THE WHITTAKER MODELS OF INDUCED REPRESENTATIONS

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If $F$ is a local non-Archimedean field, then every irreducible admissible representation $\pi$ of $GL(r, F)$ is a quotient of a representation $\xi$ induced by tempered ones. We show that $\xi$ has a Whittaker model, even though it may fail to be irreducible.

1. Introduction and notations.

(1.1) Let $F$ be a local non-Archimedean field and $\psi$ an additive character of $F$. Let $G$ be the group $GL(2, F)$ and $B$ the subgroup of triangular matrices in $G$. If $\mu_1$ and $\mu_2$ are two characters of $F^\times$ we may consider the induced representation $\xi = Ind(G, B; \mu_1, \mu_2)$. There is a nonzero linear form $\lambda$ on the space $V$ of $\xi$ such that

$$\lambda\left[\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} f\right] = \psi(x)\lambda(f), \quad f \in V.$$  

The map which sends $f$ to the function $W$, defined by

$$W(g) = \lambda[\xi(g)f],$$

is clearly bijective if $\xi$ is irreducible, that is, if $\mu_1 \cdot \mu_2^{-1} \neq \alpha_u^\pm$ (we denote by $\alpha_u$ or $a$ the module of $F$). If $\mu_1 \cdot \mu_2^{-1} = \alpha^{-1}$, the kernel of the map is one dimensional. If $\mu_1 \cdot \mu_2^{-1} = \alpha$ the map has trivial kernel. We recall the proof. Suppose more generally that $\mu_1 \cdot \mu_2^{-1} = \chi x$ with $\chi \chi = 1$ and $0 < u$. Then we may choose $\lambda$ in such a way that

$$W(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}) = H(-a)\mu_2(a)|a|^{u/2}, \quad H(a) = \int H(x)\psi(xa) \, dx,$$

where $H$ is the element of $L^1(F)$ defined by

$$H(x) = f\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right].$$

From the Fourier inversion formula, $W|B$ implies $H = 0$ and then, by continuity, $f = 0$. Thus we have proved the injectivity of the map $f \mapsto W$ and even the fact that the $W$'s are determined by their restriction to $B$.

(1.2) In this paper we extend this result (and its proof) to the group $G_r = GL(r, F), r \geq 2$. In a precise way, let $Q$ be the upper standard
parabolic subgroup of type \((r_1, r_2, \ldots, r_m), \sum r_i = r,\) in \(G_r.\) Then \(Q = MU\)
where \(U\) is the unipotent radical of \(Q\) and \(M\) isomorphic to \(\Pi GL(r_i).\) Let
\(\pi_i, 1 \leq i \leq m,\) be an irreducible representation of \(GL(r_i, F);\) suppose
\(\pi_i = \pi_{i,0} \otimes \alpha^{u_i},\) where \(\pi_{i,0}\) is irreducible, unitary, *tempered* and \(u_1 > u_2 > \cdots > u_m.\) We refer to the induced representation
\[
(1) \quad \xi = \text{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m)
\]
as an induced representation of "Langlands' type". Let now \(N_r\) be the group of upper triangular matrices with unit diagonal and let \(\theta\) or \(\theta_r\) be the character of \(N_r\) defined by
\[
(2) \quad \theta(n) = \prod_{i=1}^{r-1} \psi(n_{i,i+1}).
\]
Then there is a nonzero linear form \(\lambda\) on the space of \(\xi\) and, up to a scalar factor, only one such that
\[
(3) \quad \lambda[\xi(n)f] = \theta(n)\lambda(f).
\]
Let \(\mathfrak{W}(\xi; \psi)\) be the space spanned by the functions of the form \((1.1.1).\)
Our goal is to prove that the map \(f \mapsto \mathfrak{W}\) is bijective, even though \(\xi\) may be reducible. In fact we prove a little more: in the terminology of [B-Z] (Theorem 4.9) the representation \(\xi\) has a Kirillov model. We remark that when all \(\pi_{i,0}\) are supercuspidal, our result is a special case of Theorem 4.11 in [B-Z]. In general, one can try to reduce our result to theirs by imbedding each \(\pi_{i,0}\) in a representation induced by supercuspidal ones (cf. [Z]). For instance, denote by \(B\) the group of upper-triangular matrices in \(G_r\) and by \(\sigma\) the (unique) invariant irreducible subspace of
\[
\text{Ind}(G_r, B; \alpha^{(r-1)/2}, \alpha^{(r-1)/2-1}, \ldots, \alpha^{-(r-1)/2}).
\]
Then \(\sigma\) is a square-integrable representation (ordinary special representation). Consider now the induced representation
\[
\xi = \text{Ind}(G_5, Q; \sigma_3 \otimes \alpha^{1/2}, \sigma_2),
\]
where \(Q\) has type \((3, 2).\) Then \(\xi\) is a subrepresentation of
\[
\eta = \text{Ind}(G_5, B_5; \rho_1, \rho_2, \ldots, \rho_5)
\]
where \(\rho_3 = \alpha^{-1/2}, \rho_4 = \alpha^{1/2}.\) Since \(\rho_4 = \rho_3 \otimes \alpha,\) Theorem 4.11 of [B-Z]
does not apply to \(\eta.\) Thus our result does not follow directly from Theorem 4.11 of [B-Z]; some extra work is needed.

