FILTERING FOR PIECEWISE LINEAR DRIFT AND OBSERVATION

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ABSTRACT

The filtering problem for piecewise linear drift and observation functions is reduced to an initial-boundary value problem. The "corners" give rise to local time terms. A finite number of sufficient statistics appear, in the form of the values and one-sided derivatives of the conditional density at the "corners", or more generally in the form of weights in a representation of the conditional density by potentials. Both kinds of statistics propagate according to linear Volterra equations, and must be considered as infinite-dimensional.

The theory developed here for piecewise linear dynamics enhances the study of the general nonlinear filtering problem in a natural way: Nonlinear functions can be approximated over bounded intervals by polygons, to any degree of accuracy; by constructing or calculating the optimal filter for the approximating piecewise linear dynamics as indicated in this paper, one can conceivably obtain very good sub-optimal filters for general nonlinear dynamics. That the results extend to many dimensions is far from clear, but likely whenever the necessary local times can be defined.

1. INTRODUCTION

For several years, research in stochastic filtering theory has concentrated on discerning those cases for which the conditional density is propagated by a finite number of sufficient statistics, so that the optimal filter can be implemented by a finite dimensional dynamical system. Since it is now understood that such cases are "exceptional", it is natural to inquire what methods can be used to analyze the structure of filters in an infinite-dimensional setting.

This paper advances such a study by considering the case of piecewise-linear drift and observation functions in the model. Roughly speaking, the infinite-dimensional character of this problem is a result of the presence of a finite number n of "corners", which now act as *elastic boundaries* for the conditional density process. Solution for the latter in each interval of linearity (between two such corners), through either the Kallianpur-Striebel formula or the Zakai equation, leads to a linear parabolic PDE with linear drift and quadratic potential, both time-dependent via the observations. The fundamental kernel in each interval is expressible in terms of a pair of statistics similar to those appearing in the Kalman filter: a covariance-like $r(\cdot)$ independent of the data, and a mean-like $m(\cdot)$ driven by them. The unnormalized conditional density can then be written by Green's theorem in each interval in terms of its kernel by means of single- and double-layer potentials. The weights in these potentials are the still unknown values and one-sided derivatives of the desired density at the corners, $3n$ quantities in all. Determination of these weights rests on continuity across the corners, and on the "elastic" boundary conditions, which relate the jump in the gradient at a corner to the value there. A "jump relation" for potentials is instrumental here; by its means the boundary conditions and the representation are transformed into a system of $3n$ linear Volterra equations for the weights. A simple representation leads to $2n$ similar equations with the added virtue that they are of the second kind, so that a classical theory is available. The nonlinear filtering problem is thus reduced to solving these equations.

A simple example is discussed in detail to further illustrate the methods and the structure of the solution.

2. THE FILTER

We consider the nonlinear filtering problem of characterizing the conditional distribution of ξ_t given the observation record $\sigma(y_u : 0 \le u \le t)$ under the model

$$(1) \qquad d\xi_t = f(\xi_t)dt + dw_t \;\; ; \;\; \xi_o \perp\!\!\!\perp (w.,b.)$$

$$(2) \qquad dy_t = h(\xi_t)dt + db_t \;\; ; \;\; y_o = 0$$

where $w.$ and $b.$ are independent Brownian motion processes and $f(\cdot)$, $h(\cdot)$ are continuous, piecewise linear functions

$$(3) \qquad f'(z) = \sum_{i=0}^{n} a_i 1_{A_i}(z) \;\; , \;\; h'(z) = \sum_{i=0}^{n} h_i 1_{A_i}(z)$$

for a number n of "elbow" or "corner" points $x_1 < x_2 < \ldots < x_n$ on the real line ($x_0 \equiv -\infty$, $x_{n+1} = +\infty$), and $A_i = (x_i, x_{i+1})$.

It is known (c.f. for instance [3]) that if $F(\cdot)$ is the distribution function of ξ_o, the conditional distribution we are seeking has an (unnormalized) density given by

$$\rho(t,z) = \int_R q(t,z;0,x)dF(x)$$

in terms of the "fundamental unnormalized density" $q(t,z;0,x)$, expressible by the Kallianpur-Striebel formula as

$$(4) \qquad E^w \left[1_{\{x+w_t \in dz\}} \exp\left\{ \int_0^t f(x+w_s)dw_s - \frac{1}{2}\int_0^t f^2(x+w_s)ds + \int_0^t h(x+w_s)dy_s - \frac{1}{2}\int_0^t h^2(x+w_s)ds \right\} \right].$$

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Here E^w denotes expectation with respect to the $w.$ process. In order to eliminate the stochastic integration against the $y.$ process — and thus bring the observation path parametrically into our expressions — we have to integrate by parts. Itô's rule cannot be applied directly, however, because $h(\cdot)$ is only piecewise continuously differentiable. As in Mc Kean [4] the corners of $h(\cdot)$ give local time contributions, and we obtain the "generalized Tanaka formula"

$$h(x+w_t) = h(x) + \sum_{i=0}^{n} h_i \int_0^t 1_{A_i}(x+w_s)dw_s + \sum_{i=0}^{n} \frac{h_i - h_{i-1}}{2} \ell_{x_i}^{x+w.}(t) \tag{5}$$

and the "integration by parts formula"

$$\int_0^t h(x+w_s)dy_s = y_t h(x+w_t) - \sum_{i=0}^{n} h_i \int_0^t y_s 1_{A_i}(x+w_s)dw_s - \sum_{i=0}^{n} \frac{h_i - h_{i-1}}{2} \int_0^t y_s \ell_{x_i}^{x+w.}(ds) , \tag{6}$$

where $\ell_{x_i}^{x+w.}(t)$ is the local time spent by the process $x+w$ at $z = x_i$ up to time t.

