THE SEPARATION PRINCIPLE FOR A BAYESIAN ADAPTIVE CONTROL PROBLEM WITH NO STRICT-SENSE OPTIMAL LAW †

by

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Abstract

We study the following adaptive control problem, of the Bayesian type: to minimize $E\int_0^\infty e^{-\alpha t}\, Y_t^2 dt$ subject to $dY_t = zu_t dt + dW_t$, where z is an unobservable random variable with symmetric distribution μ, independent of the Brownian motion W. The minimization is to be over the class $\mathcal{U}$ of *wide-sense* control processes u; these are adapted to a particular "observation filtration" $\{\mathcal{F}_t\}$ that contains $\{\mathcal{F}_t^Y\}$, and take values in $[-1,1]$. We denote by $\mathcal{U}_s$ the subclass of *strict-sense* control processes $u \in \mathcal{U}$ which are adapted to $\{\mathcal{F}_t^Y\}$, the filtration generated by the process Y.

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In view of the innovations form $dY_t = \hat{z}_t u_t dt + d\nu_t$, where $\hat{z}_t = E(z|\mathcal{F}_t)$ is the least-squares estimate of z and ν is the innovations Brownian motion process, we show that the control law $u_t^* = -sgn(\hat{z}_t Y_t)$ is optimal in $\mathcal{U}$. This law is also suggested by the separation (or certainty - equivalence) principle.

Optimality is first established for a Bernoulli distribution of the type $\mu = \rho\delta_\theta + (1-\rho)\delta_{-\theta}$ with $\theta > 0$, $0 < \rho < 1$ for the unobserved random variable z. In the case $\theta = 1$, $\rho = 1/2$, it develops that the process $Z_t = \tanh^{-1}(\hat{z}_t)$ satisfies $dZ_t = -sgn(Y_t Z_t) dY_t$. This stochastic equation leads to a degenerate two-dimensional diffusion process, adapted to $\{\mathcal{F}_t\}$, whose properties are studied in detail. It is shown that Z cannot be adapted to $\{\mathcal{F}_t^Y\}$, that u^* does not belong to $\mathcal{U}_s$, and that no control law in $\mathcal{U}_s$ can be optimal.

Finally, it is shown that the same law $u^* \epsilon\, \mathcal{U}$ given by $u_t^* = -sgn(\hat{z}_t Y_t) = -sgn(Z_t Y_t)$ is optimal for more general, symmetric distributions μ with $\int_0^\infty x^2 \mu(dx) < \infty$.

1. INTRODUCTION

We consider in this paper the following stochastic control problem of the Bayesian adaptive type: to minimize the expected discounted quadratic deviation from the origin

$$E \int_0^\infty e^{-\alpha t}\, Y_t^2\, dt \tag{1.1}$$

for some $\alpha > 0$, subject to the dynamics

$$dY_t = z u_t dt + dW_t\ , \tag{1.2}$$

where

(i) z is an unobserved square-integrable random variable with known distribution μ ;

(ii) W is a standard, one-dimensional Brownian motion process, independent of z; and

(iii) u is a control process with values in $[-1,1]$, adapted to an "observation filtration" $\{\mathcal{F}_t\}$. This filtration will be specified later; it suffices to point out here that the process Y will be adapted to $\{\mathcal{F}_t\}$, and that the random variable z will be *independent* of $\mathcal{F}_\infty \triangleq \sigma(\cup_{0\le t<\infty} \mathcal{F}_t)$.

This continuous-time problem is related to the class of adaptive control questions, studied by Åström & Wittenmark (1973) in a discrete-time setting; see also the survey paper by Kumar (1985).

The difficulty in the problem at hand comes from the fact that z is an unobserved random variable. Indeed, if z is a specified real constant, it has been shown by Beneš (1974) that the control law

$$u_t^0 = -sgn(zY_t) \tag{1.3}$$

of the "bang-bang" type minimizes the expected cost (1.1); see also Ikeda & Watanabe (1977), Beneš, Shepp & Witsenhausen (1980), Karatzas & Shreve (1987), §6.5. In other words, according to (1.3) the controller has to exert full push in the direction opposite to that of $sgn(zY_t)$ - which becomes unknown, however, the moment z is taken to be an unobservable random variable.

In this latter case (i.e., when z is a non-degenerate random variable, independent of $\mathcal{F}_\infty$) it is tempting to guess that the *separation* (or *certainty-equivalence*) *principle* of replacing z in (1.3) by its least-squares estimate $\hat{z}_t = E(z|\mathcal{F}_t)$ leads to a control law

$$u_t^{opt} = -sgn(\hat{z}_t Y_t) \tag{1.4}$$

which is again optimal. This guess is further buttressed by the *innovations form* $dY_t = \hat{z}_t u_t dt + d\nu_t^u$ of the equation (1.2), where ν^u is the innovations Brownian motion process.

We shall treat the above problem as a question of *stochastic control with partial observations*, where z is the unobserved state and

Y is the observation process. Following Fleming & Pardoux (1982), we shall allow "wide-sense" control laws u, which are adapted to an observation filtration $\{\mathcal{F}_t;\ t \geq 0\}$ that contains the natural filtration $\{\mathcal{F}_t^Y\ ;\ t \geq 0\}$ with $\mathcal{F}_t^Y = \sigma(Y_s;\ 0 \leq s \leq t)$, as well as some extra information. It is in such a class that Fleming & Pardoux prove their basic existence result, and in such a class that we shall establish the optimality of the law (1.4) for non-degenerate but symmetric distributions μ with $\int_0^\infty x^2 \mu(dx) < \infty$ on the random variable z.

In particular, we shall see that in the Bernoulli case $\mu = \frac{1}{2}(\delta_\theta + \delta_{-\theta})$ for $\theta > 0$, the estimate $\hat{z}_t = E(z|\mathcal{F}_t)$ is given as $\hat{z}_t = \theta \tanh(\theta Z_t)$ in terms of a process Z that satisfies the stochastic differential equation

$$dZ_t = -sgn(Y_t Z_t) dY_t\ ,\quad Z_0 = 0\ . \tag{1.5}$$

It is convenient to study this equation under an equivalent probability measure that makes Y a Brownian motion, and the two-dimensional process (Y, Z) a *degenerate diffusion.* It shall be shown in section 6 that the equation (1.5) admits then a weak solution which is unique in the sense of the probability law, with an observation filtration $\{\mathcal{F}_t\}$ such that *both* processes Y and Z are adapted to it. We shall see (Proposition 6.2) that Z cannot be adapted to $\{\mathcal{F}_t^Y\}$, so that the optimal law of (1.4) cannot be adapted to $\{\mathcal{F}_t^Y\}$ either, and that *no "strict-sense"* (i.e., $\{\mathcal{F}_t^Y\}$ - adapted) *control law can be optimal* (Proposition 7.5).

2. WIDE-SENSE CONTROLS VIA EQUIVALENT CHANGE OF PROBABILITY MEASURE

Consider a standard, one-dimensional Brownian motion $Y = \{Y_t, \mathcal{F}_t;\ 0 \leq t < \infty\}$ with $Y_0 = y \in \mathcal{R}\backslash\{0\}$, on a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}$ which satisfies the *usual conditions*: $\mathcal{F}_{t+} = \mathcal{F}_t$, $\forall\ t \geq 0$ and $\mathcal{F}_0$ contains all the P-negligible sets in $\mathcal{F}$. The precise nature of $\{\mathcal{F}_t\}$ will be specified later (cf. Remark 4.2). Let the probability space be rich enough to support a random variable z *independent of* $\mathcal{F}_\infty$, with distribution μ that satisfies $\int_{\mathcal{R}} x^2\mu(dx) < \infty$, and denote by $\{\mathcal{G}_t\}$ the P-augmentation of the filtration $\{\sigma(z) \vee \mathcal{F}_t\ ;\ 0 \le t < \infty\}$.

2.1 Definition: We shall denote by $\mathcal{U}$ the class of *wide-sense admissible controls*, i.e. of $\{\mathcal{F}_t\}$ - progressively measurable processes $u = \{u_t;\ 0 \le t < \infty\}$ with values in $[-1,1]$.

2.2 Definition: The class $\mathcal{U}_s$ of *strict-sense admissible controls* consists of those processes $u \in \mathcal{U}$ which are adapted to $\{\mathcal{F}_t^Y\}$, the P-augmentation of the filtration generated by the Brownian motion Y. □

For every $u \in \mathcal{U}$, introduce the exponential martingale

$$(2.1) \qquad \Lambda_t^u \triangleq \exp\{z\int_0^t u_s dY_s - \frac{1}{2}z^2\int_0^t u_s^2 ds\}\ ; \quad 0 \le t < \infty$$

and the process

$$(2.2) \qquad W_t^u \triangleq Y_t - y - z\int_0^t u_s ds\ , \qquad 0 \le t < \infty\ .$$

From the Girsanov theorem (e.g. Karatzas & Shreve (1987), section 3.5) we know that the process $\{W_t^u,\ \mathcal{G}_t\ ; 0 \le t \le T\}$ is a standard Brownian motion on $[0,T]$, independent of the random variable z, under the probability measure

$$(2.3) \qquad P_T^u(A) = \int_A \Lambda_T^u dP\ , \quad A \in \mathcal{G}_T$$

for every finite $T > 0$. In other words $(\Omega, \mathcal{G}_T, P_T^u), \{\mathcal{G}_t\}, (W^u, Y)$ constitute a weak solution of the equation (1.2) on the finite horizon $[0,T]$.