At any rate, our approach is more direct and we use the results of Bernstein-Zelevinski only in an auxiliary way. In more detail, let \(P_r\) be the
subgroup of matrices \( p \) in \( G_r \) of the form

\[
p = \begin{pmatrix} g & \ast \\ 0 & 1 \end{pmatrix}, \quad g \in G_{r-1}.
\]

Call \( \tau_r \) the unitary representation of \( P_r \) induced (in Mackey's sense) by \( \theta_r \). Then \( \tau_r \) is irreducible and the right regular representation of \( P_r \) is a multiple of \( \tau_r \); the right regular representation of \( G_r \) has the same property, when restricted to \( P_r \). Thus, if \( \pi \) is an irreducible (preunitary) square-integrable representation, then denoting by \( \pi \) the corresponding unitary representation, we see that \( \pi | P_r \) is a multiple of \( \tau_r \). (Cf., for instance, [J]). Thus \( \pi \) is generic, that is, there is a linear form \( \lambda \neq 0 \) on the space \( V \) of \( \pi \) satisfying (1.2.3). Since \( \lambda \) is unique, within a scalar factor, we see that in fact \( \pi | P_r \simeq \tau_r \). Finally if \( \eta \) is an induced representation of the form

\[
\eta = \text{Ind}(G_r; \pi_1, \pi_2, \ldots, \pi_m),
\]

where the \( \pi_i \) are irreducible square-integrable, then \( \eta \) is pre-unitary and \( \eta | P_r \simeq \tau_r \) (loc. cit.). In particular \( \eta \) is irreducible. This shows that if \( \pi \) is any irreducible pre-unitary tempered representation of \( G_r \), then \( \pi | P_r \simeq \tau_r \). This is, essentially, all we need to know about tempered representations (cf. §2 below).

We also remark that the problem of finding all irreducible square-integrable representations of \( G_r \) is equivalent to the problem of finding all irreducible generic ones. Indeed, if \( \pi \) is a square-integrable representation, then \( \pi \) is generic by the above remarks, thus by Theorem 9.7 of [B-Z] (classification of all generic representations) \( \pi \) is equivalent to an induced representation of the form

\[
\xi = \text{Ind}(G_r, Q; \sigma_1, \sigma_2, \ldots, \sigma_m)
\]

