Factorization of Period Integrals

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TO THE MEMORY OF YASUKO JACQUET

We show that for certain quadratic extensions $E/F$ of number fields the period integral of a cusp form of $GL(3, E)$ over the unitary group $H_0$ in three variables is a product of local linear forms. © 2001 Academic Press

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1. GLOBAL RESULTS

Let $E/F$ be a quadratic extension of number fields. We will denote by $\sigma$ the non-trivial element of the Galois group of $E/F$ and will often write $\sigma(z) = z$. We will denote by $U_1$ the unitary group in 1 variable. We assume that every Archimedean place of $F$ splits in $E$. We let $H_0$ be the unitary group for the $3 \times 3$ identity matrix. Recall that a cuspidal automorphic representation $\pi$ of $GL(3, E_A)$ is said to be distinguished by $H_0$ if the linear form:

$$\mathcal{P}(\phi) := \int_{H_0(E_A) \backslash H_0(E)} \phi(h) \, dh,$$

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is non identically zero on the space \( \mathcal{V}(II) \) of smooth vectors of \( II \). Our main result is that if this linear form is non-zero, then it can be written as a tensor product of local linear forms. This is a rather startling result, since there is no local property of uniqueness which guarantees in advance the existence of such a decomposition.

In more detail, if \( II \) is distinguished, then the representation \( II'' \) defined by \( II''(g) = II(g^s) \) is equivalent to \( II \). Moreover, the following condition is satisfied for every place \( v_0 \) of \( F \) inert in \( E \); let \( e \) be the corresponding place of \( E \) and \( \mathcal{V}_{v_0} = \mathcal{V}(\Pi_{v_0}, H_{0,v_0}) \) be the space of linear forms on the space \( \mathcal{V}(\Pi_{v_0}) \) of smooth vectors of \( \Pi_{v_0} \) which are invariant under \( H_{0,v_0} \). Then \( \mathcal{V}_{v_0} \neq 0 \). If \( v_0 \) is a place of \( F \) which split into \( v_1 \) and \( v_2 \) in \( E \) let similarly \( \mathcal{V}_{v_0} = \mathcal{V}(\Pi_{v_0} \otimes \Pi_{v_0}, H_{0,v_0}) \) be the space of linear forms on the space \( \mathcal{V}(\Pi_{v_0} \otimes \Pi_{v_0}) \) of smooth vectors for the tensor product \( \Pi_{v_0} \otimes \Pi_{v_0} \) which are invariant under \( H_{0,v_0} \). Now \( H_{0,v_0} \) is isomorphic to the group \( \{0, \mathbb{I}\} \) and the dimension of \( \mathcal{V}_{v_0} \) is actually one.

Let \( S_0 \) be a finite set of places of \( F \) containing all the places at infinity, the even places, and the places which ramify in \( E \). Let \( S_i \) be the set of places in \( S_0 \) which are inert in \( E \) and let \( S_r \) be the set of split places. Let \( S \) be the set of places of \( E \) above a place of \( S_0 \). Let \( II \) be a distinguished representation. Suppose that \( II \) is unramified outside \( S \). We let \( \mathcal{V}^{S}(II) \) or simply \( \mathcal{V}^{S} \) be the subspace of vectors in \( \mathcal{V}(II) \) which are invariant under \( K^S := \prod_{v \in S} K_v \), \( K_v = GL(3, \mathcal{O}_{v_0}) \). We consider the elements of \( \mathcal{V}^{S}(II) \) which are pure tensors. They can be described in terms of the Whittaker models as follows. Let \( \psi \) be a non-trivial character of \( F^A \). Set \( \psi(z) = \psi(z + \overline{z}) \).

Denote by \( N \) the group of upper triangular matrices with unit diagonal, and by \( \theta \) the character of \( N(\mathcal{O}_{E,v_0}) \) defined by:

\[
\theta(n) = \psi(n_{1,2} + n_{2,3}).
\]

Similarly, define a character \( n \mapsto \theta(n\overline{n}) \) on \( N(E_{\mathbb{A}}) \) by:

\[
\theta(n\overline{n}) = \psi_E(n_{1,2} + n_{2,3}).
\]

For \( \phi \in \mathcal{V}(II) \), set

\[
\mathcal{W}(\phi) = \int_{T(\mathcal{O}_{E}) \backslash N(E_{\mathbb{A}})} \phi(n) \theta^{-1}(n\overline{n}) \, dn, \quad W(g) = \mathcal{W}(II(g) \phi).
\]