Substitution of (6) into the K.-S. formula (4) and some simple algebra provide the fundamental density in the form

$$q(t,z;0,x) = e^{h(z)y_t} u(t,z;0,x) , \tag{7}$$

$$u(t,z;0,x)dz = E^w \Bigg[1_{\{x+w_t \in dz\}} \exp\Big\{ \zeta_0^t(f+w.) - y_t \sum_{i=0}^{n} h_i 1_{A_i}(x+w)) \Big\} .$$

$$\cdot \exp\Big\{ \int_0^t \Big(\frac{1}{2} y_s^2 \sum_{i=0}^{n} (h_i^2) 1_{A_i}(x+w_s) - \frac{1}{2} h^2(x+w_s)s - y_s f(x+w_s) \sum_{i=0}^{n} h_i 1_{A_i}(x+w_s))ds - \frac{1}{2} \sum_{i=0}^{n} (h_i - h_{i-1}) \int_0^t y_s d\ell_{x_i}^{x+w.}(s) \Big\} \Bigg] , \tag{8}$$

where $\zeta_0^t(\phi.) \triangleq \int_0^t \phi_s dw_s - \frac{1}{2} \int_0^t \phi_s^2 ds$ is the Girsanov exponent.

The stochastic representation (8) helps us read off the partial differential equations (in the intervals A_i) and the boundary conditions (at the points x_i) satisfied by the function $u(.,.;0,x)$. The keys are the Feynman-Kac formula and the theory of the so-called elastic Brownian Motion [6, p. 161]. The former discerns a drift term

$$f(z) - y_t \sum_{i=0}^{n} h_i 1_{A_i}(z)$$

from the argument of the Girsanov functional, and a potential term

$$\frac{1}{2} y_t^2 \cdot \sum_{i=0}^{n} h_i^2 1_{A_i}(z) - \frac{1}{2} h^2(z) - y_t f(z) \sum_{i=0}^{n} h_i 1_{A_i}(z)$$

from the killing rate. The latter imposes an "elastic condition"

$$(u_2(t,x_i+) - u_2(t,x_i-)) + y_t(h_i - h_{i-1})u(t,x_i) = 0 \tag{9}$$

at each "corner" point x_i, $i = 1,\dots,n$.

Therefore the filtering problem reduces to the following:

Initial-boundary value problem:

To construct a continuous function $u(t,z) = u(t,z;0,x)$ on $(0,T \times R]$, which is $C^{1,2}$ in each of the intervals A_i; $i = 0,1,\dots,n$, and satisfies

(a) the forward equation

$$u_1 = \frac{1}{2} u_{22} - \frac{\partial}{\partial z} [\{f(z) - y_t h_i\}u] + (\frac{1}{2} h_i^2 y_t^2 - \frac{1}{2} h^2(z) - y_t f(z) h_i)u \tag{10}$$

in $(0,T] \times A_i$; $i = 0,1,\dots,n$,

(b) the boundary conditions (9), and

(c) the initial condition: $\lim_{t \downarrow 0} u(t,z;0,x) = \delta(z-x)$.

An alternative way to arrive at the above characterization of $u(t,z;0,x)$ proceeds via the Zakai equation for $q(t,z;0,x)$ [7]. In each of the A_i's q satisfies the stochastic equation

$$dq = (\frac{1}{2} q_{22} - f(z)q_2 - a_i q)dt + h(z)q dy_t ,$$

along with the initial condition $\lim_{t \downarrow 0} q(t,z;0,x) = \delta(z-x)$. The "Rozovsky transformation" (7) trades the (stochastic) Zakai equations for the nonstochastic ones in (10), as one can verify after a bit of stochastic calculus. Continuity of $q(t,.,;0,x)$ and $q_2(t,.;0,x)$ translate into continuity of $u(t,.;0,x)$ and the boundary conditions (9), respectively.

3. A JUMP RELATION FOR HEAT-LIKE POTENTIALS

In this section we construct the fundamental solution for parabolic equations of the form (10) and examine the behavior of certain potentials based on it.