We are now in a position to introduce the control problem of this paper.

2.3 PROBLEM: With the above notation and for a given finite $\alpha > 0$, minimize the performance criterion

$$(2.4) \quad J(u) \triangleq \lim_{T\to\infty} E_T^u \int_0^T e^{-\alpha t} \; Y_t^2 \; dt = \int_0^\infty e^{-\alpha t} \; E_t^u(Y_t^2) \; dt$$

over all wide-sense controls $u \in \mathcal{U}$. □

In other words, one is called upon to minimize the discounted expected quadratic deviation from the origin for the process Y, subject to the dynamics

$$(2.2)' \qquad dY_t = zu_t dt + dW_t^u \,, \qquad Y_0 = y$$

with W^u a standard, one-dimensional Brownian motion independent of z (under P_T^u), $|u_t| \leq 1$, and u adapted to $\{\mathcal{F}_t\}$; cf. (1.1), (1.2).

Using the Fubini theorem and the independence of z and $\mathcal{F}_\infty$, the performance criterion of (2.4) can be put in the equivalent, and very useful, form

$$(2.5) \quad \begin{aligned} J(u) &= \int_0^\infty e^{-\alpha t} \; E[Y_t^2 \Lambda_t^u] dt = E \int_0^\infty e^{-\alpha t} \; Y_t^2 E[\Lambda_t^u | \mathcal{F}_t] dt \\ &= E \int_0^\infty e^{-\alpha t} \; Y_t^2 \; F\Big(\int_0^t u_s^2 ds, \; \int_0^t u_s dY_s\Big) dt \,, \end{aligned}$$

where F is given by

$$(2.6) \quad F(t,x) \triangleq \int_{\mathcal{R}} \exp\{\varsigma x - \frac{t}{2}\varsigma^2\} \mu(d\varsigma) \; ; \quad (t,x) \in [0,\infty] \times \mathcal{R} \,.$$

2.4 Remark: It should be stressed that the "observation" filtration $\{\mathcal{F}_t\}$ is allowed to be strictly larger than $\{\mathcal{F}_t^Y\}$. At the same time, for every $u \in \mathcal{U}$ and $t \in [0, \infty)$ we notice that u_t is independent of future increments $\{Y_r - Y_t \; ; \; r \in [t, \infty)\}$, under the probability measure P. Both these qualitative properties are shared by the class of wide-sense controls in Fleming & Pardoux (1982).

2.5 Remark: From (2.2) and elementary properties of Brownian motion we obtain the bound

$$E_T^u(\sup_{0 \leq t \leq T} Y_t^2) \leq 2[y^2 + T(4 + Ez^2)] \tag{2.7}$$

for every finite $T > 0$. Put back into (2.4), this leads to the estimate

$$J(u) \leq c(1 + y^2) \; ; \qquad \forall \; u \in \mathcal{U} \tag{2.8}$$

for some constant $c > 0$ depending only on α and Ez^2.

3. FILTERING FORMULATION

It is assumed in this section that the random variable z is actually *bounded.* With this assumption, the function $F(t, x)$ of (2.6) satisfies

$$\begin{aligned} \frac{\partial^m}{\partial x^m} F(t,x) &= \int_{\mathcal{R}} \varsigma^m \exp\{\varsigma x - \frac{t}{2}\varsigma^2\}\mu(d\varsigma) \; , \\ \frac{\partial^r}{\partial t^r} F(t,x) &= \int_{\mathcal{R}} (-\frac{\varsigma^2}{2})^r \exp\{\varsigma x - \frac{t}{2}\varsigma^2\}\mu(d\varsigma) \end{aligned} \tag{3.1}$$

for every integers $m \geq 1$, $r \geq 1$, as well as the backward heat equation $\frac{\partial}{\partial t} F + \frac{1}{2} \frac{\partial^2}{\partial x^2} F = 0$; consequently, the function

$$G(t,x) \triangleq \frac{\partial}{\partial x} F(t,x) / F(t,x) \tag{3.2}$$

satisfies the equation

$$\frac{\partial}{\partial t} G + \frac{1}{2} \frac{\partial^2}{\partial x^2} G + G \frac{\partial}{\partial x} G = 0 \tag{3.3}$$

on $(0,\infty) \times \mathcal{R}$.

On the other hand, the Bayes rule (cf. Kallianpur (1980), p.282 or Karatzas & Shreve (1987), p.193) implies that the least-squares estimate

$$\hat{z}_t^u = E_t^u(z|\mathcal{F}_t) \tag{3.4}$$

of z, given the observations $\mathcal{F}_t$ up to time t, is given as

$$\hat{z}_t^u = \frac{E[z\Lambda_t^u|\mathcal{F}_t]}{E[\Lambda_t^u|\mathcal{F}_t]} = G\left(\int_0^t u_s^2 ds, \int_0^t u_s dY_s\right) . \tag{3.5}$$

Applying Itô's rule to (3.5) we obtain then, in conjunction with (3.2), the equation

$$d\hat{z}_t^u = u_t \, \frac{\partial}{\partial x} \, G\left(\int_0^t u_s^2 ds, \int_0^t u_s dY_s\right) \, d\nu_t^u \ . \tag{3.6}$$

Here

$$\nu_t^u \triangleq Y_t - y - \int_0^t u_s \hat{z}_s^u ds \ , \quad \mathcal{F}_t \ ; \quad 0 \le t < \infty \tag{3.7}$$

is the *innovations process*: for any given $T \in (0,\infty)$, $\{\nu_t^u;\ 0 \le t \le T\}$ is an $\{\mathcal{F}_t\}$ - Brownian motion under P_T^u.

3.1 Remark: By analogy with (3.4) - (3.7), for any $u \in \mathcal{U}_s$ the estimate $\check{z}_t^u \triangleq E_t^u(z|\mathcal{F}_t^Y)$ is given again by the right-hand side of (3.5), and satisfies the analogue

$$d\check{z}_t^u = u_t \, \frac{\partial}{\partial x} \, G\left(\int_0^t u_s^2 ds, \int_0^t u_s dY_s\right) \, d\check{\nu}_t^u$$

of (3.6), where now

$$\check{\nu}_t^u \triangleq Y_t - y - \int_0^t u_s \check{z}_s^u ds \ , \ \mathcal{F}_t \ ; \quad 0 \le t < \infty \tag{3.8}$$

is an $\{\mathcal{F}_t^Y\}$ - Brownian motion under P_T^u, on any given finite horizon $[0,T]$.

4. THE BERNOULLI CASE

We shall consider in this section the case of a Bernoulli random variable z with

$$P[z=\theta]=\rho\ , \qquad P[z=-\theta]=1-\rho \tag{4.1}$$

for some $\theta>0$ and $\rho \in (0,1)$. The case of a general, square-integrable random variable with symmetric distribution μ will be taken up again in section 8.

For a Bernoulli random variable as in (4.1), the expressions of (2.6), (3.2) become

$$F(t,x)=e^{-\theta^2 t/2}\,\frac{\cosh(b+\theta x)}{\cosh b}\ , \quad G(t,x)=\theta.\tanh(b+\theta x) \tag{4.2}$$

where

$$b\triangleq \tanh^{-1}(2\rho-1)\ . \tag{4.3}$$

Accordingly, for any given $u \in \mathcal{U}$, the expressions (3.5) and (2.5) take the form

$$\hat{z}_t^u=\theta.\tanh(\theta\xi_t^u) \tag{4.4}$$

and

$$\begin{aligned} J(u;\theta) &= \int_0^\infty e^{-\alpha t}E_t^u(Y_t^2)dt \\ &= \frac{1}{\cosh(\theta\xi)}\ E\int_0^\infty e^{-\int_0^t(\alpha+\frac{\theta^2}{2}u_s^2)ds}\ Y_t^2\cosh(\theta\xi_t^u)\ dt\ , \end{aligned} \tag{4.5}$$

respectively, where

$$\xi_t^u=\xi+\int_0^t u_s dY_s \qquad \xi\triangleq\frac{b}{\theta}=\frac{1}{\theta}\tanh^{-1}(2\rho-1)\ . \tag{4.6}$$

Furthermore, we obtain from (3.7), (4.4) and (4.6) the useful dynamical equations

(4.7) $$dY_t = u_t\theta\tanh(\theta\xi_t^u)dt + d\nu_t^u, \qquad Y_0 = y$$

(4.8) $$d\xi_t^u = u_t^2\theta\tanh(\theta\xi_t^u)dt + u_t d\nu_t^u, \qquad \xi_0^u = \xi$$

for the pair (Y, ξ^u), driven by the innovations process ν^u.

Now the Separation (or Certainty-Equivalence) Principle of the Introduction suggests considering

(4.9) $$u_t^* \triangleq -sgn(\hat{z}_t^* Y_t) = -sgn(Z_t Y_t)$$

as a candidate optimal law in $\mathcal{U}$, with the process $Z \equiv \xi^{u^*}$ required to satisfy the stochastic equation

(4.10) $$Z_t = \xi - \int_0^t sgn(Y_s Z_s)dY_s, \quad 0 \le t < \infty$$

in accordance with (4.6), and the process $\hat{z}^* \equiv \hat{z}^{u^*}$ given by

(4.11) $$\hat{z}_t^* = \theta.\tanh(\theta Z_t)\ , \qquad 0 \le t < \infty$$

in accordance with (4.4). It should be noted that (4.10) is the same as the equation (1.5), but for a possibly different initial condition; indeed, from (4.3) and (4.6), $\xi = 0$ if and only if $\rho = \frac{1}{2}$. One may also note that the choice (4.9) has the desirable (from the point of view of expected cost minimization) effect of "reinforcing" the discount factor on the right-hand side of (4.5), since $(u_s^*)^2 \equiv 1$.

