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ESTIMATION AND CONTROL FOR LINEAR, PARTIALLY OBSERVABLE SYSTEMS WITH NON-GAUSSIAN INITIAL DISTRIBUTION*

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The nonlinear filtering problem of estimating the state of a linear stochastic system from noisy observations is solved for a broad class of probability distributions of the initial state. It is shown that the conditional density of the present state, given the past observations, is a mixture of Gaussian distributions, and is parametrically determined by two sets of sufficient statistics which satisfy stochastic DEs; this result leads to a generalization of the Kalman–Bucy filter to a structure with a conditional mean vector, and additional sufficient statistics that obey nonlinear equations, and determine a generalized (random) Kalman gain. The theory is used to solve explicitly a control problem with quadratic running and terminal costs, and bounded controls.

Stochastic control
Kalman–Bucy filter
Zakai equation

1. Introduction

The celebrated Kalman–Bucy filter provides the solution of a state estimation problem with linear dynamics, linear observations and a Gaussian prior distribution for the initial state. The conditional distribution of the present state, given past and present observations is Gaussian with nonrandom covariance and a mean vector satisfying (as a random function of time) linear DEs, the 'Kalman filter' (see [9, 11]).

This estimation problem becomes substantially harder if any one of the assumptions in the Kalman–Bucy scheme is generalized. In the general case of arbitrary system dynamics, observation model and initial distribution it is known that the density of the conditional distribution, whenever it exists, satisfies a stochastic

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partial differential equation that is due to Stratonovich [14], Kushner [10] and Zakai [17]. However, it was only very recently that even an instance of this equation was explicitly solved for a class of genuinely nonlinear drifts and linear observations [2].

The present paper considers and solves the problem with linear dynamics and observations for a broad class of prior distributions. It is shown that the conditional distribution is a mixture of Gaussians, and is propagated by two sets of 'sufficient statistics', i.e., random processes that parametrically characterize the distribution completely. These statistics obey usually nonlinear stochastic DEs implementable in the form of a 'filter'. The controlled version of the model is also considered and a particular control problem is solved explicitly. We also check that for a Gaussian initial distribution there is only one random sufficient statistic propagating the conditional density, in accordance with the classical theory. All these results are illustrated in a block diagram for the controlled case in Fig. 1.

2. Formulation

We start with a probability space $(\Omega, \mathscr{F}, P_0; \mathscr{F}_t)$ and a Wiener process $(w_t, y_t)'$ of dimension $n+m$ defined on it, and construct on this space the solution $(x_t, \mathscr{F}_t)$ of the linear stochastic differential equation

$$dx_t = A(t)x_t\,dt + dw_t, \quad 0 \le t \le T, \qquad x(0) = x_0 \tag{2.1}$$

according to the classical Itô theory, where $A(t)$ is a continuous $(n \times n)$ matrix-valued function and x_0 a random variable independent of the Wiener future $\sigma\{w_t, y_t; t \ge 0\}$. x_0 has a distribution function $F(\cdot)$ on $\mathbb{R}^n$, with finite first and second moments. Call $P(P_x, x \in \mathbb{R}^n)$ the measure induced on $(\Omega, \mathscr{F})$ by the $\{x_t; 0 \le t \le T\}$ process (conditional on knowing the exact starting point $x \in \mathbb{R}^n$); clearly, $P(A) = \int_{\mathbb{R}^n} P_x(A)\,dF(x)$ for any $A \in \mathscr{F}$.

Now let U be a compact subset of $\mathbb{R}^n$ and $H(t)$ be a continuous $(m \times n)$ matrix. Consider a stochastic process $\{u_t; 0 \le t \le T\}$ with values in U and progressively measurable with respect to the family $\{\mathscr{F}_t^y = \sigma(y_s; 0 \le s \le t); 0 \le t \le T\}$. The class $\mathscr{A}$ of all such processes is called the class of *admissible controls*. Corresponding to each $u \in \mathscr{A}$ we now define a new measure P_u on $(\Omega, \mathscr{F})$ through the derivative

$$\frac{dP_u}{dP_0}(\omega) = L_T(u), \tag{2.2}$$

$$L_t(u) = \exp\left[\int_0^t \{u_s'\,dw_s + x_s'H'(s)\,dy_s\} - \tfrac{1}{2}\int_0^t \{|u_s|^2 + |H(s)x_s|^2\}\,ds\right]. \tag{2.3}$$

According to Girsanov [6] (see also [1, Appendix]), P_u is a probability measure, and the process

$$\begin{bmatrix} w_t^u \\ b_t \end{bmatrix} = \begin{bmatrix} w_t \\ y_t \end{bmatrix} - \begin{bmatrix} \int_0^t u_s\,ds \\ \int_0^t H(s)x_s\,ds \end{bmatrix}, \quad 0 \le t \le T, \tag{2.4}$$

is Wiener on $(\Omega, \mathscr{F}, P_u; \mathscr{F}_t)$. In differential form (2.1) in conjunction with (2.4) now reads on the new probability space as

$$\mathrm{d}x_t = A(t)x_t\,\mathrm{d}t + u_t\,\mathrm{d}t + \mathrm{d}w_t^u, \qquad x(0) = x_0, \tag{2.5}$$

$$\mathrm{d}y_t = H(t)x_t\,\mathrm{d}t + \mathrm{d}b_t, \qquad y(0) = 0. \tag{2.6}$$

The two stochastic equations above constitute a classical model for a linear, partially observable system with an element of *control* (u_t), which is allowed to depend only on the past history of the *observation process* (y_t).

