DISTINGUISHED REPRESENTATIONS AND QUADRATIC BASE CHANGE FOR $GL(3)$

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Abstract. Let $E/F$ be a quadratic extension of number fields. Suppose that every real place of $F$ splits in $E$ and let $H$ be the unitary group in 3 variables. Suppose that $\Pi$ is an automorphic cuspidal representation of $GL(3, E_\mathbb{A})$. We prove that there is a form $\phi$ in the space of $\Pi$ such that the integral of $\phi$ over $H(F) \backslash H(F_\mathbb{A})$ is non zero. Our proof is based on earlier results and the notion, discussed in this paper, of Shalika germs for certain Kloosterman integrals.

1. Introduction

Let $E/F$ be a quadratic extension of number fields. We denote by $z \mapsto \overline{z}$ the Galois conjugation in $E$. We denote by $F^+$ the group of elements of $F^\times$ which are norms and define similarly $F_v^+$ for any place $v$ of $F$ inert in $E$. We let $F_\mathbb{A}^+$ be the group of elements $z = (z_v)$ in $F_\mathbb{A}^\times$ such that $z_v \in F_v^+$ for $v$ inert. Of course $F^+ = F_\mathbb{A}^+ \cap F^\times$. We let $E^1$ be the group of elements of norm 1 in $E^\times$. We let $S$ be the variety of invertible Hermitian matrices:

$$(1) \quad S(F) = \{ s \in GL(n, E) | s^* = s \}$$

where we have set

$$s^* = \overline{s}.$$

Finally, we let $S^+(F)$ be the set of $s \in S(F)$ such that $\det s \in F^+$ and define similarly $S_v^+$ and $S^+(F_\mathbb{A})$. Let $\Pi$ be an automorphic cuspidal representation of $GL(n, E_\mathbb{A})$. For $\sigma \in S(F)$ let $H_\sigma$ be the corresponding unitary group:

$$(2) \quad H_\sigma = \{ h | h^* \sigma h = \sigma \}.$$

Following [HLR], we say that $\Pi$ is $H_\sigma$-distinguished if there exists a form $\phi \in \Pi$ such that

$$(3) \quad \int_{H_\sigma(F) \backslash H_\sigma(F_\mathbb{A})} \phi(h) \, dg \neq 0.$$

Let $Z$ be the center of $GL(n)$ so that $Z \cong GL(1)$. Implicit in this definition is the requirement that the central character $\omega$ of $\Pi$ be trivial on the set of elements of norm 1, that is, be distinguished with respect to $E^1$, the unitary group in 1 variable. According to an argument due to [HLR], if $\Pi$ is distinguished with respect to a unitary group $H_\sigma$ then $\Pi$ is invariant under Galois conjugation and thus is the base change of a cuspidal representation $\pi$ of $GL(n, F_{\mathbb{A}})$ ([AC]). It is natural to...
conjecture the converse is true: any base change representation is distinguished with respect to some unitary group. According to [HLR], this is the case if \( n = 2 \).

We consider more particularly the case where \( n \) is odd and all the Archimedean places of \( F \) split in \( E \). At a (finite) place \( v \) of \( F \) inert in \( E \) the only invariant in \( S(F_v) \) is the class of the determinant modulo \( F_v^+ \). It follows that the only invariant in \( S(F) \) is the class of the determinant modulo \( F^+ \). Thus two Hermitian matrices are equivalent if and only if their determinants are equivalent modulo the norms. As a consequence, if \( \sigma_1 \) and \( \sigma_2 \) are given in \( S(F) \), there is \( \zeta \in F^\times \) such that \( \zeta \sigma_1 \) and \( \sigma_2 \) are equivalent. In particular, two given matrices have isomorphic unitary groups so that we can speak of the unitary group without ambiguity.

We prove the converse for \( n = 3 \) under some restriction on the quadratic extension:

**Theorem 1.1.** Assume that any Archimedean place of \( F \) splits in \( E \). Let \( \Pi \) be a cuspidal automorphic representation of \( GL(3, E) \) which is a base change. Then \( \Pi \) is distinguished for the unitary group.

We may also view the above result in a more geometric light. Let \( \Phi \) be a smooth function of compact support on \( S(F) \). Let \( \chi \) be an idèle class character of \( F \). Define a function \( K_\Phi \) on \( G(E) \) by

\[
K_\Phi(g) = \int_{F^+_A/F^+} \sum_{\xi \in S(F)} \Phi(g^* \xi z g) \chi(z)d^\times z.
\]

If \( \Phi \) is supported on \( S^+(F) \) then the sum can be taken over \( S^+(F) \) since \( S^+(F) = S(F) \cap S^+(F) \). Clearly

\[
K_\Phi(\gamma z g) = K_\Phi(g) \chi(\gamma z)^{-1}
\]

for every \( \gamma \in G(E) \) and \( z \in E^\times \). An automorphic representation \( \Pi \) with central character \( z \mapsto \chi(z^\times)z^{-1} \) is distinguished if and only if

\[
\int_{Z(G(E))} K_\Phi(g) \overline{\phi}(g) \, dg \neq 0,
\]

for at least one function \( \Phi \) and one \( \phi \) in the space of \( \Pi \). Thus the problem of determining which cuspidal representations are distinguished is tantamount to finding the “projection” of the space spanned by the functions \( K_\Phi \) on the space of cusp forms. In other words, our result amounts to saying that the forms which are quadratic base change are the “automorphic spectrum” of the symmetric variety \( S \).

In general, if \( \theta \) is any involutive automorphism of a reductive group \( G \), one can define similarly the notion of an automorphic representation distinguished for the group of fixed points \( H \) of \( \theta \). As is the case here, one expects that distinguished representations have a simple characterization, in terms of the “principle of functoriality” (suitably extended to metaplectic covering groups): see [JR3]. Other examples are discussed in [J4], [M2], [FJ], [JR2], [yF1]. Of course the general theory remains to be developed. It will have many applications to special values of \( L \)-functions, cohomology and the principle of functoriality. The case at hand is a prototype case where distinguished representations have a specially simple characterization.

To prove the theorem, we use a form of the trace formula: see (18) and (23) below. This form of the trace formula has been used in other cases than the case
at hand: see [J3], [J4], [M1], [M2] and also the flawed article [yF2]. More classical contributions to the same area are [I] and [Z].

The notion of distinguished representation is specially interesting when the pair $(G, H)$ is a Gelfand pair: this means that for any place $v$ and any irreducible representation $\pi_v$ of $G_v$, the dimension of the space of linear forms on $\pi_v$ which are $H_v$ invariant is at most 1. Indeed the period integral

$$\mu(\phi) = \int_{H(F) \backslash H(F_\lambda)} \phi(h) \, dh$$

is a linear form which is $H(F_\lambda)$ invariant and one expects it is a product of local invariant forms $\mu_v$ times a number which is the special value of an $L$ function: see [G]. In the case at hand, for $n > 2$, we have an example of the opposite situation: the pair is not a Gelfand pair. Nonetheless, the trace formula that we are using suggests that the linear form $\mu$ (defined by the global integral) is an infinite tensor product of local ones. However, our result is not really sufficient to establish this assertion, because of the restrictions on the functions we are using in the trace formula.

To describe the trace formula in question we let $\Phi'$ be a smooth function of compact support on $GL(n, F_\lambda)$. We shall assume that $\Phi'$ is the product of local functions $\Phi'_v$, and that for any $v$ inert in $E$ the function $\Phi'_v$ is supported on the group

$$G_v^+ = \{g \mid \det g \in F_v^+\}.$$  

We also define $G^+(F), G^+(F_\lambda)$ similarly, as well as $Z^+(F), Z^+_v$ and $Z^+(F_\lambda)$. Thus $Z^+(F_\lambda) \simeq F_\lambda^\times$. Let $\chi$ be an idèle class character of $F$. We will set

$$K_{\Phi'}(g_1, g_2) = \int_{F^+ \backslash F_\lambda^+} \sum_{\xi \in GL(n, F)} \Phi'(g_1^{-1} \xi \sigma g_2) \chi(\xi) \, d\xi.$$  

We let $\psi$ be a nontrivial additive character of $F_\lambda/F$. We let $N$ be the group of upper triangular matrices with unit diagonal. We define an algebraic additive character of $N$, i.e., an algebraic morphism of algebraic groups from $N$ to $F$, by:

$$\theta_0(n) = \sum_i n_{i, i+1}$$  

and then set $\theta(n) = \psi(\theta_0(n))$. We compute the integral:

$$\int K_{\Phi'}(\flat n_1, n_2) \theta(n_1^{-1} n_2) \, dn_1 \, dn_2$$  

where $n_i \in N(F) \setminus N(F_\lambda)$.

To that end (compare with [F], [dG], [g,S]), we introduce a notion of orbital integral. We let $A$ be the group of diagonal matrices, $B = AN$ the group of upper triangular matrices in $G = GL(n)$, $W = W(A, G)$ the Weyl group of $A$ identified with the group of permutation matrices. We first go to a local situation and let $F$ be a local field, $\psi$ a nontrivial additive character of $F$. We define $\theta$ as above. Then an element $g \in G(F)$ is said to be relevant if the character $(n_1, n_2) \mapsto \theta_h(n_1 n_2)$ is trivial on the stabilizer $N^g$ of $g$ in the group $N \times N$. The stabilizer is defined by the equation $\flat n_1 g n_2 = g$. We will denote by $C(G(F))$ the space of smooth functions of compact support on $G(F)$. If $g$ is relevant and $\Phi' \in C(G(F))$, we define the orbital
integral $I(g, \Phi')$ by:

\begin{equation}
I(g, \Phi') = \int_{N \times (F) \setminus N(F) \times N(F)} \Phi'(tn_{1}gn_{2})\theta(n_{1}n_{2})\,dn_{1}\,dn_{2}.
\end{equation}

Recall that the elements of the form $wa$ with $w \in W, a \in A(F)$ form a set of representatives for the orbits of $N(F) \times N(F)$ on $G(F)$. We now describe a set of representatives for the relevant orbits. We say that a Levi-subgroup of $M$ is standard if it contains $A$ and a parabolic subgroup is standard if it contains $B$. We say that an element $w \in W$ is relevant if $w^2 = 1$ and for every simple root $\alpha$ (i.e. simple with respect to $B$) such that $w(\alpha)$ is negative, there is another simple root $\beta$ with $w(\alpha) = -\beta$. We denote by $R(G)$ the set of relevant elements in $W$. An element $w$ of $W$ is relevant if and only if there is a standard Levi-subgroup $M$ of $G$ such that $w$ is the longest element of $W \cap M$. We then denote by $A_w$ or $A_M$ the center of $M$. We also write $M = M_w$ and $w = w_M$. If $g = wa$ with $a \in A(F)$ is relevant then $w \in R(G)$. Assuming $w \in R(G)$ then $wa$ is relevant if and only if $a \in A_w(F)$. Suppose that $w \in R(G)$. Then let $P_w = M_wU_w$ be the parabolic subgroup which is standard and has Levi factor $M_w$. Set also $V_w = N \cap M_w$. Then $N^{wa} = N^w$ for every $a \in A_w$. Furthermore, $N^w$ is the set of pairs $(n_{1}, n_{2})$ with $n_{1} \in V_w$ and $n_{2} = w^{-1}n_{1}w$. It follows that any point of the orbit of $wa$ under $N(F) \times N(F)$ can be written uniquely in the form:

\[\xi(u_{1}, u_{2}, v) = t_{u_{1}wau_{2}}\]

with $u_{i} \in U_w(F)$ and $v \in V_w(F)$. Since the orbits of $N(F) \times N(F)$ are closed, the map $\xi$ is an isomorphism of $U_w(F) \times U_w(F) \times V_w(F)$ onto the orbit of $wa$. Recall we have fixed a nontrivial additive character $\psi$; we let $dx$ be the self-dual Haar measure on $F$. If $\alpha$ is a root let $X_{\alpha}$ be the corresponding root vector in the Lie algebra of $N$ (one entry is 1, the other entries are 0). If $U$ is a subgroup of $N$ generated by roots (i.e. whose Lie algebra is spanned by vectors $X_{\alpha}$) we set $du = \otimes dx_{\alpha}$ if $u = 1 + \sum_{\alpha}x_{\alpha}X_{\alpha}$. We take for invariant measure on the orbit the product measure $du_{1}\,dv\,du_{2}$. Thus:

\begin{equation}
I(wa, \Phi') = \int_{U_w(F) \times U_w(F) \times V_w(F)} \Phi'(t_{u_{1}wau_{2}})\theta(u_{1}u_{2})\theta(v)\,du_{1}\,dv\,du_{2}.
\end{equation}