Then

\[
\phi(g) = \sum_{\gamma \in GL(2, E) \backslash GL(2, \mathbb{A})} W \left( \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right),
\]
where $N(2)$ denotes the group of upper triangular matrices with unit diagonal in $GL(2)$. If $\phi$ is a pure tensor in $\tau^S$, then the corresponding function $W$ has the form

$$W(g) = \prod W_v(g_v),$$

where $W_v$ is in the Whittaker model $\mathcal{W}(\Pi_v)$ of $\Pi_v$ for every place $v$, and, for $v \notin S$, $W_v = W_{K_v}$ is the unique element of $\mathcal{W}(\Pi_v)$ which is invariant under $K_v$ and equal to 1 on $K_v$. Whenever convenient, we identify the space $\tau^S$ with $\mathcal{W}(\Pi_v)$. Then our precise result is the following theorem:

**Theorem 1.** There exist a constant $c \neq 0$ and, for each $v_0 \in S_0$, a non-zero element $\mathcal{P}_{v_0} \in \mathcal{H}_{v_0}$ such that, for any pure tensor $\phi$ in $\tau^S$,

$$\mathcal{P}(\phi) = c \prod_{v \in S_0} \mathcal{P}_{v_0}(W_v) \prod_{v \in S_1} \mathcal{P}_v(W_{v_1} \otimes W_{v_2}).$$

We remark that the proof of the theorem will provide us with a specific choice of the local linear forms and also a specific value for the constant $c$ in terms of $L$-functions.

Since the spaces $\mathcal{H}_v$ with $v_0 \in S_0$ are not one-dimensional in general, the existence of such a decomposition is not formal. Moreover, the theorem implies the non-trivial result that the linear form $\mathcal{P}$ is non-zero on $\tau^S$, for any $S$ satisfying the above conditions. A priori, one can only say that if it is non-zero, then it is non-zero on a space $\tau^S$, with a large enough $S$.

The paper is arranged as follows. We review the relative trace formula of [JY] in Section 3. We prove the main theorem in Section 4. In Section 5 and 7 we prove local results. Let $E/F$ be a local quadratic extension of non-Archimedean fields. If $\Pi$ is a supercuspidal representation of $GL(3, E)$ we prove in Section 5 that the dimension of $\mathcal{H}(\Pi, H_0(F))$ is at most one. In Section 6, we review the local theory for $GL(2)$. In Section 7 we show, that, at least for certain irreducible representations $\Pi$ of $GL(3, E)$, the dimension of $\mathcal{H}(\Pi, H_0(F))$ is equal to the number of irreducible representations of $GL(3, F)$ (with a given central character) which base change to $\Pi$. Conjecturally, this should always be the case. Finally, in Section 8, we state general conjectures.

We note that a recent work of E. Lapid and J. Rogawski treat a related question ([LR]). Roughly speaking, they investigate the notion of distinguished representation and period integrals for non-cuspidal automorphic representations. Finally, we refer to [yF2] for the discussion of representations distinguished by $GL(n, F)$.
We now explain our notations. In general $G$ denotes the group $GL(3)$ regarded as an algebraic group. The context indicates whether we regard $G$ as an algebraic group over $F$ or over $E$. For instance, if $v_0$ is a place of $F$ then $G_{v_0}$ denotes the group $GL(3, F_{v_0})$. Likewise if $v$ is a place of $E$ then $G_v$ denotes the group $GL(3, E_v)$. Moreover $K_{v_0}$ and $K_v$ denote the corresponding standard maximal compact subgroups. Throughout the paper $S_0$ and $S$ are finite sets of places of $F$ and $E$ respectively satisfying the previous conditions. Then $G_{S_0}$ is the product of the groups $G_{v_0}$ with $v_0 \not\in S_0$ and $G_S$ the (restricted) product of the groups $G_v$ with $v \not\in S_0$. The notations $F_{S_0}$, $F_S$, $K_{S_0}$, $K_S$, $G_{S_0}$, $G_S$, $K_{S_0}$, $K_S$, $K_S$ have a similar meaning.

