theta_j|.$$

With this and $0 < \xi_{ij} < \pi$, $\sin(\xi_{ij}) > 0$, we obtain

$$\begin{aligned} &\widetilde{\mathfrak{n}}_i' \widetilde{\mathfrak{q}}_j + \widetilde{\mathfrak{n}}_j' \widetilde{\mathfrak{q}}_i = \sin(\xi)(\|\widetilde{\mathfrak{q}}_j\| \, sgn(-\theta_j) + \|\widetilde{\mathfrak{q}}_i\| \, sgn(-\theta_i)) \; \substack{\geq \\ <} \; 0 \\ &\iff \beta = (\theta_i + \theta_j)/\xi \; \substack{\leq \\ >} \; 0; \quad 1 \le i \ne j \le 2 \, . \end{aligned} \tag{5.10}$$

Thus, we apply Lemma 3.1 and obtain Lemma 3.2.

5.7.2. *Coëfficients structure.* Next, we consider the case of linearly growing variance coëfficients defined in (3.12), and recall the tri-diagonal matrices $\mathfrak{A} = \widetilde{\Sigma} \widetilde{\Sigma}'$ as in (4.6) and $\mathfrak{R}$ as in (4.8). Consider the $(n-1)$-dimensional reflected Brownian motion $Y(\cdot)$ defined in (3.4) with $\Sigma = \widetilde{\Sigma}$ and this above $\mathfrak{R}$. We can verify such a pair $(\widetilde{\Sigma}, \mathfrak{R})$ satisfies the following element-wise equations

$$\left(2\mathfrak{D} - \mathfrak{Q}\mathfrak{D} - \mathfrak{D}\mathfrak{Q} - 2\mathfrak{A}\right)_{ij} = 0; \quad 1 \le i, j \le n - 1, \tag{5.11}$$

where $\mathfrak{D}$ is the diagonal matrix with the same diagonal elements as $\mathfrak{A}$ of (3.7), and $\mathfrak{Q}$ is the $((n-1) \times (n-1))$ matrix whose first-diagonal elements above and below the main diagonal are all $1/2$ and other elements are zeros as in (4.6). In fact, it suffices to see the cases $j = i+1, i = 2, \dots, n-1$. The equalities (5.11) are

$$0 - \frac{1}{2}(\widetilde{\sigma}_i^2 + \widetilde{\sigma}_{i+1}^2) - \frac{1}{2}(\widetilde{\sigma}_{i-1}^2 + \widetilde{\sigma}_i^2) + 2\widetilde{\sigma}_i^2 = 0,$$

or equivalently (3.12)

$$\widetilde{\sigma}_i^2 - \widetilde{\sigma}_{i-1}^2 = \widetilde{\sigma}_{i+1}^2 - \widetilde{\sigma}_i^2; \quad 2 \le i \le n-1.$$

Moreover, the equalities (5.11) are equivalent to $(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}})_{i\,j} = 0$ in (5.9). In fact, from (3.7) with $\mathfrak{D}^{1/2} = \mathfrak{C}^{-1}$ we compute

$$\begin{aligned}\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} &= \mathfrak{D}^{-1/2}U'\mathfrak{L}^{1/2}\mathfrak{L}^{-1/2}U\mathfrak{R}\mathfrak{D}^{1/2} - \widetilde{\mathfrak{N}}'\widetilde{\mathfrak{N}} \\ &= \mathfrak{D}^{-1/2}(I - \mathfrak{Q})\mathfrak{D}^{1/2} - \mathfrak{D}^{-1/2}\mathfrak{A}\mathfrak{D}^{-1/2} \\ \widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}} &= 2I - \mathfrak{D}^{-1/2}\mathfrak{Q}\mathfrak{D}^{1/2} - \mathfrak{D}^{1/2}\mathfrak{Q}\mathfrak{D}^{-1/2} - 2\mathfrak{D}^{-1/2}\mathfrak{A}\mathfrak{D}^{-1/2},\end{aligned}$$

and multiply both from the left and the right by the diagonal matrix $\mathfrak{D}^{1/2}$ whose diagonal elements are all positive:

$$\mathfrak{D}^{1/2}(\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}})\mathfrak{D}^{1/2} = 2\mathfrak{D} - \mathfrak{Q}\mathfrak{D} - \mathfrak{D}\mathfrak{Q} - 2\mathfrak{A}. \tag{5.12}$$

The equality in the relation (5.11) is equivalent to the so-called *skew-symmetric condition* introduced and studied by Harrison & Williams in (10), (20) : $\widetilde{\mathfrak{N}}'\widetilde{\mathfrak{Q}} + \widetilde{\mathfrak{Q}}'\widetilde{\mathfrak{N}} = 0$.