Since the orbit is closed, for $f \in C(G(F))$, the integrand on the right has compact support. Thus the integral converges and defines a smooth function on $A_w(F)$ (see section 2). We let $\Delta_{i}(g)$ be the minor formed with the first $i$ rows and $i$ columns of a matrix $g$. Thus $\Delta_{i}(tn_{1}a_{1}a_{2}n_{2}) = \Delta_{i}(a_{1})\Delta_{i}(g)\Delta_{i}(a_{2})$. On the support of a function $\Phi' \in C(G(F))$ the functions $\Delta_{i}(g)$ remain in a compact support of $F$, while the function $\Delta_{n}(g)$ remains in a compact support of $F^{\times}$. Thus an orbital integral $I(wa, \Phi')$ has support in a set defined by inequalities of the form:

\[|\Delta_{i}(a)| \leq C_{i} \quad \text{if } \Delta_{i}(wa) \neq 0, \quad A \leq |\det(a)| \leq B.
\]

Explicitly, for $n = 3$ the relevant elements can be defined as follows. For $M = A$, every $a$ in $A(F)$ is relevant. Let $P_1 = M_1N_1$ be the parabolic subgroup of type $(2, 1)$ and $A_1$ the center of $M_1$, that is, the group of matrices of the form $a = \text{diag}(a_1, a_2, a_3)$ with $a_1 = a_2$. The longest element of $W \cap M_1$ is

\begin{equation}
w_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{equation}
Then for \( a \in A_1(F) \) the element \( w_1 a \) is relevant. Similarly, let \( P_2 = M_2N_2 \) be the parabolic subgroup of type \((1,2)\) and \( A_2 \) the center of \( M_2 \), that is, the group of elements of the form \( a = \text{diag}(a_1, a_2, a_3) \) with \( a_2 = a_3 \). The longest element in \( W \cap M_2 \) is

\[
(9) \quad w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Then any \( w_2 a \) with \( a \) in \( A_2 \) is relevant. Finally set

\[
(10) \quad w_G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\]

Then for \( a \in Z(F) = A_{w_2}(F) \) the product \( w_G a \) is relevant. For \( GL(3) \) the orbital integrals are the functions \( I(a, \Phi), I(w_1 a, \Phi), I(w_2 a, \Phi), I(w_G a, \Phi) \) on the groups of \( F \) points of \( A, A_1, A_2, Z = A_{w_2} \) respectively.

Coming back to the global situation we define similarly the orbital integrals \( I(wa, \Phi') \) for relevant elements \( wa \) with \( w \in R(G), a \in A_w(F_\lambda) \). If \( V \) is an algebraic subgroup of \( N \) we normalize the Haar measure on \( V(F_\lambda) \) by \( \text{vol}(V(F)) \) equals \( 1 \). The integrals are infinite products of convergent integrals, almost all of which are equal to 1 because the integrand is 1 on its support which has measure 1 (see Proposition 2.1). Thus they are convergent. A simple formal manipulation gives then

\[
(11) \quad \int K_{\Phi'}(t^{\nu_1} n_1, n_2) \theta(n_1^{-1} n_2) \, dn_1 \, dn_2 = \int_{F^+_2 / F^+} \sum_w \sum_{\alpha} I(w\alpha z, \Phi') \chi(z) \, d^* z.
\]

Here the sum is for \( w \in R(G) \) and \( \alpha \in A_w(F) \). Indeed, one replace \( K_{\Phi'} \) by its expression as a sum and collects the terms belonging to one orbit of \( N(F) \times N(F) \). An irrelevant orbit contributes a zero integral. The integral on the left is over a compact set and thus converges absolutely (even when \( \Phi' \) and \( \theta \) are replaced by their absolute values). As before there are \( A > 0, B > 0 \) such that \( I(wa, \Phi') \neq 0 \) implies \( A \leq | \Delta_n(wa) | \leq B \); there is also a compact subset \( \omega \) of \( F_\lambda \) such that \( I(wa, \Phi') \neq 0 \) implies \( \Delta_i(wa) \in \omega \) for \( 1 \leq i < n \). As a result \( I(waz, \Phi') \neq 0 \) implies that \( z \) is in a compact set of \( F^{\kappa}_2 / F^+ \) and \( \alpha \) in a finite set. Thus the expression on the right is well defined.

Now let \( \Phi \) be a smooth function of compact support on \( S(F_\lambda) \). Recall \( K_{\Phi} \) defined in (4). We shall assume that \( \Phi = \prod \Phi_v \) where \( \Phi_v \) is supported on \( S_v^+ \) for \( v \) inert. Thus \( \Phi \) is supported on \( S^+(F_\lambda) \). We consider the integral

\[
\int K_{\Phi}(n) \theta(\pi n) \, dn
\]

over \( N(E) \backslash N(E_\lambda) \). Note that the product of \( \pi n \) and a suitable element of the derived group of \( N(E_\lambda) \) is in \( N(F_\lambda) \) so that the expression \( \theta(\pi n) \) is well defined. Alternatively, we define an algebraic additive character \( \theta_1 \) of \( N \) (regarded as a group over \( E \)) to \( F \) by

\[
\theta_1(n) = \sum_i (n_{i,i+1} + \pi_{i,i+1}).
\]

Then \( \theta(\pi n) = \psi(\theta_1(n)) \).
To compute this integral, we define a notion of orbital integral for the action of \( N(E_h) \) on \( S(F_h) \). To that end we again go to a local situation where \( F \) is a local field and \( E \) a quadratic extension of \( F \). The group \( G(E) \) operates on \( S(F) \) by \( s \mapsto g^*sg \). We say that an element \( s \in S(F) \) is relevant if the character \( \theta_1 \) is trivial on the stabilizer \( N(E)^s \) of \( s \) in \( N(E) \). The corresponding orbital integral is defined by

\[
J(s, \Phi) = \int_{N(E)^s \backslash N(E)} \Phi(n^*sn)\theta(\pi n) \, dn.
\]

*Mutatis mutandis*, the previous discussion applies to the present situation where \( G(F) \) is replaced by \( S(F) \) and the group \( N(F) \times N(F) \) by the group \( N(E) \) acting on \( S(F) \). Recall that the intersection of an orbit of \( N(E) \times N(E) \) on \( GL(n, E) \) with \( S(F) \) is a single orbit of \( N(E) \) acting on \( S(F) \) ([t.S]). Otherwise said, each orbit of \( N(E) \) on \( S(F) \) has a unique representative of the form \( wa \) with \( w \in W \) and \( a \in A(E) \) satisfying \( wa = \pi \). In particular, each relevant orbit has a unique representative of the form \( wa \) with \( w \in R(G) \) and \( a \in A_w(F) \) (see [JR1] or [Y4]). Consider elements \( w \in R(G) \) and \( a \in A_w(F) \). Introduce as before \( P_w = M_a U_w \) and \( V_w = M_w \cap N \). Then any element of the orbit of \( wa \) can be written in the form

\[
u^*v^*wavw
\]
with \( v \in V_w(E) \), \( u \in U_w(E) \). The stabilizer of \( w \) (or \( wa \)) is the set of \( v \in V_w(E) \) such that \( v^*wv = w \). On the other hand, it is immediate that \( v_1 = wv^*wv \) is an element of \( V_w(E) \) which satisfies the equation

\[
vw_1^*w = v_1.
\]

This equation determines an \( F \)-subgroup \( V_w^1 \) of \( V_w(E) \). Thus \( v_1 \in V_w^1(F) \). Thus in fact any point of the orbit of \( wa \) can be written uniquely in the form \( u^*wav_1u, u \in U_w(E), v_1 \in V_w^1(F) \). This gives a diffeomorphism of \( U_w(E) \times V_w^1(F) \) onto the orbit of \( wa \) in \( S(F) \). We have already defined a measure \( dv \) on \( V_w(F) \). The additive character \( z \mapsto \psi(z + \tau) \) of \( E \) gives rise to a self-dual Haar measure on \( E \) and thus, as before, to a measure on \( U_w(E) \). We can write an element \( v_1 \) of \( V_w^1(F) \) in the form

\[
v_1 = \sum_{\beta = -w} (z_\alpha X_\alpha + z_\alpha X_{-w_G, \alpha} + \sum_{\beta} x_\beta X_\beta).
\]

Here the second sum is over all positive roots \( \beta \) in \( V_w \) such that \( \beta = -w \) and the first sum is over all remaining positive roots in \( V_w \); also \( z_\alpha \in E, x_\beta \in F \). We set \( dv_1 = \otimes dz_\alpha \otimes dx_\beta \). We take for invariant measure on the orbit of \( wa \) the product measure \( du \, dv_1 \). Then

\[
(12) \quad I(wa, \Phi) = \int_{U_w(E) \times V_w^1(F)} \Phi(u^*wav_1u)\theta(u\pi)\psi(\theta_1(v_1)) \, du \, dv_1.
\]

As before the orbital integrals are absolutely convergent and define smooth functions on \( A_w(F) \). For \( GL(3) \) the orbital integrals are the functions \( J(a, \Phi), J(wa, \Phi), J(w_2a, \Phi), J(w_3a, \Phi) \) on the groups of \( F \)-points of \( A, A_1, A_2, Z = A_{w_G} \) respectively.

We can apply the same discussion to the situation where the local quadratic extension is replaced by the semisimple \( F \)-algebra \( E = F \oplus F \). Then

\[
S(E) = \{ (g, \iota g) | g \in G(F) \}
\]
and we can consider the relevant orbits of \( N(E) = N(F) \times N(F) \) on \( S(E) \). The situation is then equivalent to the situation discussed earlier for \( N(F) \times N(F) \) acting on \( G(F) \).