where the \( \sigma_i \) are "generalized special representations". But then Casselman's criterion for square-integrability shows that, in fact, \( \xi \) is itself a generalized special representation: this is a sketch of the proof of Theorem 9.3 stated in [Z] and due to I. N. Bernstein. Conversely if \( \xi \) is a representation of the form (1.2.1) then \( \xi \) has a unique irreducible quotient \( J(\pi_1, \pi_2, \ldots, \pi_m) \) ("Langlands' quotient": cf. [B-W] XI, §2). If \( \xi \) is irreducible then our result implies that \( J(\pi_1, \pi_2, \ldots, \pi_m) \) is degenerate (not generic). Since any irreducible representation \( \pi \) of \( G_r \) has the form \( J(\pi_1, \pi_2, \ldots, \pi_r) \) for appropriate \( \pi_i \), we see that if \( \pi \) is generic then \( \pi \) must be equivalent to a representation of the form (1.2.1); that is, we have another proof of Theorem 9.7 of [B-Z].
Finally we also remark that our result and its proof apply to the case $F = \mathbb{R}$ or $\mathbb{C}$ as well. Naturally $\lambda$ in (1.1.3) and (1.1.1) is then a linear form defined and continuous on an appropriate space of smooth vectors to which $f$ belongs. One needs to duplicate the estimates of §2 and check that in (3.1.2), the linear form $f \mapsto W(e)$ can be taken to be $\lambda$, that is, is continuous. Furthermore in (3.2.15) the right-hand side does not have support in the set (3.2.16) but is “of rapid decrease for $|a|_1$ large”. Rather than dealing with these minor changes now we prefer to wait for another occasion. We also remark that, taking again into account Langlands’ classification and Theorem D of [K], we get, for $GL(r, F)$, another easy proof of the difficult Theorem 6.2 of [V].

However, on the whole, our motivations are global. In [J-P-S] Theorem (13.6) and [G-J], §4 we used this result for $GL(3)$. Similar applications are expected for higher $r$’s.

(1.3) In addition to the notations already introduced we will use the following ones: $q$ will be the cardinality of the residual field of $F$, $\mathfrak{R}$ the ring of integers in $F$; $K_r$ will be the subgroup $GL(r, \mathfrak{R})$. We will denote by $Z_r$, the center of $G_r$, by $A_r$ the subgroup of diagonal matrices in $G_r$, by $B_r = A_r N_r$, the group of upper triangular matrices and, finally, by $P_r$ the subgroup of matrices of the form

\[ p = \begin{pmatrix} g & \ast \\ 0 & 1 \end{pmatrix}, \quad g \in G_{r-1}. \]

2. Estimate of tempered Whittaker functions.

(2.1) Let $\pi$ be an irreducible pre-unitary tempered representation of $G_r$. Then there is a linear form $\lambda \neq 0$ on the space $V$ of $\pi$ satisfying (1.2.3) and, within a scalar factor, only one. We denote by $\mathcal{W}(\pi; \psi)$ the space spanned by functions of the form (1.1.1) with $f$ in $V$. We recall some known facts on the elements of $\mathcal{W}(\pi; \psi)$.

(2.2) If $W$ is in $\mathcal{W}(\pi; \psi)$ then the integral

\[
\Psi(s, W, \overline{W}, \Phi) = \int_{N \backslash G_r} W(g) \overline{W}(g) \Phi[(0, 0, \ldots, 0, 1)] g | \det g |^s dg,
\]

where $\Phi$ is in the space $\mathcal{S}(F')$ of Schwartz-Bruhat functions on $F'$, converges for $\text{Res} \gg 0$ and represents a rational fraction in $q^{-s}$ without pole for $\text{Res} > 0$ ([J-P-S] Prop. (8.4)); in passing we note that this result is independent of the classification of all square-integrable representations.

(2.3) The unitary representation of $G_r$ corresponding to $\pi$ has the property that its restriction to the subgroup $P_r$ is equivalent to the
representation \( \tau \) of \( P_r \) induced (in Mackey's sense) by \( \theta_r \). It amounts to the same to say that

\[
B(W, W') = \int_{N_r \setminus G_r} W \left[ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right] \bar{W}' \left[ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right] dh
\]

defines a \( G \)-invariant form on \( \mathcal{F}(\pi; \psi) \) (cf. [J]). From this or Theorem (4.9) of [B-Z] it follows that any \( W \) is determined by its restriction to \( P_r \).

(2.4) Finally, the space of these restrictions contains the space \( \mathcal{K}_0(\pi; \psi) \) of functions \( f \) on \( G_r \), transforming on the left under \( \theta_r \), right smooth and of compact support mod \( N_r \) ([G-K] (5.2)).