The *estimation* problem is to characterize the conditional distribution $P_u(x_t \in A \mid \mathscr{F}_t^y)$, $A \in \text{Borel}_n$. If the distribution of the initial state x_0, which is 'prior' to any observations, is Gaussian, we are in the realm of Kalman filtering and it is well known (Kalman and Bucy [9], Davis and Varaiya [4]) that the conditional distribution is again Gaussian, with nonstochastic covariance matrix $R(t)$ satisfying a matrix Riccati equation and conditional mean $\hat{x}_t = E_u(x_t \mid \mathscr{F}_t^y)$ solving the stochastic equation

$$\mathrm{d}\hat{x}_t = A(t)\hat{x}_t\,\mathrm{d}t + u_t\,\mathrm{d}t + R(t)H'(t)\,\mathrm{d}\nu_t, \qquad \hat{x}_0 = E_u x_0 = E_0 x_0,$$

in which the innovations process

$$\nu_t \triangleq y_t - \int_0^t H(s)\hat{x}_s\,\mathrm{d}s, \quad 0 \leq t \leq T, \tag{2.7}$$

is Wiener on the space $(\Omega, \mathscr{F}, P_u; \mathscr{F}_t^y)$, i.e., on the past of the observations. In this case the components of the conditional mean are the only statistics required for the characterization of the conditional distribution.

In this paper we prove that for any prior distribution on x_0 with finite first and second moments, the conditional distribution of the state, given the record of the past and present observations, is a mixture of Gaussians, and is propagated by *two* sets of sufficient statistics: one is the conditional mean vector and the second conveniently determines the now random conditional covariance.

A version (2.9) of the Kallianpur–Striebel formula is instrumental in subsequent developments featuring the fundamental unnormalized version of the conditional density (see also [8]). First, since $(L_t(u), \mathscr{F}_t)$ is a P_0-martingale, it is an exercise on conditional expectations to verify the Bayes' formula,

$$E_u[f(x_t) \mid \mathscr{F}_t^y] = \frac{E_0[f(x_t)L_t(u) \mid \mathscr{F}_t^y]}{E_0[L_t(u) \mid \mathscr{F}_t^y]} \triangleq \frac{\pi_t(f)}{\pi_t(1)}, \tag{2.8}$$

for any bounded, measurable $f: \mathbb{R}^n \to \mathbb{R}^1$. Since $(x_t), (y_t)$ are independent under P_0, $\{x_s; s \leq t\}$ can be 'integrated out' to give

$$\pi_t(f) = E[f(x_t)L_t(u)] = \int_{\mathbb{R}^n} E_x[f(x_t)L_t(u)]\,\mathrm{d}F(x).$$

If we now define the density $q_t(z; x)$ through

$$q_t(z; x)\,\mathrm{d}z \triangleq E_x[1_{\{x_t \in dz\}} L_t(u)], \tag{2.9}$$

it is readily seen from (2.8) with $f = 1_A$, $A \in \text{Borel}_n$, that

$$P_u(x_t \in A \mid \mathcal{F}_t^y) = \int_A p_t(z)\,\mathrm{d}z,$$

where

$$p_t(z) = \frac{\int_{\mathbb{R}^n} q_t(z\,;x)\,\mathrm{d}F(x)}{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} q_t(z\,;x)\,\mathrm{d}z\;\mathrm{d}F(x)}. \tag{2.10}$$

Therefore, the quantity defined in (2.9) is a version of the unnormalized conditional density, conditional on also knowing the starting place $x_0 = x \in \mathbb{R}$.

3. Summary

In Section 4 we employ the Kallianpur–Striebel formula (2.9) to solve the estimation problem in the one-dimensional case with $A(t) \equiv 0$. The success of the approach depends on the possibility of carrying out the function space integration in (2.9)—a hard problem in all but a few cases (see, for instance, Beneš [2]).

The general case is attacked in Section 5 via the Zakai stochastic partial differential equation of nonlinear filtering. This approach is less direct and seems to impose some unnatural restrictions, e.g., existence of initial densities. The exact form of the conditional distribution is given parametrically, in terms of two 'sufficient statistics' ((4.17) and (5.17)). These satisfy a system of stochastic DEs similar to Kalman's filter ((4.15)–(4.16) and (5.15)–(5.16)). The special structure of this system is employed in Section 6 to solve a control problem in which control effort costs nothing but is bounded.

4. The one-dimensional case, via Kallianpur–Striebel

In this section we illustrate the usefulness of the Kallianpur–Striebel formula by performing the function space integration of (2.9) in the particular case $n = m = 1$, $A(t) \equiv 0$, $H(t) \equiv 1$. Under these assumptions (2.9) becomes

$$q_t(z\,;x)\,\mathrm{d}z = E_x\Bigg[1_{(x+w_t\in dz)}\exp\Bigg\{\int_0^t u_s\,\mathrm{d}w_s - \tfrac{1}{2}\int_0^t u_s^2\,\mathrm{d}s + \int_0^t (x+w_s)\,\mathrm{d}y_s - \tfrac{1}{2}\int_0^t (x+w_s)^2\,\mathrm{d}s\Bigg\}\Bigg]. \tag{4.1}$$

Notice that $\int_0^t (x+w_s)\,\mathrm{d}y_s = zy_t - \int_0^t y_s\,\mathrm{d}w_s$, and $\int_0^t (x+w_s)\,\mathrm{d}w_s = \frac{1}{2}(z^2 - x^2 - t)$ on the indicated set. Therefore, with the convention

$$\zeta_s^t(\phi.) \triangleq -\int_s^t \phi_\theta\,\mathrm{d}w_\theta - \tfrac{1}{2}\int_s^t \phi_\theta^2\,\mathrm{d}\theta,$$

(4.1) becomes

$$q_t(z\,;x)\,dz=\exp\left\{zy_t+\tfrac{1}{2}(z^2-x^2-t)-\tfrac{1}{2}\int_0^t u_s^2\,\mathrm{d}s\right\}$$

$$\times E_x\left[1_{(x+w_t\in\mathrm{d}z)}\exp\left\{\int_0^t(u_s-y_s)\,\mathrm{d}w_s+\zeta_0^t(x+w)\right\}\right].$$