Coming back to the global situation, we define global orbital integrals for relevant elements \( w \) and we have

\[
\int_{K_E \backslash F} K_\Phi(n) \theta(\pi n) \, dn = \int_{E \times E \backslash E} \sum_w \sum_\alpha J(w \alpha z, \Phi) \chi(z) \, d^\times z.
\]  

We can also describe the function \( K_\Phi \) as follows. To begin with, every \( \sigma \in S^+(F) \) is equivalent to the identity matrix so that we can write

\[
K_\Phi(g) = \int_{H(F_{\mathbb{A}})} \sum_{\gamma \in H(F)} \Phi(g^* \gamma z g) \chi(z) \, d^\times z.
\]

Here \( H \) is the unitary group attached to the identity matrix. There exists a smooth function of compact support \( f_\cdot \) on \( GL(n, E \mathbb{A}) \) such that

\[
\Phi(g^* g) = \int_{H(F_{\mathbb{A}}) \backslash H(F)} K_f(h, g) \, dh.
\]

We use the exact sequence

\[
1 \to E^1_{\mathbb{A}} \to E^\times_{\mathbb{A}} \to F^+_{\mathbb{A}} \to 1
\]

to define a measure on \( E^1_{\mathbb{A}} \). Then

\[
\text{vol}(E^1_{\mathbb{A}} / E^1(F)) = 1
\]

and

\[
\int_{F^+_{\mathbb{A}} / F^+} \sum_{\xi \in S(F)} \Phi(g^* \xi z g) \chi(z) \, d^\times z = \int_{E^\times_{\mathbb{A}}} \sum_{\xi \in S(F)} \Phi(g^* \xi z \xi z g) \chi(z \xi z) \, d^\times z.
\]

Set as usual

\[
K_f(x, y) = \int_{E^\times_{\mathbb{A}} \backslash E^\times_{\mathbb{A}}} \sum_{\gamma \in GL(n, E)} f(x^{-1} \gamma z y) \chi(z \xi z) \, d^\times z.
\]

Then

\[
K_\Phi(g) = \int_{H(F) / H(F_{\mathbb{A}})} K_f(h, g) \, dh,
\]

and

\[
\int_{N(E) \backslash N(E_{\mathbb{A}})} K_\Phi(n) \theta(\pi n) \, dn = \int_{N(E) \backslash N(E_{\mathbb{A}})} \left( \int_{H(F) / H(F_{\mathbb{A}})} K_f(h, n) \, dh \right) \theta(\pi n) \, dn.
\]

In our trace formula, \( n = 3 \) and the functions \( \Phi \) and \( \Phi' \) are related by the following global matching orbital integral conditions, where \( \eta = \eta_{E/F} \) is the quadratic idèle class character of \( F \) attached to \( E \):

\[
I(a, \Phi') = \eta(a_2) J(a, \Phi), \tag{14}
\]

\[
I(w_1 a, \Phi') = \eta(a_2) J(w_1 a, \Phi), \tag{15}
\]

\[
I(w_2 a, \Phi') = \eta(a_2) J(w_2 a, \Phi), \tag{16}
\]

\[
I(w G a, \Phi') = J(w G a, \Phi) = 0. \tag{17}
\]
It follows that, for all \( w \in R(G), \alpha \in A_w(F), z \in F^+_h, \)
\[
I(\omega z, \Phi') = J(\omega z, \Phi).
\]
Taking in account the equalities (11) and (13) we get
\[
(18) \quad \int K_{\Phi'}(n_1, n_2) \eta(n_1^{-1} n_2) \, dn_1 \, dn_2 = \int K_{\Phi}(n) \eta(n) \, dn.
\]
In a more precise way, we assume that \( \Phi = \prod \Phi_v, f = \prod f_w \) and \( \Phi' = \prod \Phi'_v. \)
The Haar measure \( dh \) on \( H(F_h) \) is written as a product of local Haar measures in the usual way. We let \( S \) be a finite set of places so chosen that a place \( v \) not in \( S \) is finite and split, or (finite) inert and unramified in \( E \); in addition, the residual characteristic of \( v \) is odd and the conductor of \( \psi_v \) is the ring of integers \( O_v \) of \( F_v. \)
If \( v \notin S \) is inert and \( w \) is the place of \( E \) above \( v \), we set \( K_w = \text{GL}(3, O_w) \) and assume that \( \text{vol}(H_v \cap K_w) = 1. \) We will also assume that \( S \) does contain at least one finite place of \( F \) inert in \( E. \)
Recall the function \( \Phi'_v \) is assumed to be supported on \( G^+_v \) for all places \( v \) inert in \( E. \) For \( v \notin S, \) the function \( \Phi'_v \) is bi-invariant under \( K'_v = \text{GL}(3, O_v) \) and equal to the characteristic function of \( K'_v \) for almost all \( v \notin S. \) If \( v \) is in \( S \) and inert in \( E \) then we assume that \( I(w_G a, \Phi'_v) = 0. \)
Similarly, let \( v \) be a place of \( F \) inert in \( E. \) Let us denote by \( w \) the unique place of \( E \) above \( v. \) We define \( \Phi_v \) by \( \Phi_v(s) = 0 \) if \( v \notin S^+_v \) and
\[
\Phi_v(s) = \int_{H_v} f_w(h_v g) \, dh_v
\]
if \( s \in S^+_v \) and \( s = g^* g. \) Thus \( \Phi_v \) is indeed supported on \( S^+_v. \) If \( v \in S \) we will assume that \( J(w_G a, \Phi_v) = 0. \) If \( v \notin S \) we will take \( f_w \) to be bi-invariant under \( K_w = \text{GL}(3, O_w) \) and will assume that \( \Phi'_v \) is the image of \( f_w \) under the base change homomorphism of Hecke algebras. For almost all \( v \notin S \) and inert we assume that \( \Phi'_v \) is the characteristic function of \( K'_v \) and \( f_w \) the characteristic function of \( K_w. \)
This implies that for almost all \( v \) inert and not in \( S, \) \( \Phi_v \) is the characteristic function \( \Phi_v^0 \) of \( K_v \cap S_v, \) provided we assume, as we do, that the measure of \( H_v \cap K_v \) is 1. Indeed, if \( \Phi_v(s) \neq 0 \) then \( s = g^* g \) with \( g \in H_v K_w. \) Thus in fact \( s = g^* g \) with \( g \in K_w. \) Moreover \( \Phi_v(s) = 1 \) then because \( \text{vol}(H_v \cap K_v) = 1. \) Conversely, if \( s \) is in \( S_v \cap K_w \) then \( s = g^* g \) with \( g \in K_w \) by the theory of elementary divisors for Hermitian matrices and then \( \Phi_v(s) = 1. \)
Note that in general for \( v \) inert and not in \( S \) we have
\[
\Phi_v(s) = \int \Phi_v^0(g^* g \psi_w) f_w(gw) \, dgw.
\]
For \( v \) inert in \( S \) or not in \( S, \) we will assume that the following local matching orbital integral conditions are satisfied:
\[
(19) \quad I(a, \Phi'_v) = \eta_v(a_2) J(a, \Phi_v),
\]
\[
(20) \quad I(w_1 a, \Phi'_v) = \eta_v(-a_2) J(w_1 a, \Phi_v)c(E_w/F_v, \psi_v),
\]
\[
(21) \quad I(w_2 a, \Phi'_v) = \eta_v(a_2) J(w_2 a, \Phi_v)c(E_w/F_v, \psi_v).
\]
The constant \( c(E_w/F_v, \psi_v) \) will be defined in Proposition 3.1. The essential point of this article is to show that for \( v \) in \( S \) and inert and a given \( \Phi'_v \) with \( J(w_G a, \Phi'_v) = 0 \) there is a function \( \Phi_v \) with \( J(w_G a, \Phi_v) = 0 \) satisfying these conditions (Proposition 3.2). This is the only missing element in the proof of the trace formula at hand. For \( v \) inert not in \( S, \) the constant \( c(E_w/F_v, \psi_v) \) is 1 and the “fundamental
lemma” ([JY2] and [Y2]) asserts that these conditions are indeed satisfied by our chosen functions. The product of the constants $\zeta(E_w/F_v, \psi_v)$ for $v$ inert is 1.

Finally, consider a place $v$ of $F$ (finite or infinite) which splits into two places $v_1$ and $v_2$. After choosing one of the two places, we may identify $S_v$ with the set of pairs $(g, t^g)$ with $g \in G_v$ and $H_v$ with the group of pairs $(h, t^h)$ with $h \in G_v$. The group $G_{v_1} \times G_{v_2} \simeq G_v \times G_v$ operates on $S_v$ and we have again a notion of orbital integrals. To insure the analogue of the matching conditions (19) to (21) (with $c = 1$) we take

$$\Phi'_v(g) = \Phi_v(g, t^g) = \int f_{v_1}(hg)f_{v_2}(t^h) \, dh.$$  

(22)

For all places $v \notin S$ which split we take $\Phi'_v$ to be a bi-$K'_v$ invariant function and $f_{v_1}$ to be a bi-$K_v$ invariant function. Since such a function is invariant under transposition, the function $\Phi'_v$ is simply the convolution of $f_{v_1}$ and $f_{v_2}$. For almost all such $v$ we take $f_{v_1}$ to be the characteristic function of $K_{v_1}$. Finally at each Archimedean place $w$ of $E$ (resp. $v$ of $F$), we denote by $K_w$ (resp. $K'_w$) the standard maximal compact subgroup. If $v$ is an Archimedean place of $F$ then (by assumption) it splits in $E$; we assume that each function $f_{v_i}$ is itself a convolution of two $K_{v_i}$ finite functions. It follows that $\Phi'_v$ is a $K'_v$ finite function which is a quadruple convolution product of smooth $K'_v$ finite functions of compact support. With these choices, the global matching orbital integral conditions (14) to (17) are satisfied.

We define in the usual way the cuspidal components $K_{\Phi}^{cusp}$ and $K_{\Phi'}^{cusp}$ of $K_{\Phi}$ and $K_{\Phi'}$ and define the cuspidal component of $K_{\Phi}$ by

$$K_{\Phi}^{cusp}(g) = \int K_{\Phi'}^{cusp}(h, g) \, dh.$$  

Then according to [J6] the difference:

$$\int K_{\Phi}(n)\theta(\pi n) \, dn - \int K_{\Phi}^{cusp}(n)\theta(\pi n) \, dn$$

can be represented by an absolutely convergent expression taken over the continuous spectrum (see [J6] for details). The same assertion is trivially true for the difference

$$\int K_{\Phi'}(t^{n_1}, n_2)\theta(n_1^{-1}n_2) \, dn_1 \, dn_2 - \int K_{\Phi'}^{cusp}(t^{n_1}, n_2)\theta(n_1^{-1}n_2) \, dn_1 \, dn_2.$$  

Equating the continuous parts and the discrete parts, we conclude from a standard argument that:

$$\int K_{\Phi}^{cusp}(n)\theta(\pi n) \, dn = \int K_{\Phi'}^{cusp}(t^{n_1}, n_2)\theta(n_1^{-1}n_2) \, dn_1 \, dn_2.$$  

(23)

This is the trace formula we had in mind. It will imply the theorem.

Finally we remark that for an arbitrary extension $E/F$ we would have to consider all unitary groups. If $H_\sigma$ is a set of representatives for the finitely many isomorphism classes ($n$ is odd), then for each $\sigma$ we would have to introduce a function $f_\sigma$.

The material is arranged as follows. In section 2 we study the local orbital integrals $I(\mathfrak{wa}, \Phi')$ and in section 3 the local orbital integrals $J(\mathfrak{wa}, \Phi)$ together with their matching with the integrals $I(\mathfrak{wa}, \Phi')$. The theorem is then quickly proved in section 4. In section 2, it is more convenient to discuss the qualitative asymptotic behavior of our integrals in the context of $GL(n)$. We thus define the Shalika germs for our integrals and prove the existence of the germs in the context
of $GL(n)$. As a matter of fact, one of the aims of this paper is to introduce the
correct notion of Shalika germs for our integrals. However, for the purpose at hand,
we only need very partial results on $GL(2)$ and $GL(3)$. A discussion with S. Rallis
and J. Bernstein was very helpful in formulating the definition of the germs.