To simplify derivation and notation, we consider the parabolic equation

$$v_1 = \frac{1}{2} v_{22} + (e(t) - az)v_2 - (\frac{1}{2} h^2 z^2 + g(t)z)v \tag{11}$$

with a,h real constants and $e(\cdot)$, $g(\cdot)$ continuous functions on $t \geq 0$, and seek its fundamental solution in the form

$$\Gamma(t,z;s,x) = A(t,s,x) . \mathrm{G}(r(t-s) \ , \ z - m(t,s,x)) \tag{12}$$

with $A(s,s,x) = 1$, $r(0) = 0$ and $m(s,s,x) = x$, where

$$\mathrm{G}(x,t) = (2\pi t)^{-1/2} \exp(-x^2/2t)$$

is the fundamental Gaussian kernel. Substitution of the proposed form of Γ into (11) and annihilation of the coefficients of powers of z up to order 2 yields the equations

$$\dot r(t) = 1 + 2ar(t) - h^2 r^2(t) \ ; \ r(0) = 0 \quad \text{(Riccati)} \tag{13}$$

$$\frac{\partial m}{\partial t} + (h^2 . r(t-s) - a)m + (e(t) + r(t-s)g(t)) = 0 ,$$

$$m(s,s,x) = x \ ,$$

$$\frac{\partial}{\partial t} \log A + \frac{1}{2} h^2(m^2 + r(t-s)) - a + mg(t) = 0 \ ;$$

$$A(s,s,x) = 1 \ ,$$

and therefore, with $\sigma(t) \triangleq \exp\Big\{ -h^2 \int_0^t r(u)du + at \Big\}$,

(14) $$m(t,s,x) = \sigma(t-s)\Big[x - \int_0^{t-s} \{e(s+v) + r(v)g(s+v)\}\frac{dv}{\sigma(v)}\Big]$$

(15) $$A(t,s,x) = \exp\left[-\frac{1}{2}h^2 \int_s^t \{m^2(\theta,s,x) + r(\theta-s+g(\theta)m(\theta,s,x)\}d\theta + a(t-s)\right]$$

In the special case $a = 0$, $g(t) = e(t) \equiv 0$, we have

$$r(t) = \frac{1}{h}\tanh(ht) \quad , \quad m(t,s,x) = \frac{x}{\cosh h(t-s)}$$

$$A(t,s,x) = (\cosh h(t-s))^{-\frac{1}{2}} \exp\{-\frac{1}{2} hx^2 \tanh h(t-s)\} .$$

Thus

(16) $$\Gamma(t,z;s,x) = \left[\frac{2\pi}{h}\sinh h(t-s))^{-\frac{1}{2}} \exp\left\{-\frac{h}{2}\frac{(z^2+x^2)\cosh h(t-s)-2xz}{\sinh h(t-s)}\right\}\right.$$

recovering a classical result of Szybiak [5].

Let us now establish a "jump relation" for single-layer potentials of the form

$$\pi(t,z) = \int_0^t \Gamma(t,z+s(t);u,s(u))\rho(u)du \ ,$$

similar to that derived by Friedman [2] for the heat equation.

(17) *Proposition:* For $\rho(\cdot)$ continuous and $s(\cdot)$ Lipschitz,

$$\lim_{z\to 0\pm} \int_0^t \Gamma_z(t,z+s(t);u,s(u))\rho(u)du = \mp \ \rho(t) + \int_0^t \Gamma_z(t,s(t);u,s(u))\rho(u)du \ .$$

Proof: We only need to establish the result with $z \downarrow 0$. It is seen that

$$\Gamma_z(t,z+s(t);u,s(u)) = -\left[\frac{z+s(t)-s(u)}{r(t-u)} + s(u)p(t-u)+q(t,u)\right] \Gamma(t,z+s(t);u,s(u))$$

where $p(t) = \dfrac{1-\sigma(t)}{r(t)}$,

$$q(t,u) \doteq \frac{\sigma(t-u)}{r(t-u)}\int_0^{t-u}\{e(u+v)+r(v)\,g(u+v)\}\frac{dv}{\sigma(v)} .$$

$\pi(t,z)$ is decomposed as $\pi_1(t,z) + \pi_2(t,z)$, with

$$\pi_1(t,z) \equiv -\int_0^t \frac{z+(s(t)-s(u))}{r(t-u)}\Gamma(t,z+s(t);u,s(u))\rho(u)du$$

$$\pi_2(t,z) = -\int_0^t (s(u)p(t-u) + q(t,u))\Gamma(t,z+s(t);u,s(u))\rho(u)du \ ,$$

and evidently $\lim_{z\downarrow 0} \pi_2(t,z) = \pi_2(t,0)$. So the question is whether $\lim_{z\downarrow 0} \pi_1(t,z) = -\rho(t) + \pi_1(t,)$, i.e.:

(18) $$\lim_{z\downarrow 0}\int_0^t \frac{z+s(t)-s(u)}{r(t-u)}\Gamma(t,z+s(t);u,s(u))\rho(u)du = \rho(t) + \int_0^t \frac{s(t)-s(u)}{r(t-u)}\Gamma(t,s(t);u,s(u))\rho(u)du \ .$$