2. $L^2$-NORM OF A PURE TENSOR

We keep to the notations of the introduction. We recall how to compute the $L^2$-norm of a pure tensor in a cuspidal representation $\pi$ of $GL(3, F_A)$. We assume that $\pi$ is unramified outside $S_0$. An invariant scalar product on each space $W_{v_0}((6 v_0))$ is given by:

$$(W_1, W_2) = \int_{N(2, F_{v_0}) \times GL(2, F_{v_0})} W_1 \overline{W_2} \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} dg.$$

This is a result of Bernstein ([B]) in the non-Archimedean case and of Baruch in the Archimedean case ([Ba]). More precisely, Baruch shows that the restriction of the unitary representation $\Pi_{v_0}$ to the group $P_{v_0}$ of triangular matrices with last row $(0, 0, 1)$ is irreducible (as a unitary representation). On the other hand in [JS] it is shown that the irreducible representation $\tau$ of $P_{v_0}$ induced by the character $\theta_{v_0}$ of $N_{v_0}$ occurs in this restriction. The conclusion follows. In fact this assertion is also a consequence of the global theory as we are going to see (once one knows that the above Hermitian form is defined by a convergent integral).

This being so, there is a constant $c(F, S_0)$, which depends on $F$, $S_0$, and the choice of the Haar measures, but not on $\pi$, such that:

$$||\phi||^2 = c(F, S_0) L^{S_0}(1, \pi, Ad) \prod_{v_0 \in S_0} ||W_{v_0}||^2. \tag{1}$$

Here $L^{S_0}(s, \pi, Ad)$ denotes the partial adjoint $L$-function attached to $\pi$.

Indeed, let $\Phi = \prod \Phi_{v_0}$ be a Schwartz–Bruhat function in three variables where $\Phi_{v_0}$ is the characteristic function of $O_{v_0}$ for $v_0 \not\in S_0$. Consider the Epstein–Eisenstein series:

$$E(g, \Phi, s) = \int_{F^3 \times F^3} \sum_{\xi \in F^3 - \{0\}} \Phi(t \xi g) |t|^{-3s} d^* t |\det g|^s.$$
Then, for $\Re s \gg 0$, if $\phi_1$, $\phi_2$ are pure tensors,

$$\int \phi_1 \phi(g) E(g, \Phi, s) \, dg$$

$$= L^S(s, 1) L_S^s(s, \pi, \mathrm{Ad}) \prod_{v_0 \in S_0} \int W_{1, v_0} W_{2, v_0}(g) \Phi_{v_0}[0, 0, 1, g] |\det g|^s \, dg.$$ \(\|\)

Taking the residue at $s = 1$ we obtain

$$\int \phi_1 \phi(g) \, dg \prod_{v_0 \in S_0} \int_{F_{v_0}^S} \Phi_{v_0}(x) \, dx$$

$$= \text{Res}_{s=1} L^S(s, 1) L_S^s(1, \pi, \mathrm{Ad})$$

$$\times \prod_{v_0 \in S_0} \int W_{1, v_0} W_{2, v_0} \left[ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right] \Phi_{v_0}[tk] |\det t|^3 \, d^S t \, dk \, dg.$$ \(\|\)

In the last integral $k$ is in $K_S$, $g \in GL(2, F_{v_0}^S)$, $t \in F_{S_0}^S$. Now for $v_0 \in S_0$ the integrals

$$\int_{F_{v_0}^S} \Phi_{v_0}(x) \, dx, \quad \int \Phi_{v_0}[tk] |\det t|^3 \, d^S t \, dk$$

are equal (up to a scalar factor). Moreover, the left hand side of the previous formula is an invariant Hermitian form on the space $\mathcal{H}(\Pi)$. If follows that for $v_0 \in S_0$ there is an Hermitian form $\beta_{v_0}$ on $\mathcal{H}(\Pi_{v_0})$ such that

$$\int W_{1} W_{2} \left[ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right] \Phi_{v_0}[tk] |\det t|^3 \, d^S t \, dk \, dg$$

$$= \beta_{v_0}(W_1, W_2) \int \Phi_{v_0}[tk] |\det t|^3 \, d^S t \, dk.$$ \(\|\)

It follows in turn that for every smooth function $f_{v_0}$ on $K_{v_0}$ invariant on the left under $P_{v_0} Z_{v_0} \cap K_{v_0}$, $Z$ denoting the center,

$$\int W_{1} W_{2} \left[ \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} k \right] f_{v_0}(k) \, dk \, dg = \beta_{v_0}(W_1, W_2) \int f_{v_0}(k) \, dk.$$ \(\|\)

The same relation is then true for every smooth function on $K_{v_0}$. In turn this implies that the Hermitian form

$$(W_1, W_2)$$
is invariant under $K_{v_0}$ and thus under $G_{v_0}$ as claimed. Then
\[
\int W_1 \Phi_2 \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix} \Phi_{v_0} \frac{1}{|\det t|^3} d^\times t \, dk \, dg
= (W_1, W_2) \Phi_{v_0} \frac{1}{|\det t|^3} d^\times t \, dk,
\]
and relation (1) follows.