2. Shalika germs

We let $F$ be a local field of characteristic 0 and $\psi$ a nontrivial additive character.
We often write $G$ for $G(F)$ and $C(G)$ for the space of smooth functions of compact
support on $G(F)$. We use similar notations for other groups or varieties. We denote
by $\overline{F}$ an algebraic closure of $F$.

We first describe the asymptotic behavior of our integrals in the context of
$G = GL(n)$. If $w, w'$ are in $R(G)$ we write $w \rightarrow w'$ if $A_w \supseteq A_{w'}$. This is equivalent
to $M_w \subseteq M_{w'}$ or $w \in M_{w'}$. We write $w \xrightarrow{\psi} w'$ if $w \rightarrow w'$, $w \neq w'$ and there is no
$w'' \in R(G)$ such that $w \rightarrow w'' \rightarrow w$. We can define a graph with $R(G)$ for a set
of vertices: the graph is oriented and the edges are the pairs $(w, w')$ with $w \xrightarrow{\psi} w'$.
Note that all oriented paths from a given $w$ to a given $w'$ have the same length
which we denote by $d(w, w')$. We write $w \xrightarrow{m} w'$ if $w \rightarrow w'$ and $d(w, w') = m$. For
each $w \in R(G)$ we have $c \rightarrow w \rightarrow w_G$. For $0 \leq i \leq n$ we denote by $\pi_i(g)$ the
representation of $G$ on the $i$th exterior power $V^i$ of the standard vector space $V$ of
dimension $n$. Let $e_j, 1 \leq j \leq n$, be the canonical basis of $V$. Let
$$
\epsilon_i = e_1 \wedge e_2 \wedge \cdots \wedge e_i, \quad \eta_i = e_{n-i+1} \wedge e_{n-i+2} \wedge \cdots \wedge e_n
$$
be the highest vector and the lowest vector respectively in $V^i$. In particular, the
one-dimensional vector space $V^n$ has a basis $\epsilon_n$. If $v \in V^i$ and $v' \in V^{n-i}$ we set
$$
v \wedge v' = \langle v, v' \rangle \epsilon_n.
$$
Then
$$
\Delta_i(g) = \langle \pi_i(g) \epsilon_i, \eta_{n-i} \rangle.
$$
We denote by $\Delta(G)$ the set of these functions. Note that $\Delta_0 = 1$ and $\Delta_n(g) = \det g$.

Lemma 2.1. Suppose $w \in R(G)$ and $\Delta \in \Delta(G)$. Suppose $w \neq w_G$ and $\Delta(w_G w) \neq 0$. Then $\Delta = \Delta_0$ or $\Delta = \Delta_n$. Suppose $\Delta(w) \neq 0$. Then $\Delta(m) \neq 0$ for all $m \in M_w$.

Proof. For the first assertion suppose that $\Delta = \Delta_i$ with $1 \leq i \leq n - 1$ and
$\Delta(w_G w) \neq 0$. We have
$$
\Delta(w_G w) = \pm \langle \pi_i(w) \epsilon_i, \eta_{n-i} \rangle.
$$
If $\Delta(w_G w) \neq 0$ then $\pi_i(w) \epsilon_i = \pm \eta_i$. This implies that $w$ has the form
$$
w = \begin{pmatrix}
0 & 0 & A \\
0 & B & 0 \\
A^{-1} & 0 & 0
\end{pmatrix}
$$
where $A \in GL(i)$ and $B \in GL(n - i)$. The only standard Levi subgroup which
contains $w$ is $G$. Hence $w = w_G$.

For the second assertion, let us write an element $m \in M = M_w$ as a diagonal
matrix of square blocks:
$$
m = \text{diag}(g_1, g_2, \ldots, g_r)
$$
with $g_i \in G_i \simeq GL(r_i)$. Thus
$$
M \simeq G_1 \times G_2 \times \cdots \times G_r.
$$
If $\Delta \in \Delta(G)$ then
\begin{equation}
\Delta(m) = \Delta'_1(g_1)\Delta'_2(g_2)\cdots\Delta'_r(g_r),
\end{equation}
where $\Delta'_i \in \Delta(G_i)$. In addition
\[ w = \text{diag}(w_1, w_2, \ldots, w_r) \]
where $w_i = w_{G_i}$. By the previous assertion, each $\Delta'_i$ is either the determinant function or the constant function equal to 1 on $G_i$. The lemma follows.

Recall that $N \times N$ operates on $G$:
\[ g \mapsto n_1 t n_1 g n_2 \]
if $n = (n_1, n_2)$. The following facts will not be needed but will shed some light on the definition of Shalika germs. The algebra of polynomial functions on $G(F)$ invariant under $N \times N$ is the polynomial algebra generated by $\Delta(G)$ and $\Delta^{-1}_n$. Similarly, if $w$ is relevant, we consider the closure $F_w$ of the orbit $tNwAN$ (for the Zariski topology) and the algebra of polynomial functions on $F_w$ which are invariant under $N \times N$. Let $P_w = M_w U_w$ be the standard parabolic subgroup of $G$ with Levi-factor $M_w$. Then $tNwAN = tNwAU_w$. To construct such a polynomial function on $F_w$ it suffices to construct a polynomial function on $G$ invariant under $tN$ on the left and under $U_w$ on the right and restrict it to $F_w$. The function $\Delta_w$ defined by $\Delta_w(g) = \Delta(gw)$ is an example of such a function. The algebra in question is generated by the functions $\Delta_w$ with $\Delta \in \Delta(G)$ and the function $\Delta^{-1}_n$. These functions separate the closed orbits in $F_w$ (over $\overline{F}$ for the Zariski topology or over $F$ for the ordinary topology).

If $w$ is relevant we denote by $\Omega_w$ the set of $g \in G(F)$ such that $\Delta(w) \neq 0$ implies $\Delta(g) \neq 0$.

**Lemma 2.2.** The set $\Omega_w$ is open and the map
\[ (u, v, m) \mapsto tUMv \]
from $U_w(F) \times U_w(F) \times M_w(F)$ to $\Omega_w$ is an isomorphism of analytic varieties over $F$. If $\omega_M$ is a compact subset of $M(F)$ and $\omega_G$ a compact subset of $G(F)$ then the relations
\[ tUMv \in \omega_G, \quad m \in \omega_M \]
 imply that $u$ and $v$ are in a compact subset of $U_w(F)$. If the orbit of a relevant element $w'$ intersects $\Omega_w$ then $w' \mapsto w$ and the orbit is contained in $\Omega_w$.

**Proof.** By the previous lemma, if $P = MU$ is the parabolic subgroup attached to $w$ then $\Omega_w$ is just the set of $g$ such that $\Delta(g) \neq 0$ if $\Delta$ does not vanish on $M$. We first prove the first assertion of the lemma for a parabolic subgroup $P = MU$ of type $(n_1, n_2)$. Then if $\Delta \in \Delta(G)$ does not vanish on $M$, we have $\Delta = \Delta_{n_1}$ or $\Delta = \det$ or $\Delta = 1$. Suppose that $\Delta_{n_1}(g) \neq 0$. Write
\[ g = \begin{pmatrix} m_1 & v_1 \\ v_2 & m_2 \end{pmatrix} \]
with $m_i \in M(n_i \times n_i, F)$. Since $\det(m_1) = \Delta_{n_2}(g) \neq 0$ we find $m_1 \in GL(n_1, F)$. Thus we can write $v_1 = m_1 U_1$ and $v_2 = m_2 U_2$. With
\[ u_i = \begin{pmatrix} 1 & U_i \\ 0 & 1 \end{pmatrix} \]
we have now $g = t u_1 m u_2, u_i \in U$, and $m \in M$. The first assertion of the lemma follows then from an inductive argument. The second assertion follows from the proof of the first. The third assertion follows from the uniqueness of the Bruhat decomposition.

We also set

$$Z_w = t N w A N, \quad V_w = N \cap M_w.$$ 

Then $Z_w$ is a closed subvariety of $\Omega_w$. Indeed, this follows from the previous lemma and the fact that $Z_w \cap M_w = w A V_w$ is closed in $M_w$.

**Proposition 2.1.** If $F$ is any local field (Archimedean or not) the orbital integral $I(wa, \Phi')$ converges and defines a smooth function on $A_w(F)$. Suppose that $F$ is non-Archimedean and that the conductor of $\psi$ is the ring of integers $O_F$. If $f$ is the characteristic function of $K = GL(n, O_F)$ and $a \in A_w(F) \cap K$ then $I(wa, f) = 1$.

By the previous lemma, the map

$$(u_1, u_2, v, a) \mapsto t u_1 w a v u_2$$

is a diffeomorphism from $U_w \times U_w \times V_w \times A_w$ onto a closed subvariety of $\Omega_w$. On the other hand, if $\omega$ is a compact subset of $A_w(F)$ then the same map is a homeomorphism from $U_w \times U_w \times V_w \times \omega$ onto a closed subset of $GL(n, F)$. The first assertion follows.

We prove the second assertion. We will use two lemmas:

**Lemma 2.3.** Suppose $w \in R(G)$ and $g \in M(n, O_F)$ with $\Delta(g) \in O_F^\times$ for all $\Delta$ with $\Delta(w) \neq 0$. Then we have $g = t u_1 m u_2$ with $u_i \in U_w \cap K$, $m \in M_w \cap K$ and $\Delta(m) \in O_F^\times$ for all $\Delta$ with $\Delta(w) \neq 0$.

*Proof of the lemma.* It suffices to prove our assertion when $P_w = MU$ has type $(n_1, n_2)$. We write, as before,

$$g = \begin{pmatrix} m_1 & v_1 \\ v_2 & m_2 \end{pmatrix}$$

with $m_i \in M(n_i \times n_i, F)$. Since $m_1$ is integral and $\det(m_1) = \Delta_{n_1}(g) \in O_F^\times$, we find $m_1 \in GL(n_1, O_F)$. Thus we can write $v_1 = m_1 U_1$ and $v_2 = t U_2 m_1$ where $U_i$ is integral. With

$$u_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

we have now $g = t u_1 m u_2$. Now $u_i$ is in $U \cap K$ so that $m$ is in $K \cap M$ and $\Delta(m) = \Delta(g) \in O_F^\times$ for all $\Delta$ which do not vanish on $M$.

**Lemma 2.4.** Suppose that $g$ is in $K$ and $g = t u_1 w a v u_2$ with $u_i \in U_w, v \in V_w$ and $a \in A_w(F) \cap K$. Then $u_1$ and $v$ are in $K$ as well.

*Proof of the lemma.* We can write

$$gw = t u_1 m u_2', \quad m = w w a, \quad u_2' = w u_2 w.$$

Since $\Delta(m) = \Delta(a)$ we can apply the previous lemma. It follows that $u_1, u_2'$ and $m$ are in $K$. In turn, this implies that $u_2$ and $v$ are in $K$. The lemma follows.

The second assertion of the proposition follows from the two lemmas.
From now on, we assume $F$ is non-Archimedean.