One is thus led to the following question in stochastic differential equations:

4.1 PROBLEM: Find a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions, as well as two $\{\mathcal{F}_t\}$ - adapted processes Y, Z with continuous sample paths and $(Y_0, Z_0) = (y, \xi) \in \mathcal{R}^2 \backslash \{\underset{\sim}{0}\}$, such that under P,

(i) Y is a standard, one-dimensional Brownian motion starting at y , and

(ii) the equation (4.10) is satisfied almost surely.

In other words, Problem 4.1 seeks a *weak solution* for equation (4.10). It shall be shown in section 6 that such a solution exists, and is unique in the sense of the probability law. On the other hand, we shall connect Problem 4.1 in section 5 with a two-parameter Time-Change, and with a Martingale Problem in the sense of Stroock & Varadhan (1979).

4.2 Remark: Once a solution to Problem 4.1 has been constructed, one may take $(\Omega, \mathcal{F}, P)$ as the basic probability space, and $\{\mathcal{F}_t\}$ as the "observation filtration" in the control problem of section 2 (in particular, in the Definition 2.1 of wide-sense admissible controls). With this setup the process u^* of (4.9) is then a wide-sense admissible control: $u^* \epsilon\, \mathcal{U}$.

Denoting by $\nu^* = \nu^{u^*}$ the associated innovations process of (3.7), we obtain the analogues

$$(4.12) \qquad dY_t = -\theta . sgn(Y_t Z_t) . \tanh(\theta Z_t) dt + d\nu_t^* \ , \qquad Y_0 = y$$

$$(4.13) \qquad dZ_t = \theta . \tanh(\theta Z_t) dt - sgn(Y_t Z_t) d\nu_t^* \ , \qquad Z_0 = \xi$$

of the equations (4.7), (4.8) in this case.

5. ASSOCIATED TIME-CHANGE AND MARTINGALE PROBLEMS

Let us suppose that a solution of Problem 4.1 has been constructed. Then Z is necessarily a Brownian motion, because it is a continuous local martingale with quadratic variation $< Z >_t = t$ (the P. Lévy Theorem 3.3.16 in Karatzas & Shreve (1987)). On the other hand, the processes

$$X_1(t) \triangleq \frac{1}{2}\,[\,Y_t + Z_t\,]\,, \qquad X_2(t) \triangleq \frac{1}{2}\,[\,Y_t - Z_t\,] \tag{5.1}$$

are also continuous local martingales, and we have from (4.9)

$$u_t^* = -sgn(Y_t Z_t) = \begin{Bmatrix} -1 & ; & \text{if} & |X_1(t)| > |X_2(t)| \\ 1 & ; & \text{if} & |X_1(t)| \leq |X_2(t)| \end{Bmatrix}$$

as well as $< X_1, X_2 > (t) = 0$,

$$\begin{aligned} < X_1 > (t) &= \frac{1}{4}\int_0^t (1+u_s^*)^2 ds \\ &= meas\{0 \leq s \leq t;\ |X_1(s)| \leq |X_2(s)|\} =:\ T_1(t) \\ < X_2 > (t) &= \frac{1}{4}\int_0^t (1-u_s^*)^2 ds \\ &= meas\{0 \leq s \leq t;\ |X_1(s)| > |X_2(s)|\} =:\ T_2(t)\ , \end{aligned}$$

where "meas" stands for "Lebesgue measure". From a result of F. Knight (Ikeda & Watanabe (1981), Ch. II, Theorems 7.3 and 7.3$'$ or Karatzas & Shreve (1987), Theorem 3.4.13), the processes

$$B_j(s) \triangleq X_j(T_j^{-1}(s)) - x_j\ ; \qquad 0 \leq s < \infty\ , \qquad j = 1,2$$

with $x_1 = \frac{1}{2}(y+\xi)$, $x_2 = \frac{1}{2}(y-\xi)$, are *independent* standard Brownian motions, and we have

$$X_j(t) = x_j + B_j(T_j(t)), \qquad 0 \leq t < \infty\ , \qquad j = 1,2\ .$$

It follows then that X_1, X_2 constitute a solution to the following *two-parameter time-change problem:*

5.1 PROBLEM: Find a complete probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions, and on it

(i) two independent standard Brownian motions B_1, B_2, and

(ii) two continuous, $\{\mathcal{F}_t\}$ - adapted processes X_1 and X_2, such that

(5.2)
$$X_j(t) = x_j + B_j\Big(\int_0^t 1_{Q_j}(\underset{\sim}{X}(s))ds\Big)\ , \quad 0 \le t < \infty\ , \quad j = 1, 2\ ,$$

for any given $\underset{\sim}{x} = (x_1, x_2) \in \mathcal{R}^2 \backslash \{0\}$, where $\underset{\sim}{X} = (X_1, X_2)$ and

(5.3)
$$Q_1 \triangleq \{\underset{\sim}{x} \in \mathcal{R}^2;\ |x_1| \le |x_2|\}, \quad Q_2 \triangleq \{\underset{\sim}{x} \in \mathcal{R}^2;\ |x_1| > |x_2|\}\ .$$

□

It can also be shown that any solution of Problem 5.1 induces one for Problem 4.1 as well, so that the two problems are actually *equivalent.*

Multi-parameter time-change problems, such as Problem 5.1, were introduced by Kurtz (1980). From the theory of this article, as well as that of Chapter 6 in Ethier & Kurtz (1986) (in particular, Problem 6.2), it follows that Problem 5.1 is equivalent to solving the following *martingale problem* associated with the second-order differential operator

$$\mathcal{L}f \triangleq \frac{1}{2}\Big(1_{Q_1} \frac{\partial^2 f}{\partial x_1^2} + 1_{Q_2} \frac{\partial^2 f}{\partial x_2^2} \Big)\ . \tag{5.4}$$

5.2 PROBLEM: On the canonical space $\Omega = C([0,\infty); \mathcal{R}^2)$ of continuous functions $\omega : [0, \infty) \to \mathcal{R}^2$, equipped with the σ - field $\mathcal{B} = \sigma(\omega(s);\ 0 \le s < \infty)$ and the filtration $\mathcal{B}_t = \sigma(\omega(s);\ 0 \le s \le$

t), $0 \le t < \infty$, find a probability measure $\mathbf{P}$ such that

$$\mathbf{P}[\omega(0) = \underset{\sim}{x}] = 1\,, \qquad \text{and} \tag{5.5}$$

$$\begin{gathered}\{f(\omega(t)) - \int_0^t \mathcal{L}f(\omega(s))ds,\ \mathcal{B}_t\,;\quad 0 \le t < \infty\} \\ \text{is a } \mathbf{P} - \text{martingale for every } f \in C_0^2(\mathcal{R}^2)\end{gathered} \tag{5.6}$$

hold for arbitrary but fixed $\underset{\sim}{x} \in \mathcal{R}^2 \backslash \{\underset{\sim}{0}\}$. □

It is possible to show that the Martingale Problem 5.2 is well-posed, i.e., admits exactly one solution. We shall not follow this tack; instead, we shall prove in the next section, by elementary and direct arguments, that the original Problem 4.1 admits a solution which is unique in the sense of probability law.

5.3 Remark: It is instructive to study the diffusion mechanism of the two-dimensional process $\underset{\sim}{X} = (X_1, X_2)$ in Problem 5.1. We work throughout with the convention $y = x_1 + x_2$, $\xi = x_1 - x_2$.

In the "North" and "South" quadrangles

$$\begin{aligned} Q_1^+ &\triangleq \{(y,\xi)\epsilon\mathcal{R}^2;\ y > 0, \xi < 0\}\,, \\ Q_1^- &\triangleq \{(y,\xi)\epsilon\mathcal{R}^2;\ y < 0, \xi > 0\}\,, \end{aligned} \tag{5.7}$$

respectively (cf. Figure), the process $\underset{\sim}{X}$ diffuses in the horizontal (East-West) direction according to the Brownian motion B_1. On the other hand, in the "East" and "West" quadrangles

$$\begin{aligned} Q_2^+ &\triangleq \{(y,\xi)\epsilon\mathcal{R}^2;\ y > 0, \xi > 0\}\,, \\ Q_2^- &\triangleq \{(y,\xi)\epsilon\mathcal{R}^2;\ y < 0, \xi < 0\}\,, \end{aligned} \tag{5.8}$$

respectively, the process $\underset{\sim}{X}$ diffuses in the vertical (North-South) direction, according to the *independent* Brownian motion B_2. All this follows, quite obviously, from the representations (5.2).