$x+w.$ is the Wiener process started at x. Let $\xi.$ be the Ornstein–Uhlenbeck process started at x,

$$d\xi_t=-\xi_t\,\mathrm{d}t+\mathrm{d}w_t,\quad t\geqslant 0,\qquad \xi_0=x\,;$$

the measures induced by ξ_t and by $x+w_t$ are equivalent, and by Prokhorov's formula [11, Theorem 7.7] the derivative of the first with respect to the second is

$$\frac{\mathrm{d}\mu_{\xi.}}{\mathrm{d}\mu_{x+w.}}(x+w.)=\exp\zeta_0^t(x+w.);$$

thus (4.1) becomes

$$q_t(z\,;x)\,\mathrm{d}z=\exp\left\{zy_t+\tfrac{1}{2}(z^2-x^2-t)-\tfrac{1}{2}\int_0^t u_s^2\,\mathrm{d}s\right\}$$

$$\times E_x\left[1_{(\xi_t\in\mathrm{d}z)}\exp\left\{\int_0^t(y_s-u_s)\xi_s\,\mathrm{d}s-\int_0^t(y_s-u_s)\,\mathrm{d}w_s\right\}\right].\tag{4.2}$$

The auxiliary vector process

$$h_t\triangleq\left(\xi_t,\int_0^t(y_s-u_s)\,\mathrm{d}w_s,\int_0^t(y_s-u_s)\xi_s\,\mathrm{d}s\right)'$$

is governed by the stochastic equation

$$\mathrm{d}h_t=Gh_t\,\mathrm{d}t+l\,\mathrm{d}w_t,\qquad h_0=(x,0,0)'$$

where

$$G=\begin{pmatrix}-1&0&0\\0&0&0\\y_t-u_t&0&0\end{pmatrix},\qquad l=(1,y_t-u_t,0)'.$$

The mean vector $m(t)$ and covariance matrix $R(t)=\{r_{ij}(t)\}_{1\leqslant i,j\leqslant 3}$ of the process (h_t) satisfy the equations

$$\begin{aligned}&\dot m(t)=Gm(t),\qquad m(0)=(x,0,0)',\\&\dot R(t)=R(t)G'+GR(t)+D,\qquad R(0)=0\end{aligned}\tag{4.3}$$

with

$$D=\begin{pmatrix}1&y_t-u_t&0\\y_t-u_t&(y_t-u_t)^2&0\\0&0&0\end{pmatrix}.$$

It is readily seen that

$$m(t)=(x\,\mathrm{e}^{-t},0,x\beta_t)' \quad \text{with } \beta_t \triangleq \int_0^t \mathrm{e}^{-s}(y_s-u_s)\,\mathrm{d}s.$$

The expectation in (4.2) can be written in terms of the h_t process, with $\nu=(0,-1,1)'$, as

$$E_x[1_{(h_t^1\in \mathrm{d}z)}\exp\{\nu' h_t\}]=$$

$$=(2\pi)^{-3/2}(\det R(t))^{-1/2}$$

$$\times \iint_{-\infty}^{\infty} \mathrm{d}h_2\,\mathrm{d}h_3 \exp[-\tfrac{1}{2}(h-m)'R^{-1}(t)(h-m)+v'h]_{h_1=z}\,\mathrm{d}z.$$

Writing the exponent as

$$-\tfrac{1}{2}(h-m-R\nu)'R^{-1}(h-m-R\nu)+\nu' m+\tfrac{1}{2}\nu' R\nu$$

we obtain

$$(2\pi r_{11}(t))^{-1/2}\exp\left[\nu' m(t)+\tfrac{1}{2}\nu' R(t)\nu-\frac{1}{2r_{11}(t)}\{z-(x\,\mathrm{e}^{-t}+(R(t)\nu)_1)\}^2\right]$$

whence, after solving (4.3) and doing a lot of simple algebra and calculus

$$q_t(z;x)=\eta_t\exp\left[-\frac{z^2-2\mu_t(x)}{2q(t)}-\frac{(\mu_t(x)-q(t)y_t)^2}{2q(t)(1+q(t))}-\frac{x^2-2x\beta_t}{2}\right] \tag{4.4}$$

with

$$q(t)=\tanh t\triangleq \frac{r_{11}(t)}{1-r_{11}(t)}, \qquad \mu_t(x)\triangleq \tanh t\left(y_t+\frac{x\,\mathrm{e}^{-t}+r_{13}(t)-r_{12}(t)}{1-r_{11}(t)}\right)$$

and η_t a time function, not necessarily the same throughout this paper, adapted to $\mathscr{F}_t^y$ for each $0\leqslant t\leqslant T$.

We pause for a moment to see that $\mu_t(x)$, $\tanh t$ are the Kalman filtering mean and variance, if the starting place $x_0=x$ is fixed; indeed, it is easily verified that $r_{11}/(1-r_{11})$ satisfies the Riccati equation

$$\dot{q}(t)=1-q^2(t), \qquad q(0)=0,$$

so $r_{11}/(1-r_{11})=\tanh t$, the Kalman variance. On the other hand, by applying Itô's rule to the expression for $\mu_t(x)$ and taking the equation for $R(t)$ into account, we obtain the familiar Kalman filter equation

$$\mathrm{d}\mu_t(x)=u_t\,\mathrm{d}t+[\tanh t](\mathrm{d}y_t-\mu_t(x)\,\mathrm{d}t), \qquad \mu_0(x)=x.$$