Thus we can construct an orthonormal basis of $\mathcal{V}^{-S_0}$ as follows: for each $v_0 \in S_0$, we choose an orthonormal basis $(W_{v_0})$, $x_{v_0} \in A_{v_0}$, of $\mathcal{W}(H_{v_0}, \psi_{v_0})$. We then set $A = \prod_{v_0 \in S_0} A_{v_0}$. For each $x \in A$ we define
\[
W_x(g) = \prod_{v_0 \in S_0} W_{v_0}(g_{v_0}) \prod_{v_0 \not\in S_0} W^{K_{v_0}}(g_{v_0})
\]
and then we set
\[
\phi_x(g) = \frac{1}{\sqrt{c(F, S_0) L^\infty(1, \pi, \text{Ad})}} \sum_{\gamma \in GL(2, F)} W_x \begin{bmatrix} \gamma & 0 \\ 0 & 1 \end{bmatrix} \gamma.
\]
Then $(\phi_x)$ is indeed an orthonormal basis of $\mathcal{V}^{-S_0}$.

We introduce the matrix
\[
\Phi = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\]
and a Bessel distribution on $GL(3, F_{v_0})$. We let $\mathcal{W}^{-}$ be the linear form on $\mathcal{V}^-(\pi)$ defined by:
\[
\mathcal{W}^{-}(\phi') = \int_{N(F) \backslash N(F_{v_0})} \phi'(n) \theta^{-1}(n) \, dn.
\]
We define the global Bessel distribution attached to $\pi$ as follows. If $f'$ is a smooth function of compact support on $GL(3, F_{v_0})$ we set
\[
\mathcal{B}_\pi(f') := \sum_x \mathcal{W}^{-}(\pi(f') \phi_x) \mathcal{W}^{-}(\pi(w) \phi_x).
\]

**Remark.** To make the above sum finite, we have to assume that $f = \prod_{v_0 \in S_0} f_{v_0}$ where $f_{v_0}$ is $K_{v_0}$-finite if $v_0$ is Archimedean. However with a little more effort, one can show that there is a distribution whose value on a function of this type is given by the above expression.
On the other hand, for every place $v_0 \in S_0$ we denote by \( \mathcal{W}_{v_0} \) the local Whittaker linear form (evaluation at \( e \)) and introduce the local Bessel distribution \( \mathcal{B}_{v_0} \) or simply \( \mathcal{B}_{v_0}^* \): 

\[
\mathcal{B}_{v_0}(f_{v_0}) := \sum_{\pi_{v_0}} \mathcal{W}_{v_0}(\pi_{v_0}(f^{*})) \mathcal{W}_{v_0}(w) W_{v_0}^*.
\]  

(3)

Then if $f' = \prod_{v_0 \in S_0} f_{v_0}'$ we have 

\[
\mathcal{B}_{v_0}(f') = c(\pi) \prod_{v_0 \in S_0} \mathcal{B}_{v_0}(f_{v_0}'),
\]

where we have set 

\[
c(\pi) := \frac{1}{c(F, S_0) L^{S_0}(1, \pi, \text{Ad})}.
\]  

(4)

Similar results are true for a cuspidal automorphic representation $\Pi$ of $GL(3, E_\chi)$ and a pure tensor $\phi \in F^S$: 

\[
\|\phi\|^2 = c(E, S) L^{S_0}(1, \pi, \text{Ad}) \prod_{v_0 \in S} \|W_v\|^2
\]  

(5)

with 

\[
c(\Pi) := \frac{1}{c(E, S) L^{S_0}(1, \Pi, \text{Ad})}.
\]  

(6)

3. MATCHING OF ORBITAL INTEGRALS

The main theorem will a consequence of the relative trace formula of \([JY]\). We recall the geometric side of the trace formula in question. We fix an idele class character $\omega$ of $F$.