What is not so obvious is the fact that, in addition to these two modes of diffusion, there is also *outward motion along the lines* $\{\xi = 0\}$ *and* $\{y = 0\}$, so that

$$|Y_t| + |Z_t| = 2\max(|X_1(t)|\ ,\ |X_2(t)|) \tag{5.9}$$

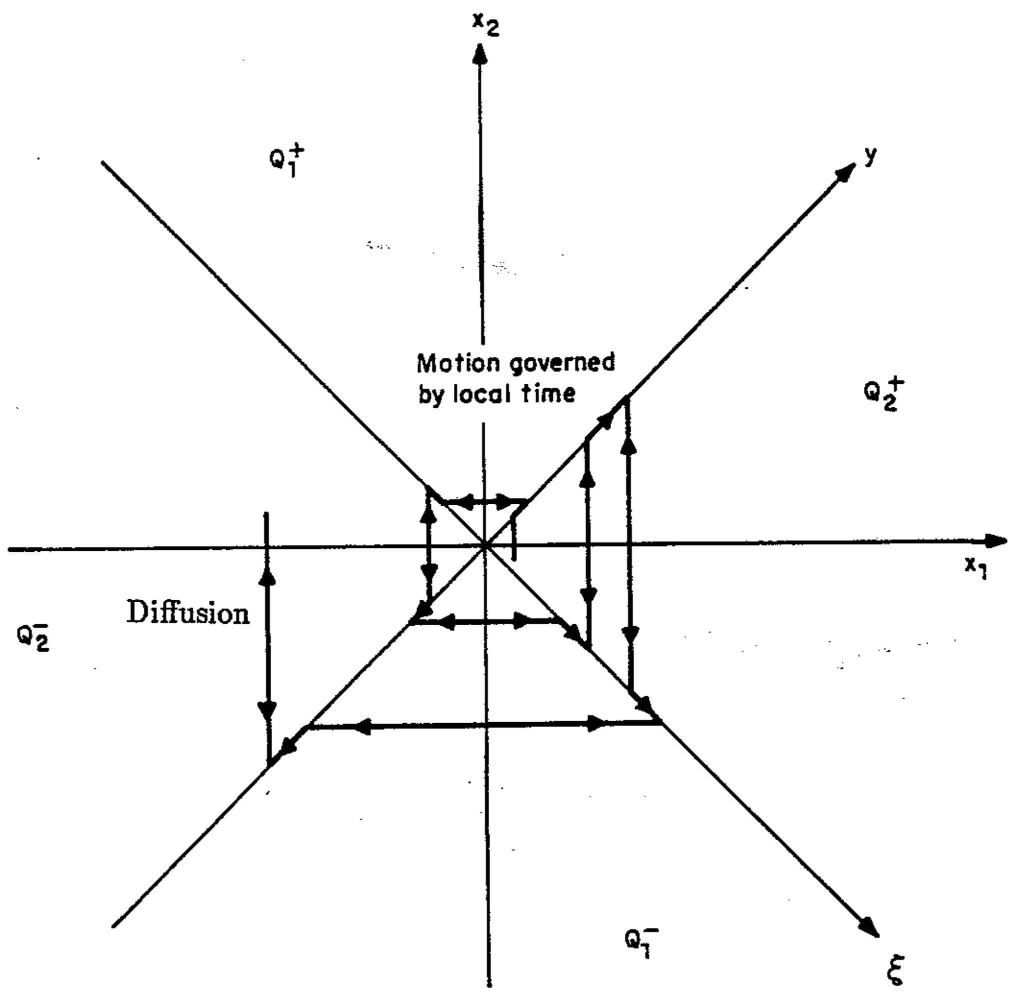
is a continuous, nondecreasing process. In order to see this, and the fact that the process of (5.9) is governed by *local times*, start with the Tanaka formulae

$$\begin{aligned} |Y_t| &= |y| + \int_0^t sgn(Y_s).dY_s + L^Y(t)\ , \\ |Z_t| &= |\xi| + \int_0^t sgn(Z_s).dZ_s + L^Z(t) \end{aligned} \tag{5.10}$$

for the local times $L^Y(\cdot)$, $L^Z(\cdot)$ at the origin of the two Brownian motions Y and Z (Karatzas & Shreve (1987), p.205). Recalling the equation (4.10), we obtain by adding up:

$$|Y_t| + |Z_t| = |y| + |\xi| + [L^Y(t) + L^Z(t)]\ , \tag{5.11}$$

justifying the claim that the process of (5.9) is indeed continuous and nondecreasing.



FIGURE

6. EXISTENCE AND UNIQUENESS

In this section we establish an existence and uniqueness result (Theorem 6.1) for Problem 4.1. We also show that the solution to Problem 4.1 cannot be strong, i.e., that *the process* Z *cannot be adapted to* $\{\mathcal{F}_t^Y\}$ (Proposition 6.2).

6.1 Theorem: For every $(y,\xi) \in \mathcal{R}^2\backslash\{\underset{\sim}{0}\}$, Problem 4.1 has a solution which is unique in the sense of the probability law; i.e., the law of (Y,Z) is uniquely determined.

Proof of existence: Take $y \neq 0$, $\xi \in \mathcal{R}$, and let B be a standard, one-dimensional Brownian motion on a complete probability space $(\Omega,\mathcal{F},P)$, with respect to a filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions. Define the continuous, $\{\mathcal{F}_t\}$ - adapted processes

$$
\begin{aligned}
Z_1(t) &\triangleq \xi + B(t) \\
Y_1(t) &\triangleq y - sgny.\int_0^t sgnZ_1(s).dB(s) \qquad (6.1)\\
&= y - \int_0^t sgn(Y_1(s)Z_1(s)).dZ_1(s)
\end{aligned}
$$

for $0 \leq t < \tau_1$, where $\tau_1 \triangleq \inf\{t \geq 0;\ Y_1(t) = 0\}$ is an $\{\mathcal{F}_t\}$ - stopping time with values in $(0,\infty)$. By analogy with (5.11) we have

$$
\begin{aligned}
|Z_1(\tau_1)| &= |Y_1(\tau_1)| + |Z_1(\tau_1)| \qquad (6.2)\\
&= (|y| + |\xi|) + [L^{Y_1}(\tau_1) + L^{Z_1}(\tau_1)] \geq |y| \,,
\end{aligned}
$$

whence also $Z_1(\tau_1) \neq 0$, almost surely. Continue now by defining

$$
\begin{aligned}
Y_2(t) &\triangleq B(t) - B(\tau_1) \\
Z_2(t) &\triangleq Z_1(\tau_1) - sgn(Z_1(\tau_1)).\int_{\tau_1}^t sgn(Y_2(s)).dB(s) \qquad (6.3)\\
&= Z_2(\tau_1) - \int_{\tau_1}^t sgn(Y_2(s)Z_2(s)).dY_2(s)
\end{aligned}
$$

for $\tau_1 \le t < \tau_2$, where $\tau_2 \triangleq \inf\{t \ge \tau_1;\ Z_2(t) = 0\}$ is an $\{\mathcal{F}_t\}$ - stopping time with $P[\tau_1 < \tau_2 < \infty] = 1$. By analogy with (5.11), (6.2) we have

$$
\begin{aligned}
|Y_2(\tau_2)| &= |Y_2(\tau_2)| + |Z_2(\tau_2)| \\
&= |Z_1(\tau_1)| + [L^{Y_2}(\tau_2) - L^{Y_2}(\tau_1)] \\
&\qquad + [L^{Z_2}(\tau_2) - L^{Z_2}(\tau_1)] > |Z_1(\tau_1)|
\end{aligned}
\tag{6.4}
$$

almost surely. Notice also from (6.3), (6.4) that τ_2 is stochastically larger than the first passage time of B to the level $2|y|$.

Continuing this way, we construct a strictly increasing sequence of stopping times $\{\tau_m\}_{m=1}^{\infty}$ such that, for every $n \ge 1$, we have almost surely

(i) on $[\tau_{2n}, \tau_{2n+1}) :\ Y_{2n}(\tau_{2n}) \ne 0,\ Z_{2n}(\tau_{2n}) = 0$

$$
Z_{2n+1}(t) \triangleq B(t) - B(\tau_{2n})\ , \qquad \tau_{2n} \le t < \tau_{2n+1}
\tag{6.5}
$$

(6.6)

$$
\begin{aligned}
Y_{2n+1}(t) &\triangleq Y_{2n}(\tau_{2n}) - sgn(Y_{2n}(\tau_{2n})).\int_{\tau_{2n}}^{t} sgn(Z_{2n+1}(s)).dB(s) \\
&= Y_{2n+1}(\tau_{2n}) - \int_{\tau_{2n}}^{t} sgn(Y_{2n+1}(s)Z_{2n+1}(s)).dZ_{2n+1}(s)\ ,
\end{aligned}
$$

$$
\tau_{2n+1} = \inf\{t \ge \tau_{2n};\ \ Y_{2n+1}(t) = 0\}
\tag{6.7}
$$

$$
|Z_{2n+1}(\tau_{2n+1})| > |Y_{2n}(\tau_{2n})|\ ,
\tag{6.8}
$$

and

(ii) on $[\tau_{2n+1}, \tau_{2n+2}) :\ Z_{2n+1}(\tau_{2n+1}) \ne 0,\ Y_{2n+1}(\tau_{2n+1}) = 0$

$$
Y_{2n+2}(t) \triangleq B(t) - B(\tau_{2n+1})\ , \qquad \tau_{2n+1} \le t < \tau_{2n+2}
\tag{6.9}
$$

(6.10)

$$Z_{2n+2}(t) \triangleq Z_{2n+1}(\tau_{2n+1}) - \\ - sgn(Z_{2n+1}(\tau_{2n+1})). \int_{\tau_{2n+1}}^{t} sgn(Y_{2n+2}(s)).dB(s) \\ = Z_{2n+2}(\tau_{2n+1}) - \int_{\tau_{2n+1}}^{t} sgn(Y_{2n+2}(s)Z_{2n+2}(s)).dY_{2n+2}(s)$$

(6.11) $$\tau_{2n+2} = \inf\{t \geq \tau_{2n+1} \; ; \quad Z_{2n+2}(t) = 0\}$$

(6.12) $$|Y_{2n+2}(\tau_{2n+2})| > |Z_{2n+1}(\tau_{2n+1})| \; .$$

It follows from (6.7), (6.8) and (6.11), (6.12) that τ_m is stochastically larger than the first passage time of B to the level $m|y|$, for every integer $m \geq 1$. Consequently, $\lim_{m\to\infty} \tau_m = \infty$, a.s.