The last can be readily solved:

$$\mu_t(x)=x\exp\left\{-\int_0^t \tanh s\,\mathrm{d}s\right\}+\int_0^t \exp\left\{-\int_s^t \tanh\theta\,\mathrm{d}\theta\right\}\{u_s\,\mathrm{d}s+\tanh s\,\mathrm{d}y_s\}$$

$$=(x+\alpha_t)/\cosh t \tag{4.5}$$

with

$$\alpha_t = \int_0^t (\cosh s\, u_s\, \mathrm{d}s + \sinh s\, \mathrm{d}y_s). \tag{4.6}$$

Substituting the expressions for $\mu_t(x)$, $q(t)$ into (4.4) we finally obtain

$$q_t(z\,;x) = \eta_t \exp\left[-\frac{\left(z - \dfrac{x+\alpha_t}{\cosh t}\right)^2 + (x \tanh t - v_t)^2}{2 \tanh t}\right] \tag{4.7}$$

where

$$v_t \triangleq \beta_t + (1 - \tanh t)(\alpha_t + y_t \cosh t). \tag{4.8}$$

By virtue of (2.10) the conditional density has the form

$$p_t(z) = \frac{(2\pi \tanh t)^{-1/2} \displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{\left(z - \dfrac{x+\alpha_t}{\cosh t}\right)^2 + (x \tanh t - v_t)^2}{2 \tanh t}\right] \mathrm{d}F(x)}{\displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{(x \tanh t - v_t)^2}{2 \tanh t}\right] \mathrm{d}F(x)}. \tag{4.9}$$

We conclude from (4.9) that (α_t, v_t) is a pair of sufficient statistics for the conditional density. From (4.6), (4.8) and Itô's rule, we see that they satisfy the equations

$$\mathrm{d}\alpha_t = (\cosh t) u_t\, \mathrm{d}t + (\sinh t)\, \mathrm{d}y_t, \qquad \alpha_0 = 0, \tag{4.6$'$}$$

$$\mathrm{d}v_t = -\frac{\alpha_t}{\cosh^2 t}\, \mathrm{d}t + \frac{1}{\cosh t}\, \mathrm{d}y_t, \qquad v_0 = 0. \tag{4.10}$$

We now introduce another pair of sufficient statistics for the conditional density $p_t(z)$, which turns out to be more convenient for purposes of implementation and control. In particular, we wish to bring the conditional mean $\hat{x}_t$ and the innovations ν_t into the picture. It is observed that

$$\hat{x}_t \triangleq \int_{-\infty}^{\infty} z p_t(z)\, \mathrm{d}z = \frac{\alpha_t + c(t, v_t)}{\cosh t} \tag{4.11}$$

where

$$c(t, v) \triangleq \frac{\displaystyle\int_{-\infty}^{\infty} x \exp\left[-\frac{(x \tanh t - v)^2}{2 \tanh t}\right] \mathrm{d}F(x)}{\displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{(x \tanh t - v)^2}{2 \tanh t}\right] \mathrm{d}F(x)}, \tag{4.12}$$

and we notice that

$$\widehat{\operatorname{var} x_t} \triangleq \int_{-\infty}^{\infty} (z - \hat{x}_t)^2 p_t(z)\, \mathrm{d}z = g(t, v_t) \tag{4.13}$$

where

$$g(t, v) \triangleq \tanh t + \cosh^{-2} t \left[\frac{\int_{-\infty}^{\infty} x^2 \exp\left[-\frac{(x \tanh t - v)^2}{2 \tanh t} \right] dF(x)}{\int_{-\infty}^{\infty} \exp\left[-\frac{(x \tanh t - v)^2}{2 \tanh t} \right] dF(x)} - c^2(t, v) \right]. \tag{4.14}$$

The conditional mean $\hat{x}_t$ satisfies the stochastic differential equation

$$d\hat{x}_t = u_t \, dt + g(t, v_t) \, d\nu_t, \quad 0 \leq t \leq T, \qquad x_0 = \int_{-\infty}^{\infty} x \, dF(x), \tag{4.15}$$

where ν_t is the innovations process $y_t - \int_0^t \hat{x}_s \, ds$. Similarly, substituting the expression $\hat{x}_t \cosh t - c(t, v_t)$ for α_t in (4.10) we get the equation

$$dv_t = \frac{c(t, v_t)}{\cosh^2 t} dt + \frac{1}{\cosh t} d\nu_t, \quad 0 \leq t \leq T, \qquad v_0 = 0. \tag{4.16}$$

We sum these results up as follows.

Theorem 4.1. *Consider the one-dimensional linear system*

$$dx_t = u_t \, dt + dw_t^u, \qquad x(0) = x_0,$$

$$dy_t = x_t \, dt + db_t, \qquad y(0) = 0,$$

on a probability space $(\Omega, \mathcal{F}, P_u; \mathcal{F}_t)$ constructed as in Section 2, with the same notation and assumptions. If the p.d.f. $F(\cdot)$ of x_0 admits finite first and second moments, then the conditional distribution $P_u(x_t \in A \mid \mathcal{F}_t^y)$ has a density $p_t(z)$ given by

$$p_t(z) = (2\pi \tanh t)^{-1/2} \int_{-\infty}^{\infty} \exp\left[-\left\{ \left\{ z - \left(\hat{x}_t + \frac{x - c(t, v_t)}{\cosh t} \right) \right\}^2 + (x \tanh t - v_t)^2 \right\} / 2 \tanh t \right] dF(x)$$

$$\times \left(\int_{-\infty}^{\infty} \exp\left[-\frac{(x \tanh t - v_t)^2}{2 \tanh t} \right] dF(x) \right)^{-1}. \tag{4.17}$$

The conditional distribution is fully characterized by the pair of sufficient statistics $(\hat{x}_t, v_t)$, obeying the filter equations (4.15)–(4.16).