(2.5) We need an estimate for the elements of \( \mathcal{F}(\pi; \psi) \). The quickest proof uses (2.2). Let \( \delta_r \) denote the module of the Borel subgroup \( B_r \) in \( G_r \). We will extend \( \delta_r \) to a function on \( G_r \) which is \( K_r \)-invariant on the right. We remark that

\[
\delta_r \left[ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} \right] = \delta_{r-1}(g) | \det g |
\]

if \( g \) is in \( G_{r-1} \). We also define a function \( \Lambda_r \) on \( G_r \) by setting

\[
\Lambda_r \left( z n \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right) = | \det g |
\]

for \( z \in Z_r, n \in N_r, k \in K_r, g \in G_{r-1} \).

**Proposition.** Suppose \( \pi \) is a tempered representation of \( G_r \) and \( W \) is in \( \mathcal{F}(\pi; \psi) \). Then, for any \( s > 0 \), there is a constant \( c_s > 0 \) such that \( | W |^2 \leq c_s \delta_r \Lambda_r^{-s} \).

**Proof.** Let \( \Phi \geq 0 \) be an element of \( \mathcal{S}(F^r) \) which is \( K_r \)-invariant on the right. Then, for \( s \gg 0 \), setting \( \eta_r = (0, 0, \ldots, 0, 1) \), we have:

\[
\Psi(s, W, \bar{W}, \Phi) = \int_{K_r} dk \int_{A_{r-1}} | W |^2 \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k \right] \delta_r^{-1} \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] | \det a |^s d^\kappa a \\
\times \int_{F^s} \Phi[\eta_r b k] | b |^s d^\kappa b.
\]

The convergence of the integral for \( \text{Res} \gg 0 \) amounts to the convergence of a power series in \( x = q^{-s} \),

\[
\Psi(s, W, \bar{W}, \Phi) = \sum_{m \geq m_0} a_m x^m,
\]
say for $0 < |x| < \varepsilon$ (cf. (4.1) and (4.2) in [J-P-S]). By (2.2), the series in (2) actually converges for $0 < |x| < 1$. But then since the integrand in (1) is $\geq 0$, the integral for $\Psi$ must actually converge for $s > 0$. In particular

$$
\int |W|^2 \left| \left[ \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right] k \right| \delta_r^{-1} \left| \left[ \begin{array}{cc} a & 0 \\ 0 & 1 \end{array} \right] \right| | \det a | \, d^\times a < +\infty
$$

(for $s > 0$) for all $k \in K_r$. Fix $k$ then and let us denote by $f(a), a \in (F^\times)^{-1} \simeq A_{r-1}$, the integrand in (3). Clearly there is an open compact subgroup, $U$ say, of $(F^\times)^{-1}$ such that $f(\alpha e) = f(a)$ for all $a$ in $(F^\times)^{-1}$, $\varepsilon$ in $U$. We deduce at once that, for all $b \in (F^\times)^{-1}$,

$$
|f(b)| \leq c \int |f(a)| \, d^\times a,
$$

c a positive constant. In other words the integrand in (3) is bounded. This is precisely what we wanted to prove. \hfill \square

3. Induced representations of Langlands' type.

(3.1) Consider a representation

$$(1) \quad \xi = \text{Ind}(G_r, Q; \pi_1, \pi_2, \ldots, \pi_m)$$

(notations as in (1.2)). A vector $f$ in the space of $\xi$ may be regarded as a function on $G_r$ with values in $\bigotimes_{i=1}^m \mathbb{C}(\pi_i; \psi)$; it may also be regarded as a scalar function on $G_r \times G_{r_1} \times \cdots \times G_{r_m}$ whose value at $(g, h_1, h_2, \ldots, h_m)$ we denote by $f(g; h_1, h_2, \ldots, h_m)$. The integral

$$(2) \quad W(g) = \int_U f(wug; e, e, \ldots, e) \tilde{\theta}(u) \, du,$$

where

$$(3) \quad w = \begin{pmatrix} 0 & 1_{r_1} \\ \vdots & \ddots & \ddots \\ 1_{r_m} & \cdots & 0 \end{pmatrix},$$