For any $0 < \eta < t$, $\int_0^{t-\eta} \xrightarrow[z\downarrow 0]{} \int_0^{t-\eta}$ (same integrands as in (18)) so we concentrate on $L \triangleq L_1 - L_2$, where

$$L_1 = \int_{t-\eta}^t \frac{z+s(t)-s(u)}{r(t-u)}\Gamma(t,z+s(t);u,s(u))\rho(u)du \ ,$$

$$L_2 = \int_{t-\eta}^t \frac{s(t)-s(u)}{r(t-u)}\Gamma(t,s(t);u,s(u))\rho(u)du$$

and on the similar expression I, I_1, I_2 with $\rho(t) \equiv 1$. Since $\dot r(0) = 1$, we have $r_0 t \le r(t) \le r_1 t$, for t sufficiently small, with r_0 positive. Letting C denote a constant (not necessarily the same throughout the paper), we have $|I_2| \le C\eta^{\frac{1}{2}}$, by virtue of the Lipschitz continuity of $s(\cdot)$. On the other hand, we decompose I_1 as $J_1 + J_2$, with

$$J_1 = \int_{t-\eta}^t \frac{z+s(t)-s(u)}{r(t-u)}\Gamma(t,z+s(t);u,s(t))du \quad , \text{ and}$$

$$J_2 = \int_{t-\eta}^t \frac{z+s(t)-s(u)}{r(t-u)}\{\Gamma(t,z+s(t);u,s(u)) - \Gamma(t,z+s(t);u,s(t))\}du \ .$$

To estimate J_2 first, consider the ratio

$$\frac{\Gamma(t,z+s(t);u,s(u))}{\Gamma(t,z+s(t);u,s(t))} = \exp(\xi) \quad , \text{ with}$$

$$\xi = \frac{s(t)-s(u)}{2r(t-u)}\sigma(t-u)\{m(t,u,s(t))+m(t,u,s(u))-2s(t)-2z\} + \frac{h^2}{2}(s(t)-s(u))\int_u^t \sigma(\theta-u)\{m(\theta,u,s(u))+m(\theta,u,s(t))+g(\theta)\}d\theta \ .$$

It can be checked that $|\xi| \le C(z+\eta)$, so by choosing z,η small we can guarantee that $|\xi| \le 1$, $|\exp\xi - 1| \le e|\xi|$. On the other hand, for z small enough, it is easy to establish a bound of the form:

$$\Gamma(t,z+s(t);u,s(u)) \le \frac{C}{\sqrt{t-u}}\exp\left[-\frac{\mu z^2}{t-u}\right] .$$

So: $$|J_2| \le C\int_0^\eta \frac{(z+u)^2}{u^{3/2}}\exp\left[-\frac{\mu z^2}{u}\right]du \le C\eta^{\frac{1}{2}} \ ,$$

by using the fact that xe^{-x}, $x^{\frac{1}{2}}e^{-x}$, e^{-x} are bounded functions on R^+.

To estimate J_1, we write it in the form $J_{11} + J_{12}$, with

$$J_{12} = \int_{t-\eta}^t \frac{s(t)-s(u)}{r(t-u)}\Gamma(t,z+s(t);u,s(t))du \quad (|J_{12}| \le C\eta^{\frac{1}{2}}) \ ,$$

$$J_{11} = \int_{t-\eta}^t \frac{z}{r(t-u)}\Gamma(t,z+s(t);u,s(t))du = \frac{1}{\sqrt{2\pi}}\int_0^{r(\eta)/z^2}\frac{1}{x^{3/2}}\exp(-\frac{1}{2x}) \cdot f(x;t,z)dx$$

after the change of variable $x = \dfrac{r(u)}{z^2}$ (with a well-defined inverse $u(x;z) = r^{-1}(xz^2)$, provided η is sufficiently small) and where

$$f(x;t,z) = \frac{A(t,t-u(x;z),s(t))}{1+2axz^2-h^2x^2z^4}\exp\left[\frac{m(t,t-u(x;z),s(t))}{xz}\left[1-\frac{m(t,t-u(x;z),s(t))}{2z}\right]\right].$$

By l'Hôpital's rule we verify that $\lim_{z\downarrow 0} f(x;t,z) = 1$; then dominated convergence and the integral

$$\int_0^\infty x^{-3/2}e^{-\frac{1}{2x}}\,dx = \sqrt{2\pi}$$

take care of $\lim_{z\downarrow 0} J_{11} = 1$. Therefore, for $\eta > 0$ sufficiently small:

$$\overline{\lim_{z\downarrow 0}}\,|I-1| \le C\eta^{1/2}.$$

Returning to the quantity L, we have $|L_2| \le C\eta^{1/2}$ and

$$\overline{\lim_{z\downarrow 0}}\,|L_1-\rho(t)| \le C(\eta^{1/2} + \sup_{t-\eta\le u\le t}|\rho(u)-\rho(t)|),$$

for all $\eta > 0$ sufficiently small; therefore,

$$\lim_{z\downarrow 0}(L-\rho(t)) = 0$$

which establishes (18).

We shall also need a jump relation for the analog

$$\int_0^t \Gamma_x(t,z+s(t);u,s(u)\rho(u)du$$

of the classical double-layer potential.