We can define now, consistently and on all of $[0, \infty)$, the pair of continuous and $\{\mathcal{F}_t\}$ - adapted processes (Y, Z) by setting

(6.13) $$(Y(t), Z(t)) \triangleq (Y_m(t), Z_m(t)) \; , \quad \text{for} \quad \tau_{m-1} \leq t < \tau_m \; .$$

From (6.6), (6.10) it follows readily that the equation (4.10) is satisfied on $[0, \infty)$.

Proof of uniqueness: Without loss of generality one may take $\xi = 0$, $y \neq 0$ and consider a solution $(\Omega, \mathcal{F}, P)$,$\{\mathcal{F}_t\}$, (Y, Z) of Problem 4.1. The process

(6.14) $$W_t \triangleq - \int_0^t sgnZ_s \, dY_s \; , \quad \mathcal{F}_t \; ; \quad 0 \leq t < \infty$$

is then standard, one-dimensional Brownian motion, and (6.14), (4.10) imply

(6.15) $$\left\{ \begin{array}{lcl} Z_t & = & \int_0^t sgnY_s.dW_s \\ Y_t & = & y - \int_0^t sgnZ_s.dW_s \end{array} \right\} \; , \quad 0 \leq t < \infty \; .$$

Clearly, in order to prove that the law of the pair (Y, Z) is uniquely determined, it suffices to show that *pathwise uniqueness holds for the equation (6.15).*

To this effect, let (Y', Z') be another solution of (6.15), on the same probability space $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$ and with respect to the same Brownian motion W:

$$(6.15)' \qquad \left\{ \begin{array}{lcl} Z_t' & = & \int_0^t sgnY_s'.dW_s \\ Y_t' & = & y - \int_0^t sgnZ_s'.dW_s \end{array} \right\} , \quad 0 \leq t < \infty .$$

By analogy with (6.7), let $\rho_1 \triangleq \inf\{t \geq 0;\ Y_t = 0 \ \text{ or } \ Y_t' = 0\}$. This is an $\{\mathcal{F}_t\}$ - stopping time with values in $(0, \infty)$, and we have $sgnY = sgnY'$ on $[0, \rho_1]$. On the other hand, by analogy with (6.2) we obtain

$$|Z_{\rho_1}| = |Y_{\rho_1}| + |Z_{\rho_1}| = |y| + [L^Y(\rho_1) + L^Z(\rho_1)] > |y| .$$

Similarly, define the $\{\mathcal{F}_t\}$ - stopping time $\rho_2 \triangleq \inf\{t \geq \rho_1;\ Z_t = 0 \text{ or } Z_t' = 0\}$ for which $\rho_1 < \rho_2 < \infty$, a.s., and show as before that $(Y, Z) = (Y', Z')$ on $[\rho_1, \rho_2]$, as well as $|Y_{\rho_2}| > |Z_{\rho_1}|$ almost surely. Continuing this procedure, one comes up with a strictly increasing sequence of $\{\mathcal{F}_t\}$ - stopping times $\{\rho_m\}_{m=1}^{\infty}$, such that $(Y, Z) = (Y', Z')$ on $[0, \rho_m]$, for every $m \geq 1$. Just as in the existence part of the proof, each ρ_m is stochastically larger than the time it takes W to hit the level $m|y|$. Consequently, $\lim_{m \to \infty} \rho_m = \infty$ a.s., which proves

$$P\ [\ (Y_t, Z_t) = (Y_t', Z_t')\ ;\ \ \forall \ \ 0 \leq t < \infty\ ] = 1 .$$

□

6.2 Proposition: Let $(\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}, (Y, Z)$ be a solution to Problem 4.1. Then

$$(6.16) \qquad Z\ ,\ sgnZ \quad \text{cannot be adapted to } \{\mathcal{F}_t^Y\} .$$

Proof: For concreteness, assume $y > 0$, $\xi \neq 0$, set $\tau_0 \equiv 0$, and define the strictly increasing sequence of $\{\mathcal{F}_t\}$ - stopping times $\{\tau_m\}_{m=1}^{\infty}$ as in (6.7), (6.11):

$$\tau_{2n+1} = \inf\{t \geq \tau_{2n};\ Y(t) = 0\}, \quad \tau_{2n+2} = \inf\{t \geq \tau_{2n+1};\ Z(t) = 0)\}$$

for $n \geq 0$. On an interval $[\tau_{2n}, \tau_{2n+1})$ we have from (6.6) and the Tanaka formula (5.10):

$$\begin{aligned} Y(t) - Y(\tau_{2n}) &= -sgnY(\tau_{2n}). \int_{\tau_{2n}}^{t} sgnZ(s).dZ(s) \\ &= -sgnY(\tau_{2n})[|Z(t)| - (L^Z(t) - L^Z(\tau_{2n}))] \ . \end{aligned} \tag{6.17}$$

In other words, Y is determined by $Y(\tau_{2n})$ and $|Z|$, during any interval of the type $[\tau_{2n}, \tau_{2n+1})$; similarly, Z is determined by $Z(\tau_{2n+1})$ and $|Y|$, during any interval of the type $[\tau_{2n+1}, \tau_{2n+2})$.

Suppose that Z were adapted to $\{\mathcal{F}_t^Y\}$; then $\{\mathcal{F}_t^Z\} \subseteq \{\mathcal{F}_t^Y\}$, and from (6.17) with $n = 0$, $\tau = \tau_1$ we would have also

$$Y(t) = -\int_0^t sgnZ(s).dZ(s) = L^Z(t) - |Z(t)|, \quad 0 \leq t < \tau \tag{6.18}$$

and thus $\{\mathcal{F}_{t\wedge\tau}^Y\} \subseteq \{\mathcal{F}_{t\wedge\tau}^{|Z|}\}$. We would thus be led to the conclusion $\{\mathcal{F}_{t\wedge\tau}^Z\} \subseteq \{\mathcal{F}_{t\wedge\tau}^{|Z|}\}$, an absurdity.

In fact, $sgnZ(\tau)$ *is independent of* $\mathcal{F}_\tau^Y$. In order to see this, notice that for any $\theta \in \mathcal{R}$ and any $\{\mathcal{F}_t^Y\}$ - adapted process ϕ with $\int_0^\tau \phi_t^2 dt < \infty$ almost surely, we have from (6.18):

$$E[sgnZ(\tau).\exp\{i\theta \int_0^\tau \phi_t\ dY(t)\}] = \tag{6.19}$$

$$= E[sgnZ(\tau).\exp\{-i\theta \int_0^\tau \phi_t\ sgnZ(t).dZ(t)\}] = 0 \ ,$$

because the change from Z to $-Z$ leaves the exponent invariant. From (6.19) we obtain easily

$$E[1_{\{sgnZ(\tau)=\pm 1\}}.\exp\{i\theta \int_0^\tau \phi_t\ dY(t)\}] = \frac{1}{2} E[\exp\{i\theta \int_0^\tau \phi_t\ dY(t)\}] \ ,$$

and the independence of $sgnZ(\tau)$, $\mathcal{F}_\tau^Y$ follows.

□

6.3 Corollary: The control process $u^* \in \mathcal{U}$ of (4.9) does *not* belong to the class $\mathcal{U}_s$ of Definition 2.2.

7. SOLUTION OF THE CONTROL PROBLEM: THE BERNOULLI CASE

Let us consider again the Bernoulli case $P[z = \theta] = \rho$, $P[z = -\theta] = 1 - \rho$ of section 4 with $\theta > 0$, $\rho \in (0,1)$, and try to study the performance of the law u^* of (4.9) in this case. According to (2.4), (4.5), and with the notation $P_T^* \equiv P_T^{u^*}$ as in (2.3) for any $T \in (0, \infty)$, this performance is given by

$$\begin{aligned}\Phi(y,\xi;\theta) &= \lim_{T\to\infty} E_T^* \int_0^T e^{-\alpha t}\, Y_t^2 dt = \int_0^\infty e^{-\alpha t} E_t^*\, (Y_t^2) dt \\ &= \frac{1}{\cosh(\theta\xi)}\, E \int_0^\infty e^{-(\alpha+\frac{\theta^2}{2})t}\, Y_t^2 \cosh(\theta Z_t) dt\,. \end{aligned} \tag{7.1}$$

This expression is uniquely determined for every $\alpha > 0$, $\theta > 0$ and $(y,\xi) \neq (0,0)$, by virtue of Theorem 6.1. The resulting function $\Phi(\cdot\,,\cdot) \equiv \Phi(\cdot\,,\cdot\,;\theta)$ (†) satisfies the symmetry properties

$$\Phi(y,\xi) = \Phi(y,-\xi) = \Phi(-y,\xi) = \Phi(-y,-\xi)\,. \tag{7.2}$$

We expect Φ to be of class C^2, in which case (7.2) leads to the properties

$$\Phi_\xi(y,0) = 0\,, \qquad \Phi_y(0,\xi) = 0 \tag{7.3}$$

$$\Phi_{y\xi}(y,\xi) = \Phi_{y\xi}(-y,-\xi) = -\Phi_{y\xi}(-y,\xi) = -\Phi_{y\xi}(y,-\xi) \tag{7.4}$$

(†) We drop the dependence on θ, whenever convenient.

as well. We also expect Φ to be of class C^∞ on $\mathcal{R}^2 \backslash \{(y,\xi);\ y\xi = 0\}$, and to satisfy the resolvent equation

(7.5)

$$\frac{1}{2}[\Phi_{yy}+\Phi_{\xi\xi}]-sgn(y\xi)[\Phi_{y\xi}+\theta\tanh(\theta\xi).\Phi_y]+\theta\tanh(\theta\xi).\Phi_\xi+y^2 = \alpha\Phi$$

suggested by (4.12), (4.13), and the growth condition

$$|\Phi(y,\xi)| \le c(1+y^2) \tag{7.6}$$

suggested by (2.8).