and $du$ is a Haar-measure on the unipotent radical $U$ of the parabolic subgroup of type $(r_m, r_{m-1}, \ldots, r_2, r_1)$, defines an element of $\mathbb{C}(\xi; \psi)$ provided it converges. We are going to show that it converges for all $f$; in fact, we are going to obtain a majorization of the function

$$(4) \quad h \mapsto \int f(wug; e, e, \ldots, e, h) \, du.$$
It will be sufficient to obtain an upper bound for the integral

$$\int |f| (wug; e, e, \ldots, e, h) \, du. \tag{5}$$

This integral, finite or infinite, is equal to

$$|\det h|^{-(r-r_m)/2} \int |f| \left[ wu \left( \begin{array}{cc} h & 0 \\ 0 & 1_{r-r_m} \end{array} \right) g; e, e, \ldots, e \right] \, du. \tag{6}$$

With notation as in (2.5), let $f_0$ be the function defined by

$$f_0(g) = \delta_q^{1/2}(q) \prod_{j=1}^m \delta_j^{1/2}(g_j) \Lambda_{r_j}(g_j)^{-s_j} |\det g_j|^\nu, \tag{7}$$

for $g$ of the form $g = qk$, $q \in Q$, $k \in K_r$ and $q$ of the form

$$q = \left( \begin{array}{ccc} g_1 & & \\ & \ddots & \\ 0 & & g_m \end{array} \right), \quad g_i \in G_{r_i}. \tag{8}$$

Here $(s_1, s_2, \ldots, s_m)$ is an $m$-tuple of positive numbers to be chosen below. Next we apply Proposition (2.5) to the (quasi-) tempered representations $\pi_i$ ($1 \leq i \leq m$) to conclude that given $g_0 \in G_r$, there is a constant $c > 0$ such that

$$|f| (gg_0; e, e, \ldots, e) \leq cf_0(g). \tag{9}$$

Thus all we need to do is to obtain an upper bound for the function

$$|\det h|^{-(r-r_m)/2} \int f_0 \left[ wu \left( \begin{array}{cc} h & 0 \\ 0 & 1_{r-r_m} \end{array} \right) \right] \, du. \tag{10}$$

This is actually equal to

$$\int f_0(wu) \, du \delta_m^{1/2}(h) \det h |\mu_{A_{r_m}}(h)|^{-s_m}. \tag{11}$$

We are thus reduced to proving that

$$\int f_0(wu) \, du < +\infty. \tag{12}$$

For that let $V$ denote the unipotent radical of the lower parabolic subgroup of $G_r$ of type $(r_1, \ldots, r_m)$. Then the integral (11) is the same as the integral

$$\int_V f_0(v) \, dv. \tag{13}$$
Next for $q$ a diagonal matrix of the form (8), we have
\[
\delta_B(q) = \delta_Q(q) \prod_{1 \leq j \leq m} \delta_{r_j}(g_j),
\]
from which we see that for $q = \text{diag}(a_1, a_2, \ldots, a_r)$
\[
f_0(q) = \delta_B^{1/2}(a) |a_1 a_2 \cdots a_{r-1}| a_{r1}^{(r_1-1)s_1+u_1} \cdot |a_{r1+1} \cdots a_{r1+r_2-1}| a_{r1+r_2}^{(r_2-1)s_2+u_2} \ldots.
\]
We have seen then that to insure the convergence of (13) it suffices to choose the $s_j > 0$ so that
\[
u_1 + (r_1 - 1)s_1 > u_1 > s_1 > u_2 + (r_2 - 1)s_2 > u_2 - s_2 > \ldots.
\]
Each inequality in (15) is either true or can be made true by making the $s_j$ positive and sufficiently small. We have now proved that the integral in (2) is indeed convergent and, moreover, obtained the inequality
\[
\int_U |f|(w; e, e, \ldots, e, h) \, du \leq c_v \delta_{r_m}^{1/2}(h) A_{r_m}(h)^{-v} |\det h|^v_m,
\]
where $v$ is any sufficiently small positive number and $w$ is given by (3).