(19) *Proposition:* For $\rho(\cdot)$ continuous and $s(\cdot)$ Lipschitz

$$\lim_{z\to 0\pm}\int_0^t \Gamma_x(t,z+s(t);u,s(u))\rho(u)du = \pm\rho(t)$$

$$+\int_0^t \Gamma_x(t,s(t);u,s(u))\rho(u)du$$

Proof: Very similar to that of (17); by using the formulas

$$K_x(t,z;s,x) = K(t,z;s,x)\left[-\frac{1}{2}h^2\int_s^t[2m(\theta,s,x)+g(\theta)]\sigma(\theta-s)d\theta + \sigma(t-s)\frac{[z-m(t,s,x)]}{r(t-s)}\right]$$

$$K_z(t,z;s,x) = -K(t,z;s,x)\frac{z-m(t,s,x)}{r(t-s)}$$

We can rewrite the left hand side of (19) in terms of $-K_z$ and additional integrands of the form $K\cdot O(t-s)$; the latter will contribute $O(\delta)$ in the critical range $(t-\delta,t)$; by (17) the $-K_z$ term will give a jump $\pm\rho(t)$.

Adapting the results of this section to our problem, we see that we are to solve in $R^+ \times A_i$ the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2} + \gamma_i(t,z)\frac{\partial u}{\partial z} - \frac{1}{2}V_i(t,z)u \tag{10i}$$

with the boundary conditions (9) and the initial condition

$$\lim_{t\to 0} u(t,z;0,x) = \delta(z-x),$$

$$\gamma_i(t,z) = h_i y_t - f(x_i) - a_i(z-x_i)$$

$$V_i(t,z) = h_i^2 z^2 + 2h_i(h(x_i)-h_i x_i+a_i y_t)z - a_i + [h(x_i)-h_i x_i]^2 + 2[f(x_i)-a_i x_i]h_i y_t - h_i^2 y_t^2$$

The identification $h = h_i$, $a = a_i$, $e(t) = h_i y_t - f(x_i) + a_i x_i$, $g(t) = h_i[h(x_i)-h_i x_i+a_i y_t]$ in (11) leads to a fundamental solution of (10i) in the form

$$K^{(i)}(t,z;s,x) = \Gamma^{(i)}(t,z;s,x)\exp\left\{-\frac{1}{2}\int_s^t k_i(u)du\right\} \tag{20}$$

where $k_i(u)$ = coefficient of z^0 in $V_i(u,z)$. Jump relations for potentials against the new kernel $K^{(i)}$ follow readily from Propositions (17) and (19).

4. REPRESENTATION

Let us fix an interval $A_i = (x_i,x_{i+1})$ and use Green's theorem to find a representation there for the solution u. With

$$\mathbf{L}_i^* = \frac{1}{2}\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\gamma_i - \frac{1}{2}V_i$$

we have the forward equation $\frac{\partial u}{\partial t} = \mathbf{L}_i^* u$, and for $K^{(i)}$ the backward $\frac{\partial K^{(i)}}{\partial s} + \mathbf{L}_i K^{(i)} = 0$, with $\mathbf{L}_i$ the formal adjoint of $\mathbf{L}_i^*$ corresponding to a drift γ_i and a potential V_i. We find in the usual way that on A_i

$$\frac{\partial}{\partial s}K^{(i)}(t,z;s,x)u(s,x) = \frac{1}{2}(K^{(i)}u_x)_x - \frac{1}{2}(K_x^{(i)}u)_x - (\gamma_i K^{(i)}u)_x$$

Integrating dx over $A_i = (x_i,x_{i+1})$ and ds from ϵ to $t-\epsilon$, one obtains

$$\int_{x_i}^{x_{i+1}}[K^{(i)}(t,z;t-\epsilon,x)u(t-\epsilon,x) - K^{(i)}(t,z;\epsilon,x)u(\epsilon,x)]dx$$

$$= \int_\epsilon^{t-\epsilon}\left\{\frac{1}{2}K^{(i)}(t,z;s,\xi)u_2(s,\xi) - \frac{1}{2}K_\xi^{(i)}(t,z;s,\xi)u(s,\xi) - \gamma_i(s,\xi)K(t,z;s,\xi)u(s,\xi)\right]_{\xi=x_i+0}^{\xi=x_{i+1}-0}ds.$$

For $z\epsilon A_i$ let $\epsilon \to 0$ to get a representation of u in terms of its values and one-sided derivatives at the boundary points x_i and x_{i+1}:

$$u = u(t,z;0,x) \tag{21}$$

$$= K^{(i)}(t,z;0,x) + \int_0^t\left\{\frac{1}{2}K^{(i)}(t,z;s,\xi)u_2(s,\xi) - \frac{1}{2}K_\xi^{(i)}(t,z;s,\xi)u(s,\xi) - \gamma_i(s,\xi)K(t,z;s,\xi)u(s,\xi)\right]_{\xi=x_i+0}^{\xi=x_{i+1}-0}ds.$$