In this section, the function Φ of (7.1) is studied in some detail. We shall establish, in particular, its properties

$$\frac{1}{2}\Phi_{\xi\xi} + \theta\tanh(\theta\xi)\Phi_\xi < 0 \tag{7.7}$$

$$sgn[\Phi_{y\xi} + \theta\tanh(\theta\xi)\Phi_y] = sgn(y\xi)\ , \tag{7.8}$$

which will allow us to cast the equation (7.5) in the form

(7.9)

$$\frac{1}{2}\Phi_{yy} + \min_{|u|\le 1}[u\{\Phi_{y\xi} + \theta\tanh(\theta\xi).\Phi_y\} + u^2\{\frac{1}{2}\Phi_{\xi\xi} + \theta\tanh(\theta\xi).\Phi_\xi\}] + y^2 = \alpha\Phi\ .$$

This is the formal **Hamilton-Jacobi-Bellman** *(HJB)* **equation** *for the stochastic control problem of section 4.* Using it in conjunction with the dynamical equations (4.7), (4.8), we shall show that the control law u^* of (4.9) is optimal for the control problem in question and that Φ is the corresponding value function, i.e.,

$$J(u^*;\theta) = \Phi(y,\xi;\theta) \le J(u;\theta)\ , \qquad \forall \quad u \in \mathcal{U}\ . \tag{7.10}$$

Let us continue this heuristic discussion by considering the function

$$V(y,\xi) \triangleq \cosh(\theta\xi).\Phi(y,\xi) \tag{7.11}$$

and noticing that, in terms of it, the equation (7.5) and the relations (7.7), (7.8) take the equivalent forms

$$\frac{1}{2}[V_{yy} + V_{\xi\xi}] - sgn(y\xi).V_{y\xi} + y^2 \cosh(\theta\xi) = \lambda V \tag{7.12}$$

$$\theta^2 V > V_{\xi\xi} \tag{7.13}$$

$$sgn(V_{y\xi}) = sgn(y\xi) \tag{7.14}$$

where $\lambda \triangleq \alpha + \theta^2/2$. Let us look at the equation (7.12) in the region Q_2^+ of (5.8), where it takes the form

$$\frac{1}{2}V_{yy} - V_{y\xi} + \frac{1}{2}V_{\xi\xi} + y^2 \cosh(\theta\xi) = \lambda V \ , \qquad \text{in} \quad Q_2^+ \ , \tag{7.15}$$

and differentiate formally with respect to y and ξ; we arrive then at the equation

$$\frac{1}{2}U_{yy} - U_{y\xi} + \frac{1}{2}U_{\xi\xi} + 2\theta y \sinh(\theta\xi) = \lambda U \ , \qquad \text{in} \quad Q_2^+ \ , \tag{7.16}$$

for the function

(7.17)

$$U(y,\xi) \triangleq V_{y\xi}(y,\xi) = \cosh(\theta\xi)[\Phi_{y\xi}(y,\xi) + \theta \tanh(\theta\xi).\Phi_y(y,\xi)]$$

which also satisfies the boundary conditions

$$U(y, 0+) = 0, \quad U(0+, \xi) = 0 \tag{7.18}$$

(recall (7.3), (7.4)).

It turns out that the equation (7.16) admits a unique solution in Q_2^+, subject to the boundary conditions (7.18).

7.1 Proposition: There is a unique solution to (7.16), (7.18); it is given in Q_2^+ by

$$\begin{aligned} \frac{\alpha^2}{2\theta^2}\, U(y,\xi) = {} & \frac{\alpha y}{\theta} \sinh(\theta\xi) - \cosh(\theta\xi) + \\ & + \frac{\cosh(\theta(y+\xi)) \sinh(\xi\sqrt{2\lambda}) + \sinh(y\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})} \ , \end{aligned} \tag{7.19}$$

and satisfies

$$(7.20) \qquad 0 \le \theta^2 U(y,\xi) < U_{\xi\xi}(y,\xi)\ , \qquad \text{in} \quad Q_2^+ \ .$$

Proof. The same change of variables as in (5.1), namely $H(x_1,x_2) \triangleq U(x_1+x_2, x_1-x_2)$, transforms (7.16) into the second-order ordinary differential equation

$$(7.16)' \qquad \frac{1}{2}h''(x_2) - \lambda h(x_2) = -2\theta(x_1+x_2).\sinh(\theta(x_1-x_2))$$

with $x_2 \in (-x_1, x_1)$; for the function $h(\cdot) \triangleq H(x_1, \cdot)$. This equation has to be solved in the interval $[-x_1, x_1]$ subject to the boundary conditions

$$(7.18)' \qquad h(\pm x_1) = 0\ ,$$

for every $x_1 \in (0,\infty)$. The general solution of $(7.16)'$ is given by

$$(7.21) \qquad \begin{aligned} H(x_1,x_2) = C(x_1).\sinh(x_2\sqrt{2\lambda}) + D(x_1).\cosh(x_2\sqrt{2\lambda}) + \\ + \frac{2\theta}{\alpha}(x_1+x_2)\sinh(\theta(x_1-x_2)) \\ - \frac{2\theta^2}{\alpha^2}\cosh(\theta(x_1-x_2)) \end{aligned}$$

for appropriate functions $C(\cdot)$, $D(\cdot)$. The latter are determined from $(7.18)'$ as

$$C(z) = \frac{\theta^2}{\alpha^2}\ \frac{1-\cosh(2\theta z)}{\sinh(z\sqrt{2\lambda})}\ , \quad D(z) = \frac{\theta^2}{\alpha^2}\ \frac{1+\cosh(2\theta z)}{\cosh\ (z\sqrt{2\lambda})}\ ,$$

and substitution of these expressions into (7.21) leads ultimately to (7.19).

For any given $x_1 > 0$, if $h(\cdot)$ takes a negative value in $(-x_1, x_1)$, it must also achieve a negative minimum in this interval; but by the maximum principle (cf. Friedman (1964), p.53) this is impossible because from $(7.16)'$ we have $\frac{1}{2}h'' - \lambda h < 0$ in $(-x_1, x_1)$. The first inequality in (7.20) has been established.

For the second, extensive computation, starting with (7.19), leads to the expressions

(7.22)
$$\frac{\alpha^2}{2\theta^2} U_\xi(y,\xi) = \alpha y \cosh(\theta\xi) - \theta \sinh(\theta\xi)\{1 - \frac{\sinh(\xi\sqrt{2\lambda})}{\sinh((y+\xi)\sqrt{2\lambda})}\} + \\ + \sqrt{2\lambda}\frac{\sinh(y\sqrt{2\lambda})}{\sinh^2((y+\xi)\sqrt{2\lambda})}[\cosh(\theta(y+\xi)) - \cosh((y+\xi)\sqrt{2\lambda})]$$

and

$$(7.23) \quad U_{\xi\xi}(y,\xi) - \theta^2 U(y,\xi) = \frac{2\theta^2}{\alpha^2} \frac{\sinh(y\sqrt{2\lambda})}{\sinh^3((y+\xi)\sqrt{2\lambda})} F(y+\xi) \ ,$$

where

$$(7.24) \quad \begin{aligned} F(z) \triangleq\ & 4\lambda + (2\lambda - \theta^2)\sinh^2(z\sqrt{2\lambda}) \\ & - 2\sqrt{2\lambda}\ [\ \theta.\cosh((\sqrt{2\lambda}-\theta)z) \\ & + (\sqrt{2\lambda}-\theta)\cosh(\theta z).\cosh(z\sqrt{2\lambda})\ ]\ . \end{aligned}$$

But this function is positive on $(0,\infty)$, as one can check easily from $F(0) = 0$ and from

(7.25)
$$F'(z) = 2\sqrt{2\lambda}(2\lambda - \theta^2)\sinh(z\sqrt{2\lambda})[\cosh(z\sqrt{2\lambda}) - \cosh(\theta z)] > 0$$

which is valid for all $z > 0$. □

We extend now the definition of U on all of $\mathcal{R}^2$ by the symmetry property $U(y,\xi) = -U(-y,\xi) = -U(y,-\xi) = U(-y,-\xi)$, to wit:

(7.26)
$$\frac{\alpha^2}{2\theta^2} sgn(y\xi).U(y,\xi) \triangleq \frac{\cosh(\theta(|y|+|\xi|)).\sinh(|\xi|\sqrt{2\lambda}) + \sinh(|y|\sqrt{2\lambda})}{\sinh((|y|+|\xi|)\sqrt{2\lambda})} \\ + \frac{\alpha}{\theta}|y|\sinh(\theta|\xi|) - \cosh(\theta\xi)\ .$$

Our program for the remainder of this section will be as follows: starting with the function U of (7.26), we shall *construct* a function Φ related to U via $U = \cosh(\theta\xi)[\Phi_{y\xi} + \theta\tanh(\theta\xi).\Phi_y]$ as in (7.17); cf. Proposition 7.2. We shall show that this function satisfies (7.5) - (7.9) (Proposition 7.3), and provides the value function for the stochastic control problem with partial observations (Theorem 7.4).