(3.2) **PROPOSITION.** Let $\xi$ be the representation (3.1.1). Then the map $f \mapsto W$ from the space of $\xi$ to $\mathcal{W}(\xi; \psi)$ defined by (3.1.2) is bijective. Moreover, if $W \in \mathcal{W}(\xi; \psi)$ then the relation $W | P_r = 0$ implies $W = 0$.

**Proof.** Our assertion is trivial for $m = 1$. Thus we may assume $m > 1$ and our assertion proved for $m - 1$. Consider then the induced representation
\[
\xi^* = \text{Ind}(G_r, Q^*; \xi', \pi_m),
\]
where
\[
\xi' = \text{Ind}(G_{r-r_m}, Q'; \pi_1, \pi_2, \ldots, \pi_{m-1}),
\]
where $Q^*$ has type $(r_1, r_2, \ldots, r_{m-1})$. Furthermore, by the induction hypothesis, we may regard $\xi'$ as acting on $\mathcal{W}(\xi'; \psi)$. Thus we may regard an element $f^*$ of $\xi^*$ as a function on $G_r$ with values in $\mathcal{W}(\xi'; \psi) \otimes \mathcal{W}(\pi_m, \psi)$, or as a scalar function on $G_r \times G_{r-r_m} \times G_m$. We denote its value at $(g, h_1, h_2)$ by $f^*(g; h_1, h_2)$. Of course the representations $\xi$ and $\xi^*$ are equivalent. If $f$, as in (3.1), is in the space of $\xi$ then the exact relation between $f$ and $f^*$ is given by
\[
f^*[g; e, e] = \int_U f\left(\begin{pmatrix} w' & 0 \\ 0 & 1_m \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & 1_m \end{pmatrix} g; e, e, \ldots, e \right) \bar{\theta}_{r-m}(v) \, dv.
\]
where

$$w' = \begin{pmatrix} 0 & 1_r' \\ \vdots & \ddots & \ddots & \vdots \\ 1_{r_{m-1}} & \cdots & 0 \end{pmatrix},$$

and $V'$ is the unipotent radical of the (upper) parabolic in $G_{r-r_m}$ of type $(r_m-1, r_{m-2}, \ldots, r_1)$. Writing (11.2) as an iterated integral, we readily find that in terms of $f^*$,

$$W(g) = \int_{v^*} f^*[w^*vg; e, e] \bar{\theta}_r(v) dv,$$

where now

$$w^* = \begin{pmatrix} 0 & 1_{r-r_m} \\ 1_r & 0 \end{pmatrix},$$

and $V^*$ is the unipotent radical of the parabolic in $G_r$ of type $(r_m, r-r_m)$. Of course the convergence of the integral (3.1.2) implies that of both integrals (3) and (5) (for all $g \in G_r$). Since the map $f \mapsto f^*$ is bijective, all of our assertions will be proved if we show

$$W|_{P_r} = 0$$

implies that $f^* = 0$.

Assume then that $W|_{P_r} = 0$. Explicitly this reads

$$\int f^*\left[w^*\begin{pmatrix} 1_{r_m} & \chi \\ 0 & 1_{r-r_m} \end{pmatrix} p; e, e\right] \psi(\text{tr}(ex)) dx = 0$$

for all $p \in P_r$. Here

$$\varepsilon = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad (r-r_m \text{ rows, } r_m \text{ columns}).$$