This representation yields integral equations for the unknown values and one-sided derivatives of u at the boundary points upon application of the jump relation (19). First let $z \to x_i$ from above to get

$$(22)\qquad \frac{1}{2}u(t,x_i)$$

$$= K^{(i)}(t,x_i;0,x) + \int_0^t \Big[\frac{1}{2}K^{(i)}(t,x_i;s,\xi)u_2(s,\xi)$$

$$-\frac{1}{2}K_\xi^{(i)}(t,x_i;s,\xi)u(s,\xi)$$

$$-\gamma_i(s,\xi)K^{(i)}(t,x_i;s,\xi)u(s,\xi)\Big]_{\xi=x_i+0}^{\xi=x_{i+1}-0} ds$$

Then let $z \to x_{i+1}$ from below to get

$$(23)\qquad \frac{1}{2}u(t,x_{i+1}) = K^{(i)}(t,x_{i+1};0,x)$$

$$+\int_0^t \Big[\frac{1}{2}K^{(i)}(t,x_{i+1};s,\xi)u_2(s,\xi)$$

$$-\frac{1}{2}K_\xi^{(i)}(t,x_{i+1};s,\xi)u(s,\xi)$$

$$-\gamma_i(s,\xi)K^{(i)}(t,x_{i+1};s,\xi)u(s,\xi)\Big]_{\xi=x_i+0}^{\xi=x_{i+1}-0} ds\ .$$

The elastic condition at x_i is

$$(9)\qquad u_2(t,x_i+0) - u_2(t,x_i-0) + y_t(h_i-h_{i-1})u(t,x_i) = 0\ ,$$

and we have $3n$ equations for $3n$ quantities $u_2(t,x_i\pm 0)$, $u(t,x_i)$, $i = 1,\ldots,n$.

It is easy to show by classical jump relation methods that if one is given $3n$ quantities $u_2(t,x_i\pm 0)$, $u(t,x_i)$ satisfying (22), (23), and (9) then the representation (23) defines a solution of problem (10). Thus we have proved the following

(24) Structure Theorem: For the "piecewise linear" dynamics of (1)-(3), the fundamental unnormalized conditional density has the form

$$q(t,z;0,x) = e^{h(z)y_t}\sum_{i=1}^n 1_{A_i}(z)\Big[K^{(i)}(t,z;0,x)$$

$$+\int_0^t \Big\{\frac{1}{2}K^{(i)}(t,z;s,\xi)u_2(s,\xi) - \frac{1}{2}K_\xi^{(i)}(t,z;s,\xi)u(s,\xi)$$

$$-\gamma_i(s,\xi)K^{(i)}(t,z;s,\xi)u(s,\xi)\Big]_{\xi=x_i+0}^{\xi=x_{i+1}-0} ds$$

with $K^{(i)}$ as in (20), and $u_2(s,x_i\pm 0)$, $u(s,x_i)$ solutions of the integral equations (22), (23), (9).

5. AN EXAMPLE

As a simple example, we consider the case $f(z) = 0$, $h(z) = |z|$, and suppose that the distribution of ξ_0 has a density $p(\cdot)$ which is an even function on R. Taking advantage of the symmetry in the problem, we are seeking to solve the initial-boundary value problem for the Zakai equation

$$d_t\rho(t,z) = = \frac{1}{2}\rho_{22}(t,z)dt + z\rho(t,z)dy_t\ ;\ t > 0, z > 0$$

$$\rho_2(t,0) = 0 \qquad ; t > 0$$

$$\rho(0,z) = p(z) \qquad ; z \geq 0$$

and then extend evenly on $z < 0$. Equivalently, one can work with the Rozovsky transformation: $\psi(t,z) = \rho(t,z).\exp(-zy_t)$; $z \geq 0$ and do the problem

$$(25)\quad \psi_1(t,z) = \frac{1}{2}\psi_{22}(t,z) + y_t\psi_2(t,z) + \frac{1}{2}(y_t^2 - z^2)\psi(t,z)\ ;\ t > 0, z > 0$$

$$\psi_2(t,0) + y_t\psi(t,0) = 0 \qquad ; t > 0$$

$$\psi(0,z) = p(z) \qquad ; z \geq 0\ .$$

It is readily seen that the fundamental solution of the transformed equation is

$$(26)\ K(t,z;s,x) = \Gamma(t,z + \int_0^t y_u du; s, x + \int_0^s y_u du)\cdot e^{\frac{1}{2}\int_s^t y_u^2 du}$$

where $\Gamma(t,z;s,x)$ is the function in (16) with $h = 1$, and that it satisfies the jump relation (Proposition (19)). Integrating Green's identify

$$\frac{\partial}{\partial x}[K\psi_2(s,x) - K_x\psi(s,x) + 2y_sK\psi(s,x)] - \frac{\partial}{\partial s}(2K\psi(s,x)) = 0$$

over $(0,t)\times R^+$, subject to the initial and boundary conditions: $\psi(0,x) = p(x)$, $\psi(t,0) \triangleq \phi(t)$, $\psi_2(t,0) = -y_t\phi(t)$, we get the integral representation

$$(27)\quad 2\psi(t,z) = \int_0^t \{K_x(t,z;s,0) - y_sK(t,z;s,0)\}\phi(s)ds + 2\int_0^\infty K(t,z;0,x)p(x)dx\ .$$