We start this program by introducing the function

(7.27)

$$\begin{aligned}\Theta(y,\xi) &\triangleq y^2 - \frac{sgn(y\xi).U(y,\xi)}{\cosh(\theta\xi)}\\ &= y^2 + \frac{2\theta^2}{\alpha^2} - \frac{2\theta}{\alpha}|y|\tanh(\theta|\xi|)\\ &\quad - \frac{2\theta^2}{\alpha^2}\,\frac{\cosh(\theta(|y|+|\xi|))\sinh(|\xi|\sqrt{2\lambda}) + \sinh(|y|\sqrt{2\lambda})}{\cosh(\theta|\xi|)\sinh((|y|+|\xi|)\sqrt{2\lambda})},\end{aligned}$$

which is continuous, of class C^∞ away from $\{(y,\xi);\ y\xi = 0\}$, and satisfies the growth condition

$$|\Theta(y,\xi)| \le K(1+y^2) \tag{7.28}$$

for some finite constant $K > 0$ depending only on α and θ.

Now consider a standard two-dimensional Brownian motion $\underset{\sim}{W} = (W_1, W_2)$ on a probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}, \{\mathcal{F}_t\}$, as well as the one-dimensional diffusion process $X^\xi(\cdot)$ given by

$$dX^\xi(t) = \theta\tanh(\theta X^\xi(t))dt + dW_2(t)\ , \quad X^\xi(0) = \xi \tag{7.29}$$

for every given $\xi \in \mathcal{R}$.

7.2 Proposition: The function $\Phi(\cdot\ ,\ \cdot) \equiv \Phi(\cdot\ ,\ \cdot\,;\theta)$ defined by

$$\Phi(y,\xi;\theta) \triangleq \mathbf{E}\int_0^\infty e^{-\alpha t}\,\Theta(y+W_1(t), X^\xi(t))\,dt \tag{7.30}$$

is of class C^2 on $\mathcal{R}^2$, of class C^∞ on $\mathcal{R}^2\backslash\{(y,\xi);\ y\xi = 0\}$, and satisfies the resolvent equation (7.5), the symmetry properties (7.2) - (7.4), and a growth condition of the type (7.6).

Proof: The symmetry properties follow directly from those of the function Θ in (7.27), and from the stochastic differential equation (7.29); the bound (7.6) is a direct consequence of (7.28).

On the other hand, it follows from Dynkin (1965), Chapter XIII that Φ has the requisite smoothness properties and satisfies the resolvent equation

$$(7.31) \qquad \frac{1}{2}[\Phi_{yy} + \Phi_{\xi\xi}] + \theta \tanh(\theta\xi).\Phi_\xi + \Theta(y,\xi) = \alpha\Phi \ .$$

In order to establish (7.5), we have thus to show

$$(7.32) \qquad U(y,\xi) = \cosh(\theta\xi).[\Phi_{y\xi}(y,\xi) + \theta \tanh(\theta\xi).\Phi_y(y,\xi)] \ ;$$

in fact, it suffices to prove (7.32) in Q_2^+, where the resolvent equation (7.31) becomes

$$(7.33) \qquad \begin{aligned} &\frac{1}{2}[\Phi_{yy} + \Phi_{\xi\xi}] + \theta \tanh(\theta\xi).\Phi_\xi - \\ &\qquad - \frac{U(y,\xi)}{\cosh(\theta\xi)} + y^2 = \alpha\Phi \ , \quad \text{in} \quad Q_2^+ \ . \end{aligned}$$

To this effect, we introduce the function

$$(7.34) \qquad V(y,\xi) \triangleq \cosh(\theta\xi).\Phi(y,\xi)$$

as in (7.11), with $\Phi(y,\xi)$ defined in (7.30), for which the equation (7.33) and the condition (7.32) become

$$(7.35) \qquad \frac{1}{2}[V_{yy} + V_{\xi\xi}] + y^2\cosh(\theta\xi) - U(y,\xi) = \lambda V \ , \qquad \text{in} \ \ Q_2^+$$

$$(7.36) \qquad U(y,\xi) = V_{y\xi}(y,\xi) \ , \qquad \text{in} \qquad Q_2^+$$

respectively. In order to derive (7.36) from the equation (7.35), differentiate in the latter successively with respect to y and ξ, to obtain

(7.37)

$$\frac{1}{2}[(V_{y\xi})_{yy} + (V_{y\xi})_{\xi\xi}] + 2\theta y \sinh(\theta\xi) - U_{y\xi}(y,\xi) = \lambda V_{y\xi} \ , \qquad \text{in} \ \ Q_2^+ \ .$$

On the other hand, the symmetry properties (7.2) - (7.4) of Φ lead to

$$V_{y\xi}(y,0+)=0, \qquad V_{y\xi}(0+,\xi)=0\ . \tag{7.38}$$

The equality $U = V_{y\xi}$ follows now from (7.37), (7.38) and from Proposition 7.1.

7.3 Proposition: The function Φ of (7.30) satisfies the properties (7.7) and (7.8), and thus also the HJB equation (7.9).

Proof: It suffices to establish (7.14), (7.13), for the function V of (7.34). The former follows directly from (7.36), (7.26) and (7.20).

As for (7.13), it suffices to prove it in Q_2^+; now from (7.35) and the "boundary conditions"

$$V_{\xi}(y,0+)=0\ , \quad V_y(0+,\xi)=0\ , \tag{7.39}$$

one can obtain the stochastic representation

$$V(y,\xi)=E\int_0^\infty e^{-\lambda t}\, G(|y+W_1(t)|,\ |\xi+W_2(t)|)\ dt\ . \tag{7.40}$$

Here we are setting

$$G(y,\xi)\ \triangleq\ y^2\cosh(\theta\xi)-U(y,\xi)\ , \tag{7.41}$$

and we *reflect* the two-dimensional Brownian motion $\underset{\sim}{W}=(W_1,W_2)$ on the faces of Q_2^+ (generator $\frac{1}{2}\frac{\partial^2}{\partial y^2}+\frac{1}{2}\frac{\partial^2}{\partial \xi^2}$ in (7.35), "reflecting" boundary conditions (7.39)). Recalling the transition probability density function

$$q(t;x,y)\ \triangleq\ p(t,x-y)+p(t,x+y)\ ;\quad x>0,\ y>0 \tag{7.42}$$

for reflecting one-dimensional Brownian motion (e.g. Karatzas & Shreve (1987), p.97) where $p(t,z) = (2\pi t)^{-\frac{1}{2}} \exp(-z^2/2t)$, we can recast (7.40) as

$$(7.43) \qquad V(y,\xi) = \int\int\int_{\mathcal{R}_+^3} e^{-\lambda t} q(t;y,u) q(t;\xi,\eta) G(u,\eta)\, du d\eta dt \; .$$

If we integrate by parts twice, and use the identity $q_{xx} = q_{yy}$ for the function of (7.42), we arrive at the expression

$$(7.44) \qquad \theta^2 V(y,\xi) - V_{\xi\xi}(y,\xi) =$$

$$= \int\int\int_{\mathcal{R}_+^3} e^{-\lambda t} q(t;y,u) q(t;\xi,\eta)[\theta^2 G(u,\eta) - G_{\eta\eta}(u,\eta)]\, du d\eta dt$$

$$- 2\int\int_{\mathcal{R}_+^2} e^{-\lambda t} q(t;y,u) p(t,\xi) G_\eta(u,0)\, du dt \; .$$

The first integrand on the right-hand side of (7.44) is positive, thanks to the consequence $\theta^2 G > G_{\xi\xi}$, in Q_2^+ of (7.20) and (7.41). As for the second integral, we have from (7.22)

$$(7.45) \qquad -G_\xi(y,0) = U_\xi(y,0) =$$

$$= \frac{2\theta^2}{\alpha^2}\left[\frac{\sqrt{2\lambda}}{\sinh(y\sqrt{2\lambda})}\{\cosh(\theta y) - \cosh(y\sqrt{2\lambda})\} + \alpha y\right] ,$$

a demonstrably positive quantity for $y \in (0,\infty)$. It develops that the right-hand side of (7.44) is positive, and (7.13) is established.

Finally, the HJB equation (7.9) follows directly from the resolvent equation (7.5), and the properties (7.7), (7.8). □

It remains now to show that the function Φ of (7.30) satisfies the comparison (7.10) - thus agreeing with the right-hand side of the expression (7.1).

7.4 Theorem. For any wide-sense control $u \in \mathcal{U}$, we have

$$(7.10) \qquad J(u;\theta) \geq \Phi(y,\xi;\theta) = J(u^*;\theta) \; .$$

In particular, the function Φ of (7.30) agrees with the right-hand side of (7.1), and is the value function for the control problem of section 4.