Special case. Suppose

$$F(x) = \int_{-\infty}^{x} p(y) \, dy, \qquad p(x) = (2\pi\sigma^2)^{-1/2} \exp\left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}.$$

Then,

$$c(t, v) = \frac{\mu + \sigma^2 v}{1 + \sigma^2 \tanh t},$$

and

$$g(t, v)=\frac{\sigma^2+\tanh t}{1+\sigma^2\tanh t}\triangleq r(t)$$

is the Kalman–Bucy variance, solving the Riccati equation $(\mathrm{d}/\mathrm{d}t)r(t)=1-(r(t))^2$, $r(0)=\sigma^2$. On the other hand, it can be shown using (4.7) and the particular form of the distribution function $F(\cdot)$ that

$$p_t(z)=(2\pi r(t))^{-1/2}\exp\left[-\frac{(z-\hat{x}_t)^2}{2r(t)}\right]$$

with

$$\hat{x}_t=\frac{\mu+\alpha_t+\sigma^2(v_t+\alpha_t\tanh t)}{\cosh t+\sigma^2\sinh t},$$

and it is not hard to verify that $\hat{x}_t$ thus defined satisfies (4.15).

5. The multivariate case

The task of performing the function space integration in the Kallianpur–Striebel formula (2.9) is equally feasible in the general setting of Section 2. For variety, however, we concentrate on a different method of getting an explicit expression for the conditional density, which makes direct use of the stochastic (and nonstochastic) partial differential equations of filtering. To this end, it is assumed throughout this section that the a priori distribution $F(\cdot)$ has a density $p(\cdot)$, and that the matrix function $H(t)$ in (2.6) is continuously differentiable on $[0, T]$.

If the starting place $x_0=x\in\mathbb{R}^n$ is known, the Kalman filtering 'conditional mean' $\mu_t(x)$ and 'conditional covariance matrix' $R(t)$ satisfy the equations

$$\mathrm{d}\mu_t(x)=A(t)\mu_t(x)\,dt+u_t\,\mathrm{d}t+R(t)H'(t)(\mathrm{d}y_t-H(t)\mu_t(x)\,\mathrm{d}t),\quad 0\leqslant t\leqslant T,\qquad \mu_0(x)=x. \tag{5.1}$$

$$\dot{R}(t)=A(t)R(t)+R(t)A'(t)-R(t)H'(t)H(t)R(t)-I_n,\quad 0\leqslant t\leqslant T,\qquad R(0)=0_n, \tag{5.2}$$

respectively, and it can be checked that the Gaussian density

$$k_t(z;x)=\{(2\pi)^n|\det R(t)|\}^{-1/2}\exp\{-\tfrac{1}{2}(z-\mu_t(x))'R^{-1}(t)(z-\mu_t(x))\}$$

satisfies the stochastic partial differential equation

$$\mathrm{d}k_t(z;x)=l_t^*k_t(z;x)\,\mathrm{d}t+k_t(z;x)(H(t)(z-\mu_t(x)))'|(\mathrm{d}y_t-H(t)\mu_t(x)\,\mathrm{d}t) \tag{5.3}$$

subject to the initial condition $k_0(z;x)=\delta(z-x)$, where l_t^* is the forward operator

$$l_t^* \triangleq \tfrac{1}{2}\Delta - (A(t)z+u_t)'\nabla - \operatorname{tr}(A(t)).$$

Consider the likelihood process

$$\Lambda_t(x) \triangleq \exp\left\{\int_0^t (H(s)\mu_s(x))'\,\mathrm{d}y_s - \tfrac{1}{2}\int_0^t |H(s)\mu_s(x)|^2\,\mathrm{d}s\right\} \tag{5.4}$$

along with the random function

$$\rho_t(z) \triangleq \int_{\mathbb{R}^n} k_t(z;x)\Lambda_t(x)p(x)\,\mathrm{d}x. \tag{5.5}$$

An application of Itô's rule to (5.5) yields, in conjunction with (5.3) and (5.4), the so-called Zakai equation (see [17]) for $\rho_t(z)$,

$$\mathrm{d}\rho_t(z) = l_t^*\rho_t(z)\,\mathrm{d}t + \rho_t(z)(H(t)z)'\,\mathrm{d}y_t, \quad 0\leq t\leq T, \qquad \rho_0(z)=p(z). \tag{5.6}$$

We propose to show that $\rho_t(z)$ as in (5.5) is a version of the unnormalized conditional density for $P_u(x_t\in A\,|\,\mathcal{F}_t^y)$, i.e., that

$$\pi_t(f) = \int_{\mathbb{R}^n} \rho_t(z)f(z)\,\mathrm{d}z \tag{5.7}$$

in the notation of (2.8). Indeed, (5.7) above can be established for any solution $\rho_t(z)$ of the Zakai equation (5.6),[1] so our claim would follow provided we showed that (5.6) admits a unique classical solution. To see the latter, we employ a device, first used by Rozovsky [13] (see Liptser and Shiryayev [11, pp. 327–328]), that has by now become standard in the study of the stochastic differential equations of filtering. The transformation

$$\psi_t(z) = \rho_t(z)\exp\{-(H(t)z)'y_t\} \tag{5.8}$$

reduces the stochastic equation (5.6) on $\rho_t(z)$ to the nonstochastic partial differential equation

$$\frac{\partial}{\partial t}\psi_t(z) = l^{v*}_t\psi_t(z) + e(z,t)\psi_t(z), \quad 0<t\leq T, \qquad \psi_0(z)=p(z) \tag{5.9}$$

for $\psi_t(z)$, with

$$l^{v*}_t = \tfrac{1}{2}\Delta + \{H'(t)y_t - (A(t)z+u_t)\}'\nabla - \operatorname{tr}(A(t)),$$
$$e(z,t) = \tfrac{1}{2}|H'(t)y_t|^2 - y_t'H(t)\{A(t)z+u_t\} - \tfrac{1}{2}|H(t)z|^2 - y_t'\dot{H}(t)z.$$

The coefficients of (5.9) depend parametrically on the observation sample path $\{y_s; s\leq t\}$ and, since they have the proper growth in z (constant diffusion, linear drift and quadratic potential terms), uniqueness of a solution follows from the maximum principle for parabolic operators (Friedman [5, Theorem 9, chapter 2]).