If $w \to w'$ we denote by $A_w'$ the set of $b \in A_w(b)$ such that $\Delta(b) = \Delta(w'w)$ for all $\Delta \in \Delta(G)$ such that $\Delta(w'w) \neq 0$. This is a single coset of a subgroup of $A_w$. More precisely, let $M' = M_w'$ and $M = M_w$. Let $A_M'$ be the algebraic subgroup of $A_M$ defined by the equations $\Delta(a) = 1$ if $\Delta(w'w) \neq 0$. Then $A_w'$ is a single coset of $A_M'(F)$. We need simple properties of this construction.

Lemma 2.1 implies that if $w \neq w_G$ then $A_w'$ is the set of $b \in A_w$ such that $\det(b) = \det(w_Gw)$. On the other hand, it is clear that $A_w' = \{1\}$ for all $w$.

Suppose that $M = M_w$ with $w \in R(G)$. Let us write an element $m \in M_w$ as a diagonal matrix of square blocks:

$$m = \text{diag}(g_1, g_2, \ldots, g_r)$$

with $g_i \in G_i \simeq \text{GL}(r_i)$ and use the notations of the proof of Lemma 2.1. In particular:

$$w = \text{diag}(w_1, w_2, \ldots, w_r)$$

where $w_i = w_{G_i}$. Similarly, every $a$ in $A_w$ has the form:

$$a = \text{diag}(a_1, a_2, \ldots, a_r)$$

with $a_i \in A_{w_i} \subset G_i$. Thus

$$A_w \simeq \prod A_{w_i}.$$  

If $w' \to w$ then $w' \in M$ and

$$w' = \text{diag}(w'_1, w'_2, \ldots, w'_r)$$

where $w'_i \in R(G_i)$ (and $w'_i \to w_i$ in $G_i$). We have then

$$A_w \simeq \prod A_{w_i}'.$$

If $\Delta(ww') \neq 0$ then, in the notations of (24), $\Delta'(w_iw'_i) \neq 0$ for each $i$. Conversely, if $\Delta'_i \in \Delta(G_i)$ is such that $\Delta'_i(w_iw'_i) \neq 0$ then there is $\Delta \in \Delta(G)$ such that

$$\Delta(m) = \Delta'_1(g_1)\Delta'_2(g_2)\cdots\Delta'_r(g_r),$$

where $\Delta'_j = \det$ on $G_j$ for $j < i$. It follows that

$$A_w \simeq \prod A_{w'_i}.$$  

**Lemma 2.5.** For $w' \to w$, the set $Y_{w'}$ of $c \in A_w$ such that $\Delta(c) = 1$ for $\Delta(ww') \neq 0$ is finite. Also $A_{w'}A_{w''} \subset A_{w'''}$.

**Proof.** Writing $M_w$ as a product of linear groups as before, we get that

$$Y_{w'} = \prod Y_{w_i'}.$$  

If $w_i = w'_i$ then

$$Y_{w_i'} = \{1\}.$$  

Thus it suffices to prove the lemma when $w = w_G$ and $w' \neq w_G$. Then $\Delta(w_Gw') \neq 0$ implies $\Delta = \det$ or $\Delta = 1$ and $Y_{w_G}$ is the set of scalar matrices $c$ with $c^n = 1$. Thus it is finite. The proof of the second assertion is similar. \hfill \qed
We can also describe the group $A^M_{M'}$, where $M' = M_{w'}$ and $M = M_w$. If $M = M'$ then the group is reduced to $\{1\}$. If $M = G$ then

$$A^G_{M'} = \{a \in A_{M'} | \det a = 1\}.$$  

In general, with the previous notations, set $M'_f = M_{w'_f}$ and $M_i = M_{w_i}$. Then

$$A^M_{M'_f} \cong \prod A^M_{M'_i}.$$  

In particular, we remark that if $\Delta$ does not vanish on $M$ then $\Delta$ defines an algebraic character of $M$ and $\Delta(ww') \neq 0$ since $ww'$ is in $M$. Moreover, the group of algebraic characters of $M$ is generated by the restriction to $M$ of those $\Delta$ which do not vanish on $M$. If we let $A^M_{M'}$ be the group of $a \in A_{M'}$ such that $\chi(a) = 1$ for any character $\chi$ of $M$ then $A^M_{M'}$ is clearly contained in $\tilde{A}^M_{M'}$. In fact, it is easy to check from the above description that $A^M_{M'}$ is just the connected component of the identity in $\tilde{A}^M_{M'}$ for the Zariski topology. Moreover, the previous lemma amounts to saying that the group $A^M_{M'}$ is the product, almost direct, of $A^M_{M'_f}$ and $A_M$.  

A system of Shalika germs will be a family of smooth functions $K_w^w$ defined over the sets $A_w^w$ for $w \to w'$ such that, for any function $f \in \mathcal{C}(G(F))$, there exist functions $\omega_w = \omega'_w \in \mathcal{C}(A_w(F))$ with:

$$I(wa, f) = \sum_{\{w'' : w \to w'''\}} \sum_{a = bc, b \in A_w^w, c \in A_{w''}} K_w^{w''}(b) \omega_{w''}(c).$$

We note that for each $w$ the set $A_w^w$ is reduced to the identity. By convention, we take $K_w^w = 1$. For $w \to w'$, the sum is over all decompositions $a = bc, b \in A_w^w, c \in A_{w''}$. It is finite by the previous lemma. In particular it is empty hence 0 if $a \not\in A_w^w A_{w'}$.  

For a given function $f$, the above relations determine the functions $\omega_w$ by a triangular system of linear equations. In particular $\omega_{wG}(a)$ is just the orbital integral $I(w_G a, f)$. When we want to emphasize the dependence of the functions on the system, we will write them as $\omega_w^{K,f}$ or $\omega_w^K$. Given a system of germs $K$ and a family of smooth functions of compact support $\omega_w \in \mathcal{C}(A_w)$, there is $f \in \mathcal{C}(G)$ such that $\omega_w = \omega_w^K f$. This follows from the following observation. Suppose $f \in \mathcal{C}(\Omega_w)$. Then $I(w'a, f) = 0$ unless the orbit of $w'a$ intersects $\Omega_w$, that is, $w' \to w$. In particular, it follows that $\omega_w^{f'} = 0$ unless $w' \to w$. Moreover $\omega_w^{f'}(a) = I(wa, f)$. Now $I(wa, f) = I(wa, f')$ where $f' \in \mathcal{C}(M_w)$, is given by

$$f'(m) = \int_{U_w \times U_w} f(t u_1 m u_2) \theta(u_1 u_2) \, du_1 \, du_2$$

and

$$I(wa, f') = \int_{V_w} f'(w a v) \theta(v) \, dv.$$ 

Since $w A_w V_w$ is closed in $M_w$, it follows that $\omega_w$ is an arbitrary element of $\mathcal{C}(A_w)$. Our assertion follows now from an easy inductive argument.  

Recall that the support of an orbital integral is contained in a set where $|\Delta(a)| \leq C$ for all $\Delta$. We note that in the definition of the germs the relation $b \in A_w^w$ amounts to $\Delta_w(w' c) = \Delta_w(wa)$. Since the functions $\Delta_w$ on $F_w$ separate the relevant orbits of $w A_w$ this remark gives some insight in the nature of the germ expansion. Given $w'$ with $w \to w'$, the contribution of $w'$ to the orbital integral $I(wa, f)$ is a function whose support is contained in a set $C_1 \leq |\Delta(a)| \leq C_2$ for all $\Delta$ with $\Delta(w'w) \neq 0$.  

On the other hand, the behavior at infinity of this contribution, that is, the behavior as \( \Delta(a) \to 0 \) for those \( \Delta \) such that \( \Delta(w'w) = 0 \), is given by the germ \( K_w^{w'} \).

The Shalika germs depend on \( \psi \).

It will be convenient to use the following notation: if \( f \) and \( g \) are functions on \( A_w \) and \( A_{w'} \) respectively then we define a new function \( f * g \) on \( A_w \) by

\[
(25) \quad f * g(a) = \sum_{\{a = bc, b \in A_{w'}, c \in A_w\}} f(b)g(c).
\]

Then the relations defining the germs read

\[
I(w., f) = \sum_{w \to w'} K_w^{w'} * \omega_{w'}.
\]

**Theorem 2.1.** There exists a system of Shalika germs. If \( H \) and \( K \) are two systems of Shalika germs, then there are functions \( t_{w''}^{w'} \in \mathcal{C}(A_w') \) such that \( t_w^{w'} = 1 \) for all \( w \) and

\[
K_w^{w'} = \sum_{w \to w'' \to w'} H_w^{w''} * t_{w''}^{w'}.
\]

**Proof.** The existence follows from the following proposition: \( \square \)

**Proposition 2.2.** Suppose \( 1 \leq m \leq d(e, w_G) \). Then there exist functions \( K_{w}^{w'} \) on \( A_w' \) for \( w \to w' \) and \( d(w', w_G) < m \) with the following properties: suppose \( f \in \mathcal{C}(G(F)) \) is given; for any \( w' \) with \( d(w', w_G) < m \), there exists a function \( \omega_{w'} \in \mathcal{C}(A_{w'}) \); for any \( w' \) with \( w' \to w' \), there exists a function \( f_{w'} \in \mathcal{C}(\Omega_{w'}) \); the following equalities are satisfied:

\[
I(w., f) = \sum_{w \to w', w \to w''} I(w., f_{w''}) + \sum_{d(w', w_G) < m, w \to w'} K_w^{w'} * \omega_{w'}.
\]

For \( m = d(e, w_G) \) this will give the existence of the Shalika germs since \( I(\omega, f_e) = 0 \) unless \( w = e \) in which case \( I(a, f_e) \) is a smooth function of compact support on \( A = A_e \). Note that in the first sum over \( w' \) the orbital integral \( I(\omega, f_{w'}) \) vanishes unless the orbit of \( \omega \) intersects \( \Omega_{w'} \), that is, unless \( w \to w' \).