Passing to the limit as $z\downarrow 0$ yields, by virtue of the jump relation, the Volterra equation

$$(28)\quad \phi(t) = \int_0^t \{K_x(t,0;s,0) - y_sK(t,0;s,0)\}\phi(s)ds + 2\int_0^\infty K(t,0;0,x)p(x)dx\ ,$$

which is of the second kind and thus uniquely solvable for a continuous function $\phi(\cdot)$. Conversely, starting with this $\phi(\cdot)$ we can define $\psi(t,z)$ by (27) and check that the equation and the initial condition are satisfied. To show that the boundary condition is also satisfied, we first let $z\downarrow 0$ in (27), to obtain from the jump relation and in conjunction with (28) that: $\psi(t,0) = \phi(t)$; $t > 0$. Secondly, we get a new integral representation of $\psi(t,z)$ in terms of $p(\cdot)$, $\phi(\cdot)$ and $\psi_2(t,\cdot)$, which compared with (27) yields

$$\int_0^t K(t,z;s,0)\lambda(s)ds = 0\ ;\ t > 0, z > 0,$$

$$\lambda(t) = \psi_2(t,0) + y_t\phi(t)\ .$$

We differentiate this expression with respect to z and then let $z\downarrow 0$, to receive by the jump relation (19):

$$\lambda(t) = \int_0^t K_z(t,0;s,0)\lambda(s)ds\quad ;\ t > 0,$$

which implies $\lambda(t) = 0$; $t > 0$, in accordance with Lemma 7 of [1].

Conclusion: the (unnormalized) conditional density for this filtering problem is

$$\rho(t,z) = e^{zy_t}\Big[\frac{1}{2}\int_0^t \{K_x(t,z;s,0) - y_s K(t,z;s,0)\}\phi(s)ds + K(t,z;0,x)p(x)ds\Big] \quad ; t > 0, z \geq 0$$

$$= \rho(t,-z) \quad ; t > 0; z < 0$$

with $\phi(\cdot)$ the unique solution of the integral equation (28).

6. ANOTHER APPROACH

For $n > 1$, or asymmetric initial data the system of integral equations (22), (23) for the weights in the potentials representing $u(t,z)$ in the various intervals A_i is "of the first kind" in an unpleasant way that is not easily made to be "of the second kind" by Abel's transformation. To find an alternative equivalent set of equations more amenable to the classical method, we represent the solution of equation (10) in $R^+ \times A_i$ as a superposition of two single layer potentials against the fundamental solution $K^{(i)}$ and the weight functions ϕ_i^+, ϕ_{i+1}^- placed at the two end-points of the interval A_i; $0 \leq i \leq n$ $(\phi_0^+(t) = \phi_{n+1}^-(t) \equiv 0)$;

$$(29) \qquad u^{(i)}(t,z) = \int_0^t K^{(i)}(t,z;s,x_i)\phi_i^+(s)ds + \int_0^t K^{(i)}(t,z;s,x_{i+1})\phi_{i+1}^-(s)ds + K^{(i)}(t,z;0,x) .$$

The above expression satisfies equation (10) and the requisite initial condition. Matching the values of $u^{(i)}(t,\cdot)$, $u^{(i-1)}(t,\cdot)$ across x_i we obtain the Volterra equation of the first kind:

$$(30) \qquad \int_0^t K^{(i)}(t,x_i;s,x_i)\phi_i^+(s)ds + \int_0^t K^{(i)}(t,x_i;s,x_{i+1})\phi_{i+1}^-(s)ds + K^{(i)}(t,x_i,0,x) =$$
$$= \int_0^t K^{(i-1)}(t,x_i;s,x_{i-1})\phi_i-1(s)ds + \int_0^t K(i-1)(t,x_i;s,x_i)\phi_i^-(s)ds + K^{(i-1)}(t,x_i;0,x) .$$

This equation can be reduced to one of the second kind via the Abel transformation. For convenience, let us introduce the notation: $K_i^+(t,s) = K^{(i)}(t,x_i;s,x_i)$, $K_{i+1}^-(t,s) = K^{(i)}(t,x_i;s,x_{i+1})$, $K_{i-1}^+(t,s) = K^{(i-1)}(t,x_i;s,x_{i-1})$, $K_i^-(t,s) = K^{(i-1)}(t,x_i;s,x_i)$ and denote generically any of the integrals in (30) by $\int_0 K(t,s)\phi(s)ds$. Now we Abel-transform, i.e. replace t by τ, multiply by $(t-\tau)^{-1/2}$, integrate τ over $(0,t)$ and interchange the order of s and τ integrations to obtain the expression

$$\int_0^t L(t,s)\phi(s)ds$$

with

$$L(t,s) = \int_0^t \frac{\ell(\tau,s)ds}{(t-\tau)^{1/2}(\tau-s)^{1/2}} \quad , \; \ell(t,s) = (t-s)^{1/2}.K(t,s) .$$