Let $F^+$ be the set of elements of $F^\times$ which are norm of an element of $E^\times$ and, for each inert place $v_0$ of $F$, let $F^+_v$ be the set of $x \in F^+_v$ which are a norm of an element of $E_v^\times$. We set $F^+_S = \prod_{v_0 \in S} F^+_v \prod_{v_0 \in S} F^+_{v_0}$. Similarly, let $F^+_\mathcal{A}$ be the set of ideles $x$ such that $x_{v_0} \in F^+_v$ for every inert place $v_0$. Then $F^+ = F^+_S \cap F^+_\mathcal{A}$.

Let $\mathfrak{F}$ be the variety of Hermitian matrices in $GL(3, E)$ and $\mathfrak{F}^+(F)$ the set of elements of $\mathfrak{F}(F)$ whose determinant is in $F^+$. For each inert place $v_0$, let $\mathfrak{E}_{v_0}$ be the set of $s \in \mathfrak{E}_{v_0}$ whose determinant is in $F^+_{v_0}$. Note that $GL(3, E_v) \cap \mathfrak{E}_{v_0}$ is contained in $\mathfrak{E}_{v_0}$ if $v_0$ is odd, inert, and unramified. Let $\mathfrak{E}^+_S$ be the set of $s \in \mathfrak{E}_S$ such that $s_{v_0} \in F^+_S$. Let also $\mathfrak{E}^+_\mathcal{A}$ be the set of elements of $\mathfrak{E}(F^+_\mathcal{A})$ whose determinant is in $F^+_\mathcal{A}$. We define a distribution
$J(\bullet)$ on $\mathbb{Z}_A^+$ as follows. If $\Phi$ is a smooth function of compact support on $\mathbb{Z}_A^+$ we set:

$$J(\Phi) := \int_{N(EA)/(N(E)F^+)} \int_{F_A^+(F^+)} \left( \sum_{z \in \mathbb{Z}_A^+(F)} \Phi(nz/n) \right) \omega(n) \, d\theta(n) \, dn. \quad (7)$$

On the other hand, let $G^+(F)$ be the group of $g \in GL(3, F)$ such that $\det g \in F_A^+$. Define similarly, for any inert place $v_0$, the group $G_{v_0}^+$ of $g_{v_0} \in GL(3, F_{v_0})$ with $\det g_{v_0} \in F_{v_0}^+$. Finally, let $G^+(F_A)$ be the set of $g \in GL(3, F_A)$ such that $\det g$ is in $F_A^+$. We define a distribution $J'(\bullet)$ on $G^+(F_A)$:

$$J'(f') := \int_{N(EA)/(N(E)F^+)} \int_{F_A^+(F^+)} \left( \sum_{z \in \mathbb{Z}_A^+(F)} f'(n_1z/n_2) \right) \times \omega_1(n_1) \, d\theta(n_1) \, dn_1 \, dn_2. \quad (8)$$

We have a notion of matching orbital integrals; if $\Phi$ and $f'$ have matching orbital integrals, then:

$$J(\Phi) = J'(f'). \quad (9)$$

In a precise way, we assume that $\Phi$ and $f'$ are products of local functions which themselves have matching orbital integrals. This means the following. If $v_0$ is an inert place, we say that $\Phi_{v_0}$ and $f'_{v_0}$ have matching orbital integrals, if, for any diagonal matrix $a$ whose determinant is a norm:

$$\int_{N(E)} \Phi_{v_0}(na'n) \theta_{v_0}(n) \, dn = \gamma(a, \psi_{v_0}) \int_{N(E)} f'_{v_0}(n_1a'n_2) \theta_{v_0}(n_1n_2) \, dn_1 \, dn_2. \quad (10)$$

Here the transfer factor $\gamma(a, \psi_{v_0})$ is defined by:

$$\gamma(\text{diag}(a_1, a_2, a_3), \psi) = \omega_{E/F}(a_2).$$

One can show that, this relation implies, in turn, that there are similar relations between the other relevant orbital integrals [JY4]. In an earlier paper [JY4], we have shown that given $\Phi_{v_0}$ there is a function $f'_{v_0}$ satisfying the above conditions and conversely. For instance, if $v_0$ is odd, $E_v/F_{v_0}$ is unramified and the character $\psi_{v_0}$ has for conductor the ring of integers, then, if $\Phi_{v_0}$ is the characteristic function of $K_{v_0}$ in $\mathbb{Z}_{v_0}$ we may take for $f'_{v_0}$ the characteristic function of $K_{v_0}$. 
If \( f_v \) is a smooth function of compact support on \( GL(3, E_v) \) and \( \Phi_{y_v} \) is the function on \( \mathbb{Z}_v^+ \) defined by

\[
\Phi_{y_v}(g, \overline{g}) = \int f_v(g, h_{v}) \, dh_{v},
\]
we say that \( f_v \) and \( f'_v \) have matching orbital integrals provided \( \Phi_{y_v} \) and \( \Phi'_{y_v} \) do. If \( f_v \) is a Hecke function and \( f'_v \) is its image by the base change homomorphism, then \( f'_v \) is supported on \( G_v^+ \) and \( f_v \) and \( f'_v \) have matching orbital integrals.