We first prove the assertion for \( m = 1 \). The above relation then reads

\[
I(w., f) = \sum_{w \to w', w \to w''} I(w., f_{w''}) + K_w^{w'} * \omega_{w'}.
\]

Let \( \Omega \) be the complement of \( Z_{w_G} \). Restriction of a function on \( G \) to \( Z_{w_G} \) gives us an exact sequence

\[
0 \to \mathcal{C}(\Omega) \to \mathcal{C}(G) \to \mathcal{C}(Z_{w_G}) \to 0.
\]

We denote by \( \mathcal{C}(C, \theta) \) the subvector space of \( \mathcal{G}(G) \) spanned by the functions of the form

\[
f(t^n_n1_gn_2) - \overline{\theta(n_1 n_2)}f(g)
\]

and by \( \mathcal{C}(G)_\theta \) the quotient space (coinvariants). An element of the dual vector space may be viewed as a distribution on \( G \), relatively invariant under the character \( \theta \otimes \theta \) of \( N \times N \). We denote by \( \mathcal{C}(G)^\# \) this dual. Examples of such elements are the orbital integrals. We write \( f_1 \simeq f_2 \) if \( f_1 \) and \( f_2 \) have the same image in \( \mathcal{C}(G)_\theta \). In particular, we have then \( I(\omega, f_1) = I(\omega, f_2) \) for all \( \omega \). Conversely, by the theorem of density of Bernstein ([B]), if this condition is satisfied then \( f_1 \simeq f_2 \).
It is a fundamental observation of Casselman that the functor of coinvariants is exact. Thus we have an exact sequence
\[ 0 \to \mathcal{C}(\Omega)_\theta \to \mathcal{C}(G)_\theta \to \mathcal{C}(Z_{w_G})_\theta \to 0. \]

Our first task is to determine the dual space \( \mathcal{C}(Z_{w_G})^{*\theta} \). The map
\[ \xi: (n_1, n_2, a) \mapsto t^1 n_1 w_G an_2 \]
is surjective and submersive from \( N \times N \times A \) to \( Z_{w_G} \). Thus there is a surjective map \( \alpha \mapsto f_\alpha \) from \( \mathcal{C}(N \times N \times A) \) to \( \mathcal{C}(Z_{w_G}) \) such that
\[ \int_{N \times N \times A} \alpha(n, a) T(\xi(n, a)) \, dn \, da = \int_{Z_{w_G}} f_\alpha(z) T(z) \, dz, \]
for \( T \in \mathcal{C}(Z_{w_G}) \). Here \( dz \) is a measure on \( Z_{w_G} \), product of an invariant measure on the orbit of \( wa \) and the measure \( da \). It follows that for any \( T \in \mathcal{C}(Z_{w_G})^{*\theta} \) there is a distribution \( T^* \) on \( A \) such that
\[ \int \alpha(n, a) \theta \otimes \theta(n) \, dn \, da = \int f_\alpha(z) \, dT^*(z). \]
Suppose \( \alpha_1 \in \mathcal{C}(A) \) has support in the complement of \( A_{w_G} \). Choose a smooth map \( a \mapsto n_a \) from the support of \( \alpha_1 \) to \( N^{w_G a} \) with \( \theta \otimes \theta(n_a) \neq 1 \), a function \( \alpha_0 \in \mathcal{C}(N) \) with \( \int \theta \otimes \theta(n) \, \alpha_0(n) \, dn = 1 \). Then the function
\[ \alpha(n, a) = (\alpha_0(n) - \alpha_0(n_a n))\alpha_1(a) \]
belaongs to \( \mathcal{C}(N \times N \times A) \). For such a function \( f_\alpha = 0 \); thus the value of \( T^* \) on the function
\[ a \mapsto (1 - \bar{\theta}(n_a))\alpha_1(a) \]
vaneshes. Thus \( T^* \) is supported on \( A_{w_G} \), that is, for \( f \in \mathcal{C}(Z_{w_G}) \),
\[ T(f) = \int_{A_{w_G}} I(w_G a, f) \, dT^*(a). \]
We conclude that a function \( f \) on \( G \) has a zero image in \( \mathcal{C}(Z_{w_G})_\theta \) if and only if all its orbital integrals \( I(w_G a, f) \) are 0. This also follows from the theorem of density of Bernstein. The above exact sequence asserts there is then a function \( f_0 \in \mathcal{C}(\Omega) \) such that
\[ I(wa, f) = I(wa, f_0) \]
for all \( w \) and all \( a \in A_w \).

We apply this observation as follows: let \( G_1 \) be the set of \( g \in G \) with \( \det g = \det w_G \). We can also define the orbital integrals of a function \( f_0 \) on \( G_1 \). If \( w_G a \) where \( a \in A_{w_G} \) is in \( G_1 \) then the scalar matrix \( a \) verifies \( a^n = 1 \). In particular, the orbital integral \( I(w_G j, f_0) \) is defined as a function on the set of \( j \in F \) which are \( n \)-roots of \( 1 \). We can choose \( f_0 \) in such a way that \( I(w_G, f_0) = 1 \) and \( I(w_G j, f_0) = 0 \) if \( j \neq 1 \). Given \( f \) define \( f_1 \) by
\[ f_1(g) = \sum_{g = h z, z \in A_{w_G}, h \in G_1} f_0(h) I(w_G z, f). \]
Note that the sum is empty, hence 0, if \( \det g \notin F^{n} \det w_G \). This is a smooth function of compact support. Moreover, for \( a \in A_{w_G} \):
\[ f_1(t^1 n_1 w_G a n_2) = \sum_{(j: j^n = 1)} f_0(t^1 n_1 w_G j^{-1} n_2) I(w_G ja, f). \]
Hence

\[ I(w_G a, f_1) = \sum_j I(w_G j^{-1}, f_0) I(w_G j a, f) = I(w_G a, f). \]

Thus there is \( f_2 \in \mathcal{C}(\Omega) \) such that

\[ f \simeq f_2 + f_1. \]

Next we can find an increasing sequence of open subsets \( \Omega_i, 0 \leq i \leq r \), invariant under \( N \times N \), such that

\[ \Omega_0 = \bigcup_{w \to w_G} \Omega_w, \]

\( \Omega_r = \Omega \) and each difference \( Z_i = \Omega_{i+1} - \Omega_i \) has the form \( {}^tNwAN \) for a suitable irrelevant \( w \). This follows from the fact that \( N \) has finitely many orbits in \( B \setminus G \).

We have again exact sequences

\[ 0 \to \mathcal{C}(\Omega_i) \to \mathcal{C}(\Omega_{i+1}) \to \mathcal{C}(Z_i) \to 0. \]

An argument similar to the one used before shows that the last term on the right is zero. It follows that

\[ \mathcal{C}(\Omega_{i+1}) = \mathcal{C}(\Omega_i), \]

Inductively we have then

\[ \mathcal{C} \left( \bigcup_{w : d(w, w_G) = 1} \Omega_w \right) = \mathcal{C}(\Omega). \]

Thus we may assume that \( f_2 \) has support in the union of the sets \( \Omega_w \) with \( d(w, w_G) = 1 \). Using now a partition of unity we obtain that

\[ f \simeq \sum_{\{w : d(w, w_G) = 1\}} f_w + f_1, \]

with \( f_w \in \mathcal{C}(\Omega_w) \). Next, we compute the orbital integrals of \( f \). For \( w' \neq w_G \) we have

\[ I(w'a, f) = \sum_{d(w, w_G) = 1} I(w'a, f_w) + I(w'a, f_1). \]

To compute \( I(w'a, f_1) \) we write in all possible ways \( g = {}^t n_1 w' a n_2 \) in the form \( g = g_1 c \) with \( g_1 \in G_1, c \in A_{w_G} \). We get

\[ {}^t n_1 w' a n_2 = {}^t n_1 w' b n_2 c \]

with \( c \in A_{w_G} \) and \( \det b = \det w_G w' \). This amounts to \( a = bc \) with \( b \in A_{w_G} \). We find then

\[ I(w'a, f_1) = \sum_{a = bc} I(w'b, f_0) I(w_G c, f). \]

We obtain our assertion for \( m = 1 \) with

\[ K_{w'}^{w_G}(b) = I(w'b, f_0) \]

and \( \omega_G(z) = I(w_G z, f) \) for \( z \in A_{w_G} \).

To continue, we assume the assertion of the proposition true for \( m \) and prove it for \( m + 1 \). With the notations of the proposition, we consider an element \( w' \) such
that \( w' \xrightarrow{m} w_G \). We recall that the open set \( \Omega_{w'} \) is isomorphic to \( U_{w'} \times U_{w'} \times M_{w'} \). In particular, there is a function \( h_{w',r} \in \mathcal{C}(M_{w'}) \) such that
\[
I(wa, f_{w'}) = I(wa, h_{w'})
\]
for all \( w \to w' \). Here we extend in an obvious way the notion of orbital integral to the Levi subgroups of \( GL(n) \). The assertion of the proposition for \( m = 1 \) is true for the factors of \( M_{w'} \); hence, in an obvious sense, it is true for the group \( M_{w'} \) as well. Thus there are functions \( K_{w,r}^{w'} \) on \( A_{w,r}^{w'} \) for \( w \to w' \) with the following property: for every \( w'' \xrightarrow{1} w' \) there is a function \( \omega_{w'',w',r} \in \mathcal{C}(\Omega_{w''} \cap M_{w'}) \) such that
\[
I(w', h_{w',r}) = \sum_{w'' \xrightarrow{1} w'} I(w', h_{w'',w',r}) + K_{w,r}^{w'} \ast \omega_{w',w'',w',r}.
\]
Here the function \( \omega_{w',w'',w',r} \in \mathcal{C}(A_{w',r}) \) is given by
\[
\omega_{w',w'',w',r}(a) = I(w'' a, f_{w''}) = I(w'a,h_{w',w'',w',r}).
\]
We can find a function \( w'' f_{w''} \) with support in \( \Omega_{w''} \) such that \( I(w'' a, f_{w''}) = I(w'a,h_{w',w'',w',r}) \). We obtain then the assertion of the proposition for \( m + 1 \) by setting:
\[
f_{w''} = \sum_{w'' \xrightarrow{1} w'} w'' f_{w''}
\]
for each \( w'' \) with \( w'' \xrightarrow{m+1} w_G \). Note that in the above sum \( w'' \xrightarrow{m} w_G \). The proposition and the existence of the germs are established.

We pass to the proof of the uniqueness. Consider first a system \( K_{w}^{w'} \) of Shalika germs and a system \( t_{w}^{w'} \) of functions in \( \mathcal{C}(A_{w}^{w'}) \) (with \( t_{w}^{w} = 1 \)). Then the relations
\[
K_{w}^{w'} = \sum_{w'' \xrightarrow{1} w'} H_{w''}^{w} \ast t_{w''}^{w'}
\]
form a triangular system of relations which can be solved for the functions \( H_{w}^{w'} \). Moreover:
\[
I(w', f) = \sum_{w'' \xrightarrow{1} w'} H_{w''}^{w} \ast t_{w''}^{w'} \ast \omega_{w''}^{K,f}.
\]
It follows that the functions \( H_{w}^{w'} \) are also a system of germs. More precisely,
\[
\omega_{w}^{H,f} = \sum_{w'' \xrightarrow{1} w'} t_{w''}^{w'} \ast \omega_{w''}^{K,f}.
\]
To prove the uniqueness we consider two systems of germs \( H \) and \( K \) as in the theorem. It is clear that there are functions \( t_{w}^{w'} \) on \( A_{w}^{w'} \) with \( t_{w}^{w} = 1 \) for all \( w \) such that
\[
K_{w}^{w'} = \sum_{w'' \xrightarrow{1} w'} H_{w''}^{w} \ast t_{w''}^{w'}
\]
for any pair \( w \to w' \). Indeed this relation reads
\[
K_{w}^{w'} = t_{w}^{w'} + H_{w}^{w'} + \sum_{w'' \xrightarrow{1} w'' \neq w, w'' \xrightarrow{1} w'} H_{w''}^{w} \ast t_{w''}^{w'}
\]
so that the functions $t$ are determined by a triangular system of equations. Similarly we have

$$\omega^H_w f = \sum_{w \to w'} t^w_{w'} * \omega^K_{w'}.$$  

It is clear that the functions $t$ are smooth. What we have to see is that they have compact support. This is clear for $w = w'$ since $t^w_{w} = 1$. Thus we may assume that for $d(w, w') \leq m$ the functions $t^w_{w'}$ have compact support and prove the same assertion for the functions $t^w_{w'}$ with $d(w, w') = m + 1$. Thus we fix a pair $w^{m+1} \to w'$. To that end, let us consider only functions $f \in C(\Omega_w)$. Then as before $I(w., f) = 0$ unless $w \to w'$ and $\omega^K_{w} f = \omega^H_{w} f = 0$ unless $w \to w'$. In particular,

$$\omega^K_{w}, f = \omega^H_{w}, f = I(w', f)$$  

is an arbitrary function $\omega_{w'} \in C(A_{w'})$. Moreover,

$$\omega^H_w = \sum_{w \to w_1 \to w', w_1 \neq w'} t^{w}_w * \omega^K_{w_1} + t^{w'}_{w} * \omega_{w'}.$$

Since the functions $t^w_{w'}$ in the formula have compact support by the induction hypothesis we conclude that the last term has compact support on $A_w$. Its restriction to $A_{w'}$ also has compact support; it can be written as

$$a \mapsto \sum_{c \in Y} t^{w'}_{w}(ac^{-1})\omega_{w'}(c)$$

where $Y = Y_{w'}$ is the finite subset of $A_{w'}$ introduced earlier. We can choose $\omega_{w'}$ in such a way that $\omega_{w'}(1) = 1$ and $\omega_{w'}(c) = 0$ for $c \in Y$, $c \neq 1$. The conclusion follows.