[1] By an adaptation of the forward and backward PDE method used by Pardoux in [12].

We now calculate the expression in (5.5). Introducing the fundamental matrix $\Phi(t)$ as a solution of the matrix equation

$$\dot{\Phi}(t)=\{A(t)-R(t)H'(t)H(t)\}\Phi(t), \quad 0\leq t\leq T,$$

$$\Phi(0)=I_n=\text{the identity matrix for } n \text{ dimensions},$$

we verify that (5.1) is solved by

$$\mu_t(x)=\Phi(t)(x+\alpha_t) \quad \text{with } \alpha_t=\int_0^t \Phi^{-1}(s)\{u_s\,\mathrm{d}s+R(s)H'(s)\,\mathrm{d}y_s\}, \tag{5.10}$$

and that $\Lambda_t(x)=\eta_t\exp\{-\frac{1}{2}(x'S(t)x-2x'v_t)\}$ with the conventions

$$\begin{aligned} S(t)&\triangleq\int_0^t \Phi'(s)H'(s)H(s)\Phi(s)\,\mathrm{d}s,\\ v_t&\triangleq\int_0^t \Phi'(s)H'(s)\{\mathrm{d}y_s-H(s)\Phi(s)\alpha_s\,\mathrm{d}s\}. \end{aligned} \tag{5.11}$$

Therefore,

$$\rho_t(z)=\eta_t\int_{\mathbb{R}^n}\exp\{-\tfrac{1}{2}(z-\Phi(t)(x+\alpha_t))'R^{-1}(t)(z-\Phi(t)(x+\alpha_t))\}\Lambda_t(x)p(x)\,\mathrm{d}x. \tag{5.12}$$

From (5.12) it is seen that the conditional mean

$$\hat{x}_t=\int_{\mathbb{R}^n} z p_t(z)\,\mathrm{d}z=\frac{\int_{\mathbb{R}^n} z\rho_t(z)\,\mathrm{d}z}{\int_{\mathbb{R}^n}\rho_t(z)\,\mathrm{d}z}$$

is given by

$$\hat{x}_t=\Phi(t)[c(t,v_t)+\alpha_t]$$

with

$$c(t,v)\triangleq\frac{\int_{\mathbb{R}^n} x\exp\{-\frac{1}{2}(x'S(t)x-2x'v)\}p(x)\,\mathrm{d}x}{\int_{\mathbb{R}^n}\exp\{-\frac{1}{2}(x'S(t)x-2x'v)\}p(x)\,\mathrm{d}x}, \tag{5.13}$$

while the conditional covariance matrix is

$$\widehat{\mathrm{cov}(x_t)}=\frac{\int_{\mathbb{R}^n}(z-\hat{x}_t)(z-\hat{x}_t)'\rho_t(z)\,\mathrm{d}z}{\int_{\mathbb{R}^n}\rho_t(z)\,dz}\equiv G(t,v_t)$$

where

$$G(t,v)\triangleq R(t)+\Phi(t)\left[\frac{\int_{\mathbb{R}^n} xx'\exp\{-\frac{1}{2}(x'S(t)x-2x'v)\}p(x)\,\mathrm{d}x}{\int_{\mathbb{R}^n}\exp\{-\frac{1}{2}(x'S(t)x-2x'v)\}p(x)\,\mathrm{d}x}-c(t,v)c'(t,v)\right]\Phi'(t). \tag{5.14}$$

It can also be checked that, in analogy with (4.15) and (4.16), the two statistics $(\hat{x}_t, v_t)$ satisfy the pair of stochastic differential equations

$$\mathrm{d}\hat{x}_t = A(t)\hat{x}_t\,\mathrm{d}t + u_t\,\mathrm{d}t + G(t, v_t)H'(t)\,\mathrm{d}\nu_t, \quad 0 \leqslant t \leqslant T, \qquad \hat{x}_0 = \int_{\mathbb{R}^n} xp(x)\,\mathrm{d}x \tag{5.15}$$

and

$$\mathrm{d}v_t = (H(t)\Phi(t))'(H(t)\Phi(t))c(t, v_t)\,\mathrm{d}t + (H(t)\Phi(t))'\,\mathrm{d}\nu_t, \quad 0 \leqslant t \leqslant T, \qquad v_0 = 0. \tag{5.16}$$

We formulate these conclusions in the following theorem.

Theorem 5.1. *Consider the system* (2.5), (2.6) *under the assumptions of Section* 2. *Let the a priori state distribution* $F(\cdot)$ *have a density* $p(\cdot)$ *and let* $H(t)$ *be continuously differentiable. The conditional distribution* $P_u(x_t \in A \mid \mathscr{F}_t^y)$, $A \in \mathrm{Borel}_n$ *then has a density*

$$p_t(z) = \int_{\mathbb{R}^n} \{(2\pi)^n |\det R(t)|\}^{-1/2} \exp[-\tfrac{1}{2}\{z - (\hat{x}_t + \Phi(t)(x - c(t, v_t)))\}'R^{-1}(t) \times \{z - (\hat{x}_t + \Phi(t)(x - c(t, v_t)))\}] \times \exp\{-\tfrac{1}{2}(x'S(t)x - 2x'v_t)\}p(x)\,\mathrm{d}x \Big/ \int_{\mathbb{R}^n} \exp\{-\tfrac{1}{2}(x'S(t)x - 2x'v_t)\}p(x)\,\mathrm{d}x, \tag{5.17}$$

propagated by the pair $(\hat{x}_t, v_t)$ *of sufficient statistics; the latter constitute the 'filter'* (5.15)–(5.16) *depicted in Fig.* 1.