If \( y_v \) splits into \( v_1, v_2 \) then we may identify \( S_{y_v} \) to the set of pairs \((s, t)\) with \( s \in GL(3, F_{v_1}) \) and \( H_{y_v} \) to the set of pairs \((h, t)\) with \( h \in GL(3, F_{v_2}) \).

Thus we may identify \( GL(3, F_{v_0}) \), \( S_{y_v} \) and \( H_{y_v} \). Then we take the condition of matching orbital integrals to be:

\[
f'_{y_v}(g) = \Phi_{y_v}(g). \tag{11}
\]

We say that \( f_{y_v} \) and \( f_{v_1} \times f_{v_2} \) have matching orbital integrals if:

\[
f'_{y_v}(g) = \Phi_{y_v}(g) = \int f_{v_1}(gh) \, f_{v_2}(h^{-1}) \, dh.
\]

For Hecke functions this means that \( f_{y_v} \) is the convolution of \( f_{v_1} \) and \( f_{v_2} \).

Identity (9) follows readily from the condition of matching.

To the functions \( f \) and \( f' \) we attach kernels in the usual way:

\[
K_f(x, y) := \int_{E_v^+} \sum_{\xi \in GL(3, F_v)} f(x^{-1} \xi y) \, \Omega(z) \, dz,
\]
\[
K_{f'}(x, y) := \int_{F_v^+} \sum_{\xi \in G^+(F)} f'(x^{-1} \xi y) \, \Omega(z) \, dz.
\]

Then

\[
J(\Phi) = \int_{N(F) \backslash N(F_{v_0}) \times H(F) \backslash H(F_{v_0})} K_f(n, h) \, \theta^{-1}(nm) \, dn \, dh,
\]
\[
J'(f') = \int_{N(F) \backslash N(F_{v_0}) \times N(F) \backslash N(F_{v_0})} K_{f'}(n_1, n_2) \, \theta^{-1}(n_1) \, \theta(n_2) \, dn_1 \, dn_2.
\]

From identity (9) follows the equality of the integral of the two kernels. In turn, this implies the equality of the integrals of the corresponding spectral kernels and, finally, the equality of the integrals of the kernels attached to a cuspidal representation \( \pi \) and its base change.
Now let $\Omega$ be the base change of $\omega$, that is, $\Omega(z) = \omega(z\sigma(z))$. Let $\pi$ a cuspidal representation of $GL(3, F_A)$ with central character $\omega$ and let $\Pi$ its base change. Let $K^{\Pi}_{\phi}$ and $K^{\pi}_{\phi'}$ be the kernels attached to the representations $\Pi$ and $\pi$ respectively:

$$K^{\Pi}_{\phi}(x, y) = \sum \Pi(f) \phi_i(x) \overline{\phi_i(y)},$$

$$K^{\pi}_{\phi'}(x, y) = \sum \pi(f') \phi'_i(x) \overline{\phi'_i(y)},$$

where the sums are over orthonormal bases of $\mathcal{V}(\Pi)$ and $\mathcal{V}(\pi)$ respectively.

Then, if $f$ and $f'$ have matching orbital integrals,

$$\int \int K^{\Pi}_{\phi}(n h) \theta^{-1}(n \bar{m}) \ dn \ dh = \int \int K^{\Pi}_{\phi}(n_1, n_2) \theta^{-1}(n_1) \theta(n_2) \ dn_1 \ dn_2. \hspace{1cm} (12)$$

Recall that on the space of $\mathcal{V}(\Pi)$ we have introduced the following linear forms:

$$\mathcal{W}(\phi) = \int \phi(n) \theta^{-1}(n \bar{m}) \ dn, \hspace{1cm} \mathcal{P}(\phi) = \int \phi(h) \ dh.$$