The proof of existence gives a way to compute inductively the germs $K^w_{w'}$ in terms of the germs $K^w_{G}$. Indeed, if $w \neq w_G$ then $M = M_w$ can be written as a product of linear groups $G_i$. For $w$ we get

$$w = \text{diag}(w_1, w_2, \ldots, w_r)$$

with $w_i = w_{G_i}$. If $w' \to w$ then

$$w' = \text{diag}(w'_1, w'_2, \ldots, w'_r)$$

with $w'_i \to w_i$ in $G_i$. Now write $a$ in $A_{w'}$ as

$$a = \text{diag}(a_1, a_2, \ldots, a_r)$$

with $a_i \in A_{w_i}$. We find (for a suitable system of germs)

$$K^w_{w'}(a) = \prod K^w_{w'_i}(a_i).$$  

(26)

For $n = 2$ the relations defining the germs read:

$$I(a, f) = \omega(a) + \sum_{a = bc} K^w_{b}(b)\omega_{w}(c),$$

(27)

$$I(wa, f) = \omega_{w}(a),$$

(28)

where

$$w = w_G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

(29)
The sum is for $b \in A$ with $\det b = -1$ and $c \in A_{wG}$. The set $A_w$ is the set of matrices of the form

$$\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}. $$

The germ $K_{wG}$ is supported on a set of matrices of the above form with $|z| \leq C$. It is defined within the addition of a smooth function of compact support.

**Proposition 2.3.** The germ $K = K_{wG}$ is given, for $|z|$ small enough, by

$$K\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix} = \psi\begin{pmatrix} 2 \\ z \end{pmatrix} \gamma\begin{pmatrix} 2 \\ z \end{pmatrix} \left| \frac{1}{2z} \right|^{1/2}$$

where $\gamma$ is the Weil constant.

We recall the definition of the Weil constant:

$$\int \hat{\Phi}(x) \psi\left(\frac{ax^2}{2}\right) dx = |a|^{-1/2} \gamma(a, \psi) \int \hat{\Phi}(x) \psi\left(-\frac{x^2}{2a}\right) dx,$$

where $\hat{\Phi}$ is the Fourier transform of $\Phi \in C(F)$. In particular, if $O$ is any open neighborhood of 0 in $F$ then

$$\int_O \psi\left(\frac{ax^2}{2}\right) dx = |a|^{-1/2} \gamma(a, \psi)$$

for $|a|$ large enough.

**Proof.** Let $O$ be an ideal in $F$ which will be taken as small as needed. We may take it so small that $\psi = 1$ on $O$, $1 \not\equiv -1 \mod O$, $2^{-1}O$ is contained in the maximal ideal of $O_F$ and the square root function $u \mapsto \sqrt{1+u}$ is defined by the Taylor formula on $O$; moreover, we may assume that $\sqrt{1+u} \in 1 + 2^{-1}O$ for $u \in O$. Let $f$ be the characteristic function of the set $wK_O$ where $K_O$ is the congruence subgroup determined by $O$. We have

$$I(wG, f) = \int_F f\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \psi(x) dx = \text{Meas}(O).$$

Similarly:

$$I(-wG, f) = \int_F f\begin{pmatrix} 0 & -1 \\ -1 & -x \end{pmatrix} \psi(x) dx = 0.$$

On the other hand,

$$I\begin{pmatrix} z & 0 \\ 0 & -z^{-1} \end{pmatrix}, f = \int f\begin{pmatrix} z & zx_1 \\ zx_2 & -z^{-1} + x_1x_2z \end{pmatrix} \psi(x_1 + x_2) dx_1 dx_2.$$

This is 0 unless $z \in O$. We change $x_1$ to $x_1/z$ and set

$$x_2 = \frac{1 + zu}{zx_1}$$

where $u \in O$. The integral simplifies at once to

$$\left|z\right|_F^{-1} \int_{x_1 \equiv 1 \mod 0} \psi\left(\frac{x_1 + x_1^{-1}}{z}\right) dx_1 \int_O du.$$

We can set $x_1 = 1 + v$ with $v \in O$ and then set

$$y = \frac{v}{\sqrt{1 + v}}.$$
We find \( dx_1 = dv = dy \) if \( O \) is small enough. The integral becomes
\[
|z|^{-1}_F \text{Meas}(O) \int_O \psi \left( \frac{2 + y^2}{z} \right) \, dy.
\]
If \( z \) is sufficiently small this becomes
\[
\frac{1}{|2z|^{1/2}_F} \psi \left( \frac{2}{z} \right) \gamma \left( \frac{2}{z}, \psi \right) I(w_G, f).
\]
Our assertion follows.

We pass to the group \( G = GL(3) \). Then the germs relations are
\[
\begin{align*}
I(., f) &= \omega_e + K_e^{w_1} * \omega_{w_1} + K_e^{w_2} * \omega_{w_2} + K_{eG}^{w} * \omega_{wG}, \\
I(w_1., f) &= \omega_{w_1} + K_{w_1}^{w} * \omega_{wG}, \\
I(w_2., f) &= \omega_{w_2} + K_{w_2}^{w} * \omega_{wG}, \\
I(w_G., f) &= \omega_{wG}.
\end{align*}
\]

The germs \( K_{w_1}^{w} \) ad \( K_{w_2}^{w} \) are defined within the addition of a smooth function of compact support. The set \( A_{w_1}^{w_1} \) is the subset of \( A \) defined by
\[
\Delta_2(b) = -1, \quad \Delta_3(b) = -1,
\]
in other words, the set of elements of the form:
\[
b = \begin{pmatrix}
z & 0 & 0 \\
0 & -z^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]
By formula (26) and the uniqueness, we may assume
\[
K_e^{w_1}(b) = K \begin{pmatrix}
z & 0 & 0 \\
0 & -z^{-1} & 0
\end{pmatrix}
\]
where \( K \) is the \( GL(2) \) germ. Similarly, \( A_{w_2}^{w_2} \) is the set of matrices of the form
\[
b = \begin{pmatrix}
1 & 0 & 0 \\
0 & z & 0 \\
0 & 0 & -z^{-1}
\end{pmatrix}.
\]
We may assume
\[
K_e^{w_2}(b) = K \begin{pmatrix}
z & 0 & 0 \\
0 & -z^{-1} & 0
\end{pmatrix}.
\]
The set \( A_{wG}^{w} \) is the set of \( a \in A_{w_1} \) with \( \det(a) = 1 \). The set \( A_{wG}^{w} \) is the set of \( a \in A \) with \( \det(a) = -1 \).

Finally, suppose that the orbital integrals of \( f \) on \( w_G A_{wG} \) vanish; then the above relations simplify to
\[
\begin{align*}
I(., f) &= \omega_e + K_e^{w_1} * \omega_{w_1} + K_e^{w_2} * \omega_{w_2}, \\
I(w_1., f) &= \omega_{w_1}, \quad I(w_2., f) = \omega_{w_2}, \quad I(w_G., f) = 0.
\end{align*}
\]
Moreover, the functions \( \omega_e, \omega_{w_1}, \omega_{w_2} \) are arbitrary smooth functions of compact support.
3. Matching

Now we let $E$ be a quadratic extension of our local field $F$. The group $G(E)$ operates on $S(F) \times S(F)$ by $s \mapsto g^*sg$. Mutatis mutandis, the discussion of the previous section applies to the present situation where $G(F)$ is replaced by $S(F)$ and the group $N(F) \times N(F)$ by the group $N(E)$ acting on $S(F)$. As before we can define the Shalika germs $L_w^\varphi$ for the integrals $J(wa, \Phi)$ and show they exist and are essentially unique.

We compute the Shalika germ for $GL(2)$. It is defined within the addition of a smooth function of compact support. The defining relations read

$$J(a, \Phi) = \omega(a) + \sum_{a = bc} L_w^\varphi(b) \omega_w(c),$$

(36)

$$I(wa, f) = \omega_w(a).$$

(37)

Proposition 3.1. The Shalika germ $L = L_w^\varphi$ is given for $|z|$ small enough by

$$L \left( \begin{array}{cc} z & 0 \\ 0 & -z^{-1} \end{array} \right) = \left| \frac{1}{2z} \right|_{F}^{1/2} \psi \left( \frac{2}{z} \right) \gamma \left( \frac{2\tau}{z}, \psi \right)$$

where $E = F(\sqrt{\tau})$. In particular

$$L \left( \begin{array}{cc} z & 0 \\ 0 & -z^{-1} \end{array} \right) = c(E/F, \psi) \eta_{E/F}(z) K \left( \begin{array}{cc} z & 0 \\ 0 & -z^{-1} \end{array} \right),$$

where

$$c(E/F, \psi) = \gamma(\tau, \psi) \gamma(1, \psi)^{-1} \eta_{E/F}(2).$$

For future reference, we remark that we may (and will) assume that the relation between $L$ and $K$ holds for all $z$.

Proof. As before we consider a small enough ideal $O$ in $E$ and the characteristic function $\Phi$ of $wK_O \cap S(F)$. We have

$$I(w, \Phi) = \text{Meas}(O \cap F)$$

and $I(-w, \Phi) = 0$. Now

$$I \left( \begin{array}{cc} z & 0 \\ 0 & -z^{-1} \end{array} \right), \Phi = \int_{E} \Phi \left( \begin{array}{cc} z & yz \\ 0 & -z^{-1} + y \sqrt{\tau} \end{array} \right) \psi(y + y) dy.$$}

Here $dy$ is the self-dual Haar measure on $E$. This is 0 unless $z \in O \cap F$. We change $y$ to $y/z$. The integral becomes

$$\left| z \right|_{F}^{-2} \int \psi \left( \frac{y + y}{z} \right) dy$$

where $y \in 1 + O, y \in 1 + zO$. We can then set

$$y = \sqrt{1 + x} \left( \sqrt{1 + x^2 \tau} + x \sqrt{\tau} \right)$$

where $u \in O \cap F$ and $x \in O'$, where $O'$ is an open neighborhood of zero in $F$ which does not depend on $z$. Then

$$dy = \left| z \right|_{F} dudx \left| \tau \right|_{F}^{1/2}.$$
Here $du = dx$ is the self-dual Haar measure on $F$. In the formula for $y$, the first square root belongs to $1 + 2^{-1}z(O \cap F)$. It follows that the integrand does not depend on $u$ (if $z$ is small enough). Thus the integral becomes

$$\text{Meas}(O \cap F)|z|^{-1}|\tau|^{1/2} \int_{O'} \psi \left( \frac{2\sqrt{1 + x^2\tau}}{z} \right) dx.$$

We can set

$$2\sqrt{1 + x^2\tau} = 2 + t^2\tau$$

with $dx = dt$ and $t \in O''$, where $O''$ is another neighborhood of zero. In fact, we can take

$$t = \frac{x}{\sqrt{1 + x^2\tau + 1}}.$$

Then the integral takes the form

$$I(w, \Phi)|z|^{-1}|\tau|^{1/2} \int_{O''} \psi \left( \frac{t^2\tau}{z} \right) dt.$$

Then for $|z|$ small enough the integral takes the value

$$I(w, \Phi) \left| \frac{1}{2z} \right|^{1/2} \psi \left( \frac{2}{z} \right) \gamma \left( \frac{2\tau}{z}, \psi \right)$$

and the first assertion follows.