Thus, it follows from (5.9), (5.11) and (5.12) that the reflected Brownian motion Z defined in (4.7), under the assumption of Proposition 3, satisfies that any two dimensional process (Z_i, Z_j) *never* attains the corner $(0,0)'$ for $1 \le i < j \le n-1$ i.e.

$$\mathbb{P}(Z_i(t) = Z_j(t) = 0, \exists t > 0, \exists (i,j),\ 1 \le i \ne j \le n) = 0. \tag{5.13}$$

Using this fact, we construct a weak solution to (4.1) from the reflected Brownian motion. This final step is explained as an application to the financial Atlas model in the last part of Section 4.2.2.

5.7.3. *Proof of Lemma 4.1.* Let us fix arbitrary $T \in [0,\infty)$. Since $\beta(\cdot)$ is strictly positive, we cannot have simultaneously $\alpha(t) = \beta(t) + \gamma(t) = 0$, and $\gamma(t) \ge 0$. Because the continuous function $\beta(\cdot)$ attains the minimum on $[0,T]$, we obtain

$$\begin{aligned}&\{\, t \in [0,T] \,:\, \alpha(t) = 0 \,\} = \{\, t \in [0,T] \,:\, \alpha(t) = 0\,, \gamma(t) < 0 \,\} \\ &\subset \{\, t \in [0,T] \,:\, \gamma(t) \le - \min_{0 \le s \le T} \beta(s) < 0 \,\}.\end{aligned} \tag{5.14}$$

Let us define $t_0 := \inf\{\, t \in [0,T] \,:\, \alpha(t) = 0 \,\}$ following the usual convention that if the set is empty, $t_0 := \infty$. If $t_0 = \infty$, then $\alpha(t) > 0$ for $0 \le t < \infty$ and hence it follows from the assumptions $\gamma(0) = 0$ and $\int_0^T \mathbf{1}_{\{\alpha(t) > 0\}} d\,\gamma(t) = 0$ for $0 \le T < \infty$ that $\gamma(\cdot) \equiv 0$. On the other hand, if $t_0 < \infty$, then it follows from the same argument as in (5.14) that $\gamma(t_0) < -\min_{0 \le s \le t_0} \beta(s) < 0$. However, this is impossible, since $\alpha(s) > 0$ for $0 \le s < t_0$ by the definition of t_0 and hence the continuous function $\gamma(\cdot)$ is flat on $[0, t_0)$, i.e.,

$0 = \gamma(0) = \gamma(t_0-) = \gamma(t_0)$. Thus, $t_0 = \infty$ and $\gamma(\cdot) \equiv 0$. Therefore, the conclusions of Lemma 4.1 hold.

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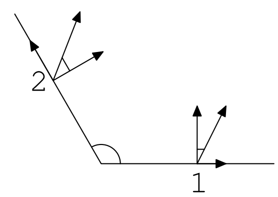

FIGURE 1. Direction of reflections when $\theta_1 + \theta_2 < 0$, $\theta_2 < \theta_1 < 0$, $\|\widetilde{\mathfrak{q}}_2\| > \|\widetilde{\mathfrak{q}}_1\|$.

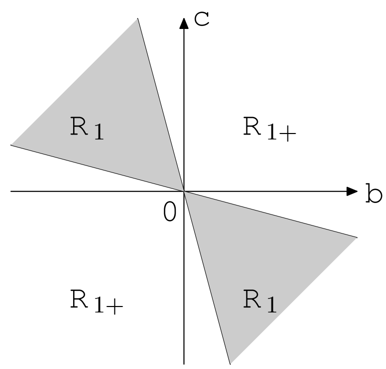

FIGURE 2. Polyhedral regions in Remak 2.9. $b := x_2 - x_3$, $c := x_3 - x_1$.