Remarks. (1) The drift in (5.16) for the statistic v_t is nonlinear, and is a gradient.

(2) The form of (5.15) and (5.16), in particular the fact that the control process (u_t) only appears in the former, suggests that, for purposes of control, the process (u_t) could only depend on $\hat{x}_t$. In other words, we guess that the statistic $\hat{x}_t$ may be 'sufficient' for control.

In the next section we exhibit an instance where the above guess is true. Here we propose to show that the class $\mathscr{S}$ of *separated control* processes of the form $u_t = u(t, \hat{x}_t)$, $u : [0, T] \times \mathbb{R}^n \to U$ measurable, is a subclass of the admissible controls: $\mathscr{S} \subseteq \mathscr{A}$. In fact, the system of equations

$$\mathrm{d}\hat{x}_t = [\{A(t) - G(t, v_t)H'(t)H(t)\}\hat{x}_t + u(t, \hat{x}_t)]\,\mathrm{d}t + G(t, v_t)H'(t)\,\mathrm{d}y_t, \quad 0 \leqslant t \leqslant T, \tag{5.15$'$}$$

$$\mathrm{d}v_t = (H(t)\Phi(t))'H(t)[\Phi(t)c(t, v_t) - \hat{x}_t]\,\mathrm{d}t + (H(t)\Phi(t))'\,\mathrm{d}y_t, \quad 0 \leqslant t \leqslant T, \tag{5.16$'$}$$

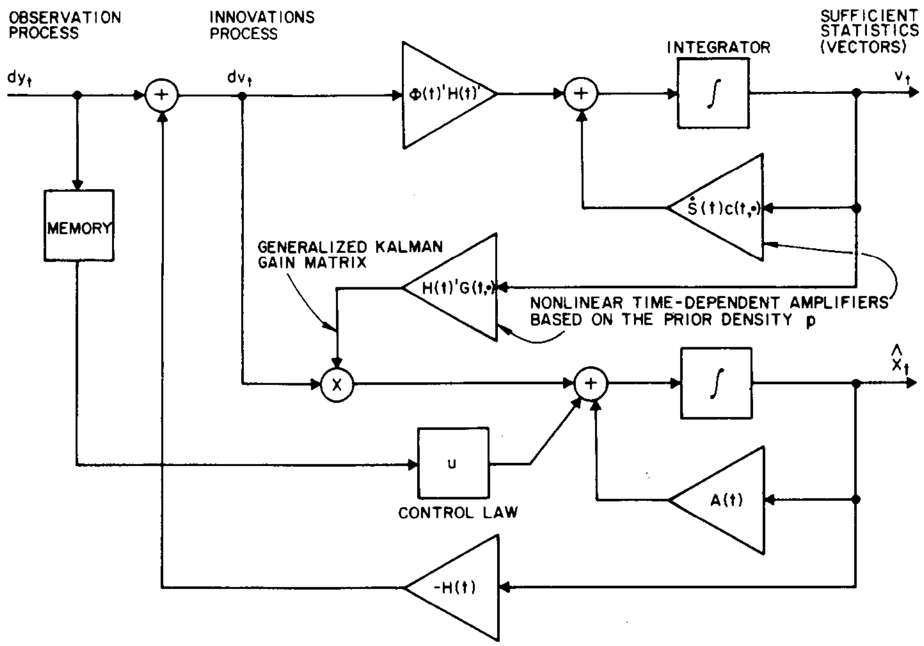

Fig. 1. Block diagram for filter based on equations (5.15) and (5.16).

on the probability space $(\Omega, \mathcal{F}, P_u; \mathcal{F}_t)$ is solvable in the strong sense that $(\hat{x}_t, v_t)$ is $\mathcal{F}_t^y$-measurable for all $0 \leq t \leq T$ (see [15] or [18] for the one-dimensional case). Therefore $u_t = u(t, \hat{x}_t)$ is $\mathcal{F}_t^y$-measurable, $0 \leq t \leq T$, and the resulting control process is admissible.

6. A control problem

Consider the system of one state dimension

$$dx_t = u_t \, dt + dw_t^u, \qquad x(0) = x_0,$$

$$dy_t = x_t \, dt + db_t, \qquad y(0) = 0,$$

treated in Section 2, with control set $U = [-1, 1]$. As a sample control problem, let us minimize a cost functional of the form

$$J(u) = E_u\left(\int_0^T x_t^2 \, dt + x_T^2\right). \tag{6.1}$$

We notice immediately that $J(u) = \hat{J}(u)$ where

$$\hat{J}(u) = E_u\left(\int_0^T g(t, v_t) \, dt + g(T, v_T) + \int_0^T (\hat{x}_t)^2 \, dt + (\hat{x}_T)^2\right). \tag{6.2}$$

On the basis of intuition, and of similar results in the case of a Gaussian initial distribution (Beneš and Karatzas [3]), it is natural to expect that the bang-bang law,

$$u_t^* = -\operatorname{sgn} \hat{x}_t,$$

is optimal. However, an attempt to prove the optimality of this law by classical (dynamic programming) arguments would have to overcome the difficulty that the Bellman equation for this problem is degenerate, since (4.15)–(4.16) for the two sufficient statistics $(\hat{x}_t, v_t)$ are driven by the same Wiener process (ν_t). We provide an optimality argument that avoids the use of partial differential equations.