For the second assertion, we recall the formulas:

$$\gamma(ab, \psi) = \gamma(a, \psi) \gamma(b, \psi) \gamma(1, \psi)^{-1}(a, b), \quad \gamma(ab^2, \psi) = \gamma(a, \psi)$$

and the fact that $(a, \tau) = \eta_{E/F}(a)$. The assertion follows.

For the group $GL(3)$ we simply record the minimum information that we need: if $J(w_1a, \Phi) = 0$ then

$$J(a, \Phi) = \omega_e(a) + \sum_{a = bc} L^w_e(b)\gamma_{w_1}(c) + \sum_{a = bc} L^w_e(b)\omega_{w_2}(c),$$

$$J(w_1a, \Phi) = \omega_{w_1}(a), \quad J(w_2a, \Phi) = \omega_{w_2}(a), \quad J(w_Ga, \Phi) = 0.$$

Moreover the functions $\omega_e, \omega_{w_1}, \omega_{w_2}$ are arbitrary smooth functions of compact support. The germs $L^w_e$ can be computed as before in terms of $L$. Taking into account the previous proposition we see that the first relation can be written as

$$J(a, \Phi) = \omega_e(a) + c(E/F, \psi) \sum \eta(-b_2)K^w_e(b)\omega_{w_1}(c) + c(E/F, \psi) \sum \eta(b_2)K^w_e(b)\omega_{w_2}(c),$$

or, multiplying by $\eta(a_2)$ and taking into account the relation $a_2 = b_2c_2$:

$$\eta(a_2)J(a, \Phi) = \eta(a_2)\omega_e(a) + c(E/F, \psi) \sum \eta(-c_2)K^w_e(b)\omega_{w_1}(c) + c(E/F, \psi) \sum \eta(c_2)K^w_e(b)\omega_{w_2}(c).$$

Since the functions $\omega_e$ are arbitrary the first assertion of the following proposition follows at once:
Proposition 3.2. If \( \Phi' \in \mathcal{C}(G(F)) \) has vanishing orbital integrals on \( w_G A_{w_0} \) there is \( \Phi \in \mathcal{C}(S(F)) \) with vanishing orbital integrals on \( w_G A_{w_0} \) such that the matching conditions (19) to (21) are satisfied. Moreover, if \( \Phi' \) is supported on \( G(F)^+ \) then we can take \( \Phi \) supported on \( S(F)^+ \).

For the second assertion, one simply multiplies \( \Phi \) by the characteristic function of \( S(F)^+ \).

Our last task will be to show that, essentially, we will not lose information by restricting ourselves to the kind of function considered in the Introduction. To that end we consider a unitary generic representation \( \pi \) of \( G = GL(3, F) \) (or \( G = GL(n, F) \) with \( n \) odd). As before let \( G^+ \) be the group of \( g \in G \) such that \( \det g \in F^+ \). Set also \( A^+ = A \cap G^+ \) and \( Z^+ = Z \cap G^+ \). We let \( \lambda \) be a nonzero linear form on the space of smooth vectors of \( \pi \) such that \( \lambda(\pi(n)v) = \theta(n)\lambda(v) \) for all \( n \in N(F) \) and all vectors \( v \). We define a distribution \( \Theta \) on \( G^+ \) by
\[
\Theta(f) = \sum \lambda(\pi(f)v_i)\overline{\lambda(v_i)},
\]
if \( f \) is supported on \( G^+ \). Here \( v_i \) is any orthonormal basis of \( \pi \) (contained in the space of smooth vectors). Clearly \( \Theta \) transforms on the left and on the right under the character \( \theta \) of \( N \).

Lemma 3.1. There is at least one function \( f \) which is supported on \( G^+ \) and vanishes on \( A^+ N \) such that \( \Theta(f) \neq 0 \).

Proof. Since \( n \) is odd, we have \( G(F) = G^+ Z \) and the restriction of \( \pi \) to \( G^+ \) is irreducible. Suppose there is no function \( f \) with the required property. Then the support of \( \Theta \) is contained in the group \( NA^+ \). Since \( N \) is normal in \( NA^+ \) and \( Z^+ \) is the stabilizer of the character \( \theta \) of \( N \) the support of \( \Theta \) is in fact contained in \( NZ^+ \). Since \( \Theta \) transforms under a character \( \omega \) of \( Z^+ \), then in fact
\[
\Theta(f) = c \int_{NZ^+} f(nz)\theta(n)\overline{\omega(z)} \, dn \, dz,
\]
for a suitable constant \( c \). Now \( \Theta \) and the integral on the right, call it \( M(f) \), are distributions of positive type. Thus \( c \geq 0 \). Suppose \( c > 0 \). Since \( \pi \) restricted to \( G^+ \) is irreducible, the unitary representation \( \pi \) (restricted to \( G^+ \)) can be reconstructed from the distribution \( \Theta \) by considering the positive semidefinite form
\[
(f_1, f_2) \mapsto \Theta(f_1 \ast f_2),
\]
where we have set \( f^*(g) = \overline{f}(g^{-1}) \). The corresponding Hilbert space is the space on which \( \pi \) operates, the action of \( G^+ \) corresponding to right translations on \( f_1 \). The same construction applied to \( M \) produces the unitary representation of \( G^+ \) induced by the character \( \theta \) of \( NZ^+ \). If \( c > 0 \) then \( \pi \) must be equivalent to this induced representation which is absurd. Thus \( c = 0 \); that is \( \Theta = 0 \). However, the irreducibility of \( \pi \) under \( G^+ \) implies there is a \( f \) supported on \( G^+ \) such that \( \Theta(f) \neq 0 \). Thus we get a contradiction. \( \square \)

4. Proof of Theorem 1

We go back to the global situation and the notations of the introduction. If \( \phi \) is a cusp form on \( GL(3, F_\lambda) \) let us set
\[
\lambda(\phi) = \int \phi(n)\overline{\Theta}(n) \, dn.
\]
Then
\begin{equation}
\int K_{\Phi}^{\text{cusp}}(n_1, n_2) \theta(n_1^{-1}n_2) \, dn_1 \, dn_2 = \sum_{\pi} \sum_{i} \lambda(\pi(\Phi^i)\phi_i) \overline{\lambda}(\phi_i),
\end{equation}
where we have set $\Phi^i(g) = \Phi'(-w_G g)$. The outer sum is over all unitary cuspidal representations $\pi$ whose central character is equal to $\chi$ on $Z^+ (F_A)$ and which have a nonzero vector invariant under $K^{S\pi} = \prod_{v \notin S} K_v^\prime$. For each such $\pi$, the inner sum is over an orthonormal basis of the space of vectors invariant under $K^{S\pi}$. Since $\Phi'$ and $\Phi''$ have support in $G^+(F_A)$, the representations $\pi$ and $\pi \otimes \eta$ give the same contribution. For $v$ in $S$ let $\lambda_v$ be a linear form on the space of $\pi_v$ transforming under $\theta_v$. Then the above expression can be rewritten as:
\begin{equation}
\sum_{\pi} \hat{\pi}^{S}(\Phi'^S) c(\pi, S) \prod_{v \in S} \Theta_{\pi_v}(\Phi''_v),
\end{equation}
where $\hat{\pi}^{S}(\Phi'^S)$ is the Hecke eigenvalue associated to the function $\Phi'^S$ (we set $\Phi' = (\prod_{v \notin S} \Phi'_v)\Phi'^S$). The constant $c(\pi, S)$ depends only on $\pi$ and the choice of the $\lambda_v$. Finally, for $v \in S$:
\begin{equation}
\Theta_{\pi_v}(\Phi''_v) = \sum \lambda_v(\pi_v(\Phi''_v)W_i)\overline{\lambda_v}(W_i);
\end{equation}
the sum is over an orthonormal basis of the space of vectors invariant under $K_v^\prime$. Again, each one of these objects gives the same contribution for $\pi$ and $\pi \otimes \eta$. If $v$ is in $S$, there is $\Phi'_v$ such that $\Theta_{\pi_v}(\Phi'_v) \neq 0$. Indeed, this is clear if $v$ splits and is finite because there is then no constraint on the function $\Phi'_v$. If $v$ is Archimedean, the only constraint is that $\Phi'_v$ be a quadruple of $K_v^\prime$ finite functions and our assertion is still true. If $v$ is inert, then it is a finite place and we demand that the orbital integrals of $\Phi'_v$ vanish on $w_G A_w(F_v)$. This is certainly the case if $\Phi'_v$ is supported on $G_v^+$ and vanishes on $B(F_v)$. But by Lemma 3.1 we can choose such a function in such a way that $\Theta_{\pi_v}(\Phi'_v) \neq 0$.

Similarly, if $\phi$ is a cuspidal automorphic representation of $GL(3, E_\mathcal{A})$, we set
\begin{equation}
\mu(\phi) = \int \phi(h) \, dh, \quad \Lambda(\phi) = \int_{N(E) \backslash N(E_\mathcal{A})} \phi(n) \overline{\theta}(n) \, dn.
\end{equation}
Then
\begin{equation}
\int K_{\Phi}^{\text{cusp}}(n) \theta(n) \, dn = \sum_{\Pi} \sum_{i} \mu(\Pi(f)\phi_i) \overline{\Lambda}(\phi_i).
\end{equation}
Let $T$ be the set of places of $E$ above the places of $S$. The outer sum is over all representations $\Pi$ having a nonzero vector fixed under $K^T = \prod_{v \notin T} K_v$ and central character $z \mapsto \chi(z \pi)$; for each such $\Pi$ the inner sum is over an orthonormal basis of the space of vectors invariant under $K^T$. We can also factor out the Hecke eigenvalue to obtain
\begin{equation}
\sum_{\Pi} \hat{\Pi}^{T}(fT) \sum_{i} \mu(\Pi(fT)\phi_i) \overline{\Lambda}(\phi_i).
\end{equation}
Suppose that $\Pi$ is the base change of a representation $\pi$. Then
\begin{equation}
\hat{\pi}^{S}(\Phi'^S) = \hat{\Pi}^{T}(fT)
\end{equation}
and $\Pi$ is then the base change of $\pi$ and $\pi \otimes \eta$ and of no other representation. It follows from the identity of the spectral contributions (see (23)) that

$$\sum_{\nu} \mu(\Pi(f_T)\phi_\nu)\Lambda(\phi_\nu) = 2c(\pi, s) \prod_{v \in S} \Theta_{\pi_v}(\Phi'_v).$$

Choosing the $\Phi'_v$, $v \in S$, in such a way that the right-hand side is $\neq 0$ we see that $\mu \neq 0$ on the space of $\Pi$. This concludes the proof of the theorem. 

\[\square\]

References