Replacing $p$ by

$$\begin{pmatrix} g_1 & 0 \\ 0 & 1_{r-r_m} \end{pmatrix} p,$$
where $g_1 \in G_{r_m}$, and changing variables, we can write this condition in the form

\[
(10) \quad \int f^* \left[ w^* \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p; e, g_1 \right] \psi(\text{tr}(e g_1 x)) \, dx = 0,
\]

for all $p \in P_r$, $g_1 \in G_{r_m}$. We can also replace $g_1$ by $hg_1$ where $h \in P_{r_m}$. Note that $\varepsilon h = \varepsilon$. Thus if we set, for $h \in G_{r_m}$,

\[
(11) \quad F(h) = \int f^* \left[ w^* \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} p; e, hg_1 \right] \psi(\text{tr}(e g_1 x)) \, dx,
\]

then we see that the function $F$ defined on $G_{r_m}$ has a zero restriction to $P_{r_m}$. At this point we may assume $u_m = 0$. We are going to show that $F$ is actually zero. To see that we first need a majorization of $F$. Using (3) to express $f^*$ in terms of $f$ we obtain at once from (3.1.16):

\[
(12) \quad |F(h)| \leq c_0 \delta_{r_m}^{1/2}(h) \Lambda_{r_m}(h)^{-v},
\]

again for $v > 0$ sufficiently small, and all $h \in G_{r_m}$.

Thus, for $W'' \in \mathcal{O}(\pi_m; \psi)$, we have the inequality

\[
(13) \quad \int_{N_{r_m} \setminus G_{r_m}} |F W''| \left[ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right] \, dh 
\]

\[
\leq c_v \int_{N_{r_m} \setminus G_{r_m}} |W''| \left[ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right] \delta_{r_m}^{1/2} \left[ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right] \Lambda_{r_m} \left[ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right]^{-v} \, dh.
\]

We claim now that both integrals are finite. It suffices to check that the integral

\[
(14) \quad \int |W''| \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \delta_{r_m-1}^{-1}(a) \delta_{r_m}^{1/2} \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \det a^{-v} d^\infty a
\]

is finite for any $v > 0$. Now by (2.5) we have

\[
(15) \quad |W''| \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \leq c_v \delta_{r_m}^{1/2} \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] \det a^{-v}.
\]

Moreover the support of $W''(0, 0, 1)$ is contained in the set $C$ defined by the conditions

\[
(16) \quad a = \text{diag}(a_1 a_2 \cdots a_{r-1}, a_2 \cdots a_{r-1}, \ldots, a_{r-1}), \quad |a_i| \leq c_i,
\]

for suitable $c_i$. Since

\[
\delta_{r_m} \left[ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \right] = \delta_{r_m-1}(a) \det a
\]
we are reduced to considering the integral \( \int_{c} \det a^{1-v} d^{x} a \). This is indeed finite, provided \( 0 < v < 1 \). Our argument shows in fact that, if in

\[
(17) \quad \int_{N_{m-1} \setminus G_{m-1}} FW' \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) dh,
\]

we replace \( F \) by its expression (11), then the resulting double integral converges. Thus (17) can be written as

\[
(18) \quad \int \psi(\text{tr}(\epsilon gx)) \, dx \int f^* \left[ w^* \left( \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} \right) p; e, \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) g \right] \cdot W' \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) dh.
\]

Next, since we have taken \( u_m = 0 \), the representation \( \pi_m \) of \( G_r \) is pre-unitary. Thus (2.1.2) defines an \textit{invariant} Hermitian form on \( \mathcal{W}(\pi_m; \psi) \). Hence the inner integral in (18) can also be written as

\[
\int f^* \left[ w^* \left( \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} \right) p; e, \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) g^{-1} \right] dh.
\]

Since \( W' \) is arbitrary we can replace \( W' \) by any of its right translates. We get that

\[
(19) \quad \int W' \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) dh \int \psi(\text{tr}(\epsilon gx)) \, dx
\]

\[
\cdot \int f^* \left[ w^* \left( \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} \right) p; e, \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) \right] = 0
\]

for all \( g \in G_{r_m} \) and all \( p \in P_r \). Here \( W' \) can be taken arbitrary in \( \mathcal{W}_0(\psi) \) (cf. (2.2.1)). Thus we finally get

\[
(20) \quad \int \psi(\text{tr}(\epsilon gx)) \, dx \int f^* \left[ w^* \left( \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} \right) p; e, e \right] = 0,
\]

again for all \( g \in G_{r_m} \) and \( p \in P_r \). But \( \text{tr}(\epsilon gx) = yx_1 \), where \( y \) is the last row of \( g \in G_{r_m} \) and \( x_1 \) is the first column of \( x \). Thus we get at first for all \( y \in F^r_{r_m} \) nonzero, and then for all \( y \), the relation