The identity

$$-\frac{\partial}{\partial\tau}\left[2\tan^{-1}\left[\frac{t-\tau}{\tau-s}\right]\right]^{1/2} = (t-\tau)^{-1/2}(\tau-s)^{-1/2}$$

helps us integrate by parts:

$$L(t,s) = \pi.\ell(s,s) + 2\int_s^t \tan^{-1}\left[\frac{t-\tau}{\tau-s}\right]^{1/2} .\ell_1(\tau,s)ds ,$$

$$L_1(t,s) = \frac{1}{t-s}\int_s^t \left[\frac{t-\tau}{\tau-s}\right]^{-1/2} \ell_1(\tau,s)ds .$$

For the kernels under considerations, $\ell(t,s)$ and $\ell_1(t,s)$ are bounded functions, and therefore so are $L(t,s)$ and $L_1(t,s)$. Consequently, one can formally differentiate:

$$\frac{d}{dt}\int_0^t L(t,s)\phi(s)ds = L(t,t).\phi(t) + \int_0^t L_1(t,s)\phi(s)ds ,$$

where $L(t,t) = \lim_{s\uparrow t} L(t,s)$; for the kernels under consideration, it can be checked that

$$L_i^+(t,t) = L_i^-(t,t) = \left[\frac{\pi}{2}\right]^{1/2} , \quad L_{i+1}^-(t,t) = L_{i-1}^+(t,t) = 0 .$$

Abel-transforming equation (30) and then differentiating, we obtain the equivalent equation

$$\left[\frac{\pi}{2}\right]^{1/2}(\phi_i^+(t)-\phi_i^-(t)) + \int_0^t L_{i1}^+(t,s)\phi_i^+(s)ds + \int_0^t L_{(i+1)1}^-(t,s)\phi_{i+1}^-(s)ds -$$
$$(31) \qquad - \int_0^t L_{(i-1)1}^+(t,s)\phi_{i-1}^+(s)ds - \int_0^t L_{i1}^-(t,s)\phi_i^-(s)ds =$$
$$= \frac{\partial}{\partial t}\int_0^t (t-s)^{-1/2}\{K^{(i-1)}(s,x_i;0,x) - K^{(i)}(s,x_i;0,x)\}ds .$$

On the other hand, one can evaluate the limiting values of $v_2(t,z)$ as z converges to x_i from either A_i or A_{i-1} by means of the jump relation, as

$$u_2^{(i)}(t,x_i+) = -\phi_i^+(t) + \int_0^t K_z^{(i)}(t,x_i;s,x_i)\phi_i^+(s)ds + \int_0^t K_z^{(i)}(t,x_i,s,x_{i+1})\phi_{i+1}^-(s)ds + K_z^{(i)}(t,x_i;0,x) ,$$

$$u_2^{(i-1)}(t,x_i-) = \phi_i^-(t) + \int_0^t K_z^{(i-1)}(t,x_i;s,x_{i-1})\phi_{i-1}^+(s)ds + \int_0^t K_z^{(i-1)}(t,x_i;s,x_i)\phi_i^-(s)ds + K_z^{(i-1)}(t,x_i;0,x) .$$

The boundary condition (9) then becomes:

$$\phi_i^+(t) + \phi_i^-(t) - \int_0^t K_z^{(i)}(t,x_i;s,x_i)\phi_i^+(s)ds - \int_0^t K_z^{(i)}(t,x_i;s,x_{i+1})\phi_{i+1}^-(s)ds$$
$$(32) \qquad + \int_0^t K_z^{(i-1)}(t,x_i;s,x_{i-1})\phi_{i-1}^+(s)ds$$
$$= K_z^{(i)}(t,x_i;0,x) - K_z^{(i-1)}(t,x_i;0,x) +$$
$$y_t(h_i-h_{i-1})\left\{\int_0^t K^{(i)}(t,x_i;s,x_i)\phi_i^+(s)ds + \int_0^t K^{(i)}(t,x_i;s,x_{i+1})\phi_{i+1}^-(s)ds + K^{(i)}(t,x_i;s,x)\right\}.$$

The system of Volterra equations (31), (32) of the second kind is then uniquely solvable for $2n$ continuous functions $\{\phi_i^{\pm}(t), 1 \le i \le n\}$. Thus we have proved the following

Second Structure Theorem: The fundamental unnormalized conditional density for the "piecewise-linear" filter in (1)-(3) is given by

$$q(t,z;0,x) = e^{h(z)y_t} \sum_{i=0}^{n} 1_{A_i}(z) \left\{ \int_0^t K^{(i)}(t,z;s,x_i)\phi_i^{+}(s)ds + \right.$$

$$\left. + \int_0^t K^{(i)}(t,z;s,x_{i+1})\phi_{i+1}^{-}(s)ds + K^{(i)}(t,z;0,x) \right\}$$

where the $2n$ continuous weight functions $\{\phi_i^{\pm}(t);\ 1 \le i \le n\}$ are obtained as the unique solution to the system of integral equations (31), (32), and $\phi_0^{+}(t) = \phi_{n+1}^{-}(t) \equiv 0$.

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