On a space $(\Omega, \mathscr{F}, P_*; \mathscr{F}_t^y)$, consider the processes $(\hat{x}_t^*, v_t^*)$ satisfying the pair of stochastic equations

$$\mathrm{d}\hat{x}_t^* = -\operatorname{sgn} \hat{x}_t^*\, \mathrm{d}t + g(t, v_t^*)\, \mathrm{d}\nu_t^*, \qquad x_0^* = \int_{-\infty}^{\infty} x\, \mathrm{d}F(x),$$

$$\mathrm{d}v_t^* = (\cosh^{-2} t) c(t, v_t^*)\, \mathrm{d}t + (\cosh^{-1} t)\, \mathrm{d}\nu_t^*, \qquad v_0^* = 0.$$

The process (u_t^*), $u_t^* = -\operatorname{sgn} \hat{x}_t^*$ is admissible, as mentioned in Remark 2 at the end of Section 5. Consider also any admissible process $(u_t) \in \mathscr{A}$, along with the pair of processes (x_t^u, v_t^u) on an appropriate probability space $(\Omega, \mathscr{F}, P_u; \mathscr{F}_t^y)$,

$$\mathrm{d}\hat{x}_t^u = u_t\, \mathrm{d}t + g(t, v_t^u)\, \mathrm{d}\nu_t^u, \qquad \hat{x}_0^u = \int_{-\infty}^{\infty} x\, \mathrm{d}F(x),$$

$$\mathrm{d}v_t^u = (\cosh^{-2} t) c(t, v_t^u)\, \mathrm{d}t + (\cosh^{-1} t)\, \mathrm{d}\nu_t^u, \qquad v_0^u = 0.$$

Theorem 6.1. *For any admissible control process* $(u_t) \in \mathscr{A}$

$$\boldsymbol{J}(u^*) \leq \boldsymbol{J}(u). \tag{6.3}$$

Proof. By a lemma of Ikeda and Watanabe [7] there exists a probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P}; \tilde{\mathscr{F}}_t)$ and a quintuple of real-valued, $\tilde{\mathscr{F}}_t$-adapted processes $(\tilde{v}_t^u, \tilde{x}_t^u, \tilde{v}_t^*, \tilde{x}_t^*, \tilde{\nu}_t)$, such that $(\tilde{\nu}_t, \tilde{\mathscr{F}}_t, \tilde{P})$ is Wiener and

(i) $(\tilde{v}_t^u, \tilde{x}_t^u, \tilde{\nu}_t)$ has the same law as $(v_t^u, \hat{x}_t^u, \nu_t^u)$.

(ii) $(\tilde{v}_t^*, \tilde{x}_t^*, \tilde{\nu}_t^*)$ has the same law as $(v_t^*, \hat{x}_t^*, \nu_t^*)$.

On this new probability space,

$$\mathrm{d}\tilde{v}_t^* = (\cosh^{-2} t) c(t, \tilde{v}_t^*)\, \mathrm{d}t + (\cosh^{-1} t)\, \mathrm{d}\tilde{\nu}_t, \qquad \tilde{v}_0^* = 0,$$

$$\mathrm{d}\tilde{v}_t^u = (\cosh^{-2} t) c(t, \tilde{v}_t^u)\, \mathrm{d}t + (\cosh^{-1} t)\, \mathrm{d}\tilde{\nu}_t, \qquad \tilde{v}_0^u = 0,$$

and for some $\tilde{\mathscr{F}}_t$-adapted process $(\tilde{u}_t)$ with values in $[-1, 1]$,

$$d\tilde{x}_t^* = -\operatorname{sgn} \tilde{x}_t^* \, dt + g(t, \tilde{v}_t^*) \, d\tilde{\nu}_t, \qquad \tilde{x}_0^* = \int x \, dF(x),$$

$$d\tilde{x}_t^u = \tilde{u}_t \, dt + g(t, \tilde{v}_t^u) \, d\tilde{\nu}_t, \qquad \tilde{x}_0^u = \int x \, dF(x).$$

The processes $(\tilde{v}_t^u, \tilde{v}_t^*)$ satisfy the same stochastic equation, with smooth coefficients, driven by the same Wiener process $(\tilde{\nu}_t)$; consequently,

$$\tilde{P}(\tilde{v}_t^u = \tilde{v}_t^*, 0 \leq t \leq T) = 1.$$

Now, by a comparison theorem for solutions of stochastic differential equations (Ikeda and Watanabe [7, Theorem 1.1]),

$$\tilde{P}(|\tilde{x}_t^u| \geq |\tilde{x}_t^*|, 0 \leq t \leq T) = 1,$$

and a fortiori

$$\hat{J}(u) = \tilde{E}\left[\int_0^T g(t, \tilde{v}_t^u) \, dt + g(T, \tilde{v}_T^u) + \int_0^T |\tilde{x}_t^u|^2 \, dt + |\tilde{x}_T^u|^2\right]$$

$$\geq \tilde{E}\left[\int_0^T g(t, \tilde{v}_t^*) \, dt + g(T, \tilde{v}_T^*) + \int_0^T |\tilde{x}_t^*|^2 \, dt + |\tilde{x}_T^*|^2\right] = \hat{J}(u^*),$$

which proves (6.3) and the optimality of the law u^*.

Note. In this special case it is possible to verify the admissibility of the control process (u_t^*), $u_t^* = -\operatorname{sgn} \hat{x}_t^*$ directly. Indeed, it is a straightforward exercise to check pathwise uniqueness for the system of equations,

$$d\hat{x}_t^* = -[g(t, v_t^*)\hat{x}_t^* + \operatorname{sgn} \hat{x}_t^*] \, dt + g(t, v_t^*) \, dy_t, \qquad \hat{x}_0^* = \int_{-\infty}^{\infty} x \, dF(x);$$

$$dv_t^* = \frac{1}{\cosh^2 t}[c(t, v_t^*) - \cosh(t)\hat{x}_t^*] \, dt + \frac{1}{\cosh t} \, dy_t, \qquad v_0^* = 0.$$

Strong existence is then guaranteed by the existence of a weak solution and pathwise uniqueness (see Yamada and Watanabe [16]).

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