\[
(21) \quad \int \psi(yx_1) f^* \left[ w^* \left( \begin{pmatrix} 1_{r_m} & x \\ 0 & 1_{r-r_m} \end{pmatrix} \right) p; e, e \right] dx = 0.
\]
Since the integral (21) is absolutely convergent, we may apply Fourier inversion to conclude that

\[
\int f^* \begin{bmatrix}
1_r & 0 & x \\
0 & 1 & 0 \\
0 & 0 & 1_{r-r_m-1}
\end{bmatrix} w^* \begin{bmatrix}
p; & e, & e
\end{bmatrix} dx = 0,
\]

for all \( p \in P_r \).

We shall now prove that, for any \( j \) with \( 1 \leq j \leq r - r_m - 1 \), the relation

\[
\int f^* \begin{bmatrix}
1_r & 0 & x \\
0 & 1 & 0 \\
0 & 0 & 1_{r-r_m-j}
\end{bmatrix} w^* \begin{bmatrix}
p; & e, & e
\end{bmatrix} dx = 0,
\]

for all \( p \in P_r \), implies the same relation with \( j \) replaced by \( j + 1 \). For this let

\[
p' = \begin{pmatrix} g & 0 \\
0 & 1_{r-r_m-j} \end{pmatrix},
\]

where \( g \) is an element of \( G_{r_m+j} \) of the form

\[
g = \begin{pmatrix}
1_r & 0 & 0 \\
0 & 1_j & 0 \\
z & 0 & 1
\end{pmatrix},
\]

\( z \) being a row of length \( r_m \). Our hypothesis on \( j \) implies \( p' \in P_r \). Thus we can replace \( p \) by \( p'p \) in (23). Then, after a simple computation, we get

\[
\int f^* \begin{bmatrix}
1_j & v & 0 \\
0 & 1_{r-r_m-j} & 0 \\
0 & 0 & 1_{r_m}
\end{bmatrix} w^* \begin{bmatrix}
1_r & 0 & x \\
0 & 1_j & 0 \\
0 & 0 & 1_{r-r_m-j}
\end{bmatrix} p; e, e \ dx = 0.
\]

Here \( v \) is the \( r_m \times j \) matrix given by

\[
v = \begin{bmatrix} 0 \\
-z \end{bmatrix}.
\]

Since \( f^* \) belongs to the space of \( \xi^* \), this reduces to the relation

\[
\int f^* \begin{bmatrix}
1_r & 0 & x \\
0 & 1_j & 0 \\
0 & 0 & 1_{r-r_m-j}
\end{bmatrix} w^* \begin{bmatrix}
p; & e, & e
\end{bmatrix} \psi(-zx_1) \ dx = 0;
\]
as before $x_i$ is the first column of $x$. If we again use Fourier inversion, we arrive at (23) with $j$ replaced by $j + 1$.

Thus we have now proved that $f^*[w*p; e, e] = 0$ for all $p \in P_r$. Replacing $p$ by

\[
\begin{pmatrix}
1 & 0 \\
0 & g
\end{pmatrix} p, \quad g \in P_{r-r_m},
\]

we get

\[(26)\]

\[f^*[w*p; g, e] = 0\]

for all $g \in P_{r-r_m}, p \in P_{r_m}$. Since the function $g \mapsto f^*[w*p; g, e]$ belongs to $\mathcal{W}(\xi'; \psi')$, at this point we can apply the second part of our induction hypothesis to the representation $\xi'$ to conclude that

\[(27)\]

\[f^*[w*p; g, e] = 0\]

for all $p \in P_{r_m}$ and now for all $g \in G_{r-r_m}$. But then (27) implies that $f^*[uw*q; e, e] = 0$ for all $q$ in the parabolic subgroup of type $(r_m, r-r_m)$ and all $u$ in the unipotent radical of $Q^*$. By continuity we get $f^*[g; e, e] = 0$ for all $g$, that is, $f = 0$. \hfill \square

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