Proof: For any $u \in \mathcal{U}$ we consider the continuous, $\{\mathcal{F}_t\}$ - adapted process

$$(7.46) \qquad e^{-\alpha t}\Phi(Y_t, \xi_t^u) + \int_0^t e^{-\alpha s} \; Y_s^2 \, ds, \quad \mathcal{G}_t \; ; \qquad 0 \le t \le T \; .$$

Applying Itô's rule to (7.46), in conjunction with (4.7), (4.8) and (7.5), we obtain the semimartingale decomposition

$$(7.47) \qquad \Phi(y, \xi) + M_t^u + B_t^u \; , \qquad 0 \le t < \infty$$

for the process of (7.46), where

(7.48)

$$M_t^u \triangleq \int_0^t [\Phi_y(Y_s, \xi_s^u) + u_s \Phi_\xi(Y_s, \xi_s^u)] e^{-\alpha s} d\nu_s^u \; , \quad B_t^u \triangleq \int_0^t e^{-\alpha s} \beta_s^u ds$$

and

$$(7.49) \qquad \begin{aligned} \beta_t^u \; &\triangleq \frac{1}{2}\Phi_{yy}(Y_t, \xi_t^u) + Y_t^2 - \alpha\Phi(Y_t, \xi_t^u) + \\ &+ u_t\{\Phi_{y\xi}(Y_t, \xi_t^u) + \theta \tanh(\theta \xi_t^u).\Phi_y(Y_t, \xi_t^u)\} + \\ &+ u_t^2\{\frac{1}{2}\Phi_{\xi\xi}(Y_t, \xi_t^u) + \theta \tanh(\theta \xi_t^u).\Phi_\xi(Y_t, \xi_t^u)\}. \end{aligned}$$

It follows from (7.9) that β^u is a *nonnegative*, $\{\mathcal{F}_t\}$ - progressively measurable process, and from (7.5) that

(7.50)

$$\begin{aligned} \beta_t^* \; \triangleq \; \beta_t^{u^*} &= \frac{1}{2}\Phi_{yy}(Y_t, Z_t) + Y_t^2 - \alpha\Phi(Y_t, Z_t) - \\ &- sgn(Y_t Z_t)\{\Phi_{y\xi}(Y_t, Z_t) + \theta \tanh(\theta Z_t).\Phi_y(Y_t, Z_t)\} + \\ &+ \frac{1}{2}\Phi_{\xi\xi}(Y_t, Z_t) + \theta \tanh(\theta Z_t).\Phi_\xi(Y_t, Z_t) \end{aligned}$$

is evanescent. On the other hand, $\{M_t^u, \; 0 \le t \le T\}$ is a $(P_T^u, \{\mathcal{F}_t\})$ - local martingale, for any finite $T > 0$.

Let us fix such a T, and introduce the stopping times

$$\tau_n \triangleq \inf\{t \in [0,\infty)\ ;\ |Y_t| \geq n \text{ or } |\xi_t^u| \geq n\}\ , \quad \forall\ n \geq 1 \tag{7.51}$$

for any given $u \in \mathcal{U}$. We have then $E_T^u(M_{T\wedge\tau_n}^u) = 0$, and from (7.47), (7.46):

$$\Phi(y,\xi) + E_T^u \int_0^{T\wedge\tau_n} e^{-\alpha t}\beta_t^u dt = E_T^u \int_0^{T\wedge\tau_n} e^{-\alpha t}\, Y_t^2 dt \tag{7.52}$$
$$+E_T^u[e^{-\alpha(T\wedge\tau_n)}\Phi(Y_{T\wedge\tau_n}, \xi_{T\wedge\tau_n}^u)]\ .$$

From (7.6) it follows that

$$\begin{aligned} e^{-\alpha(T\wedge\tau_n)}|\Phi(Y_{T\wedge\tau_n}, \xi_{T\wedge\tau_n}^u)| &\leq ce^{-\alpha(T\wedge\tau_n)}(1 + Y_{T\wedge\tau_n}^2) \\ &\leq c(1 + \sup_{0\leq t\leq T} Y_t^2)\ , \end{aligned} \tag{7.53}$$

a.s., where the last random variable is P_T^u - integrable, by virtue of (2.7). From the Dominated Convergence Theorem and (7.53) we have then

$$\begin{aligned} \lim_{n\to\infty} E_T^u[e^{-\alpha(T\wedge\tau_n)}\Phi(Y_{T\wedge\tau_n}, \xi_{T\wedge\tau_n}^u)] &= e^{-\alpha T}E_T^u\Phi(Y_T, \xi_T^u) \\ &\leq c(1+T)e^{-\alpha T} \end{aligned} \tag{7.54}$$

(c denotes here a generic constant in $(0,\infty)$ that depends only on α and θ, not necessarily the same throughout). We can let $n \to \infty$ in (7.52), appeal to the Monotone Convergence Theorem and (7.54) in order to obtain

$$\begin{aligned} \Phi(y,\xi) + E_T^u \int_0^T e^{-\alpha t}\beta_t^u dt &= E_T^u \int_0^T e^{-\alpha t} Y_t^2 dt \\ &\quad + e^{-\alpha T}E_T^u\Phi(Y_T, \xi_T^u)\ , \end{aligned} \tag{7.55}$$

and then let $T \to \infty$ in this expression to get, by similar arguments:

$$\begin{aligned} \Phi(y,\xi) \leq \Phi(y,\xi) + \int_0^\infty e^{-\alpha t}E_t^u(\beta_t^u)dt &= \\ = \int_0^\infty e^{-\alpha t}E_t^u(Y_t^2)dt &= J(u;\theta)\ . \end{aligned} \tag{7.56}$$

The inequality in (7.56) holds as an equality if $u \equiv u^*$, and (7.10) follows.

7.5 Proposition: No strict-sense control can be optimal; i.e., for any $u \in \mathcal{U}_s$ (cf. Definition 2.2) we have

$$J(u;\theta) > \Phi(y,\xi;\theta)\ . \tag{7.57}$$

Proof: Let $u \in \mathcal{U}$ be optimal: $J(u;\theta) = \Phi(y,\xi)$. Then (7.56) implies $\int_0^\infty e^{-\alpha t} E_t^u(\beta_t^u)dt = 0$, whence $\beta_t^u(\omega) = 0$ for $meas \otimes P$ - a.e. $(t,\omega) \in [0,\infty) \times \Omega$. But from (7.7) - (7.9), (7.49) and the fact that Y, ξ^u are Brownian motions, this means

$$u_t(\omega) = -sgn(Y_t(\omega)\xi_t^u(\omega))\ ,\quad meas \otimes P - a.e.\ \ (t,\omega)$$

and (4.6) gives

$$\xi_t^u = \xi - \int_0^t sgn(Y_s \xi_s^u) dY_s\ , \qquad 0 \le t < \infty \tag{7.58}$$

almost surely.

Now suppose that $u \in \mathcal{U}_s$, i.e., u is adapted to $\{\mathcal{F}_t^Y\}$. From (4.6) again, it follows that ξ^u is *also* adapted to $\{\mathcal{F}_t^Y\}$; but this is impossible, by virtue of (7.58) and Proposition 6.2. □

Nevertheless, the infimum of the left-hand side of (7.57) over $u \in \mathcal{U}_s$ is equal to the right-hand side of this expression.

7.6 Proposition: $\Phi(y,\xi;\theta) = \inf_{u \in \mathcal{U}_s} J(u;\theta)$.

Proof. For any $u \in \mathcal{U}_s$, recall the $\{\mathcal{F}_t^Y\}$ - adapted Brownian motion process $\tilde{\nu}^u$ of (3.8), as well as the analogues

$$dY_t = u_t \theta \tanh(\theta \xi_t^u)dt + d\tilde{\nu}_t^u\ , \qquad Y_0 = y \tag{7.59}$$

$$(7.60)\qquad d\xi_t^u = u_t^2\theta \tanh(\theta\xi_t^u)dt + u_t d\tilde{\nu}_t^u\ , \qquad \xi_0^u = \xi$$

of the dynamical equations (4.7) and (4.8), for the pair of $\{\mathcal{F}_t^Y\}$ - adapted processes (Y, ξ^u). According to P.L. Lions (1983) the function

$$\tilde{\Phi}(y,\xi;\theta) \triangleq \inf_{u \in \mathcal{U}_s} J(u;\theta)$$

coincides with the unique viscosity solution of the HJB equation (7.9) associated with (7.59), (7.60) and (4.5). But we have shown in this section that (7.9) admits a *classical* solution, namely the function Φ of (7.30); thus, $\tilde{\Phi} = \Phi$. □

8. SOLUTION OF THE CONTROL PROBLEM: THE GENERAL CASE

Finally, let us return to the case of a general *symmetric* and square-integrable distribution μ on the random variable z, i.e., $\mu(A) = \mu(-A)$, $\forall\, A \in \mathcal{B}(\mathcal{R})$. Then the function F of (2.6) takes the form

$$(8.1)\qquad F(t,x) = 2\int_0^\infty e^{-\theta^2 t/2} \cosh(\theta x)\mu(d\theta)\ ,$$

and the expected cost $J(u)$ of (2.5) becomes

$$(8.2)\qquad J(u) = 2\int_0^\infty J(u;\theta)\mu(d\theta)$$

in the notation of (4.5).

From Theorem 7.4 applied for each $\theta > 0$ to the Bernoulli distribution $\mu_\theta \triangleq \frac{1}{2}(\delta_\theta + \delta_{-\theta})$ - i.e., with $\rho = \frac{1}{2}$ and $\xi = 0$ - we obtain then after integrating with respect to μ:

$$(8.3)\quad J(u) \geq J(u^*) = 2\int_0^\infty J(u^*;\theta)\mu(d\theta) = 2\int_0^\infty V(y,0;\theta)\mu(d\theta)\ .$$

This establishes the optimality of u^* in the general case as well.

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