Thus the integral of $K^{\Pi}_{\phi}$ can be written as

$$\sum_{\phi_i} \mathcal{W}(\Pi(f) \phi_i) \overline{\mathcal{P}(\phi_i)}.$$

On the other hand:

$$\int_{N(F_A) \backslash N(F)} \overline{\mathcal{P}(\phi') \theta(n') \ dn'} = \int_{N(F_A) \backslash N(F)} \phi'(n') \theta^{-1}(n') \ dn'$$

$$= \int_{N(F_A) \backslash N(F)} \phi'(n' w') \theta^{-1}(n') \ dn'$$

$$= \int_{N(F) \backslash N(F_A)} \phi'(n' w) \theta^{-1}(n') \ dn'$$

$$= \mathcal{W}(\pi(w) \phi').$$

Thus the integral of $K^{\pi}_{\phi'}$ can also be written:

$$\sum_{\phi'_i} \mathcal{W}(\pi(f') \phi'_i) \overline{\mathcal{W}(\pi(w) \phi'_i)}.$$
We arrive at the identity:

$$\sum_{\phi} \mathcal{H}(\Pi(f) \phi_i) \overline{\mathcal{H}(\phi_i)} = \sum_{\phi'} \mathcal{H}'(\pi(f') \phi'_i) \overline{\mathcal{H}'(\pi(w) \phi'_i)},$$  \hspace{1cm} (13)

whenever $f$ and $f'$ have matching orbital integrals.

**Remark.** If $v_0$ is an Archimedean place of $F$, by hypothesis, the place $v_0$ splits into $v_1$ and $v_2$. We assume that the functions $f_{v_1}$ and $f_{v_2}$ are in fact $K_{v_1}$-finite so as to have only finite sums in the above identity. In fact, both sides may be viewed as distributions and then the identity is true without restriction at the infinite places.

### 4. PROOF OF THE MAIN THEOREM

We now prove the theorem stated in the first section. Thus we let $\Pi$ be a distinguished cuspidal representation of $GL(3, E_{\mathbb{A}})$ with central character $\Omega$. It is thus the base change of a unique cuspidal representation $\pi$ of $GL(3, F_{\mathbb{A}})$ with central character $\omega$. We let $S_0$ and $S$ be as before. If $f$ is a smooth function of compact support on $G_S$ and $K_S$-finite, we set:

$$\mathcal{B}_\Pi(f) = \sum_{\phi} \mathcal{H}(\Pi(f) \phi_i) \overline{\mathcal{H}(\phi_i)},$$

the sum over an orthonormal basis $(\phi_i)$ of $\mathcal{V}_S$. The sum does not depend on the choice of the orthonormal basis. We think of this linear form as being the relative Bessel distribution attached to $\Pi$.

Recall

$$G_{S_0}^+ := \prod_{v_0 \in S_0} G_{v_0} \prod_{v_0 \in S_0} G_{v_0}^+.$$  

We have defined the global Bessel distribution attached to $\pi$. We can compute its value on a function $f'$ smooth and of compact support, on the group $G_{S_0}^+$:

$$\mathcal{B}_\pi(f') = \sum_{\phi'_i} \mathcal{H}'(\pi(f') \phi'_i) \overline{\mathcal{H}'(\pi(w) \phi'_i)},$$

where the sum is over an orthonormal basis of $\mathcal{V}_{S_0}(\pi)$. Actually, as before, to make the sum finite we have to assume that $f' = \prod_{v_0 \in S_0} f_{v_0}$ where $f_{v_0}$ is $K_{v_0}$-finite for $v_0$ infinite. We think of $\mathcal{B}_\pi$ as the Bessel distribution attached to $\pi$. 
If $f$ and $f'$ have matching orbital integrals then it follows from the previous section that:

$$R_\nu(f) = B_\nu(f').$$ \hfill (14)

For $v_0 \in S$, we define a distribution $\mathcal{B}_v$ on $GL(3, E_v)$ as follows: given a smooth function of compact support $f_v$, we choose a function $f'_{v_0}$ with matching orbital integrals and we set:

$$\mathcal{B}_v(f) = B_{v_0}(f'_{v_0}).$$

We must check that the right hand side is independent of the choice of $f'_{v_0}$. But if $f''_{v_0}$ is another choice then $f'_{v_0}$ and $f''_{v_0}$ have the same orbital integrals, or, what amounts to the same, all the orbital integrals of the difference vanish. However, the orbital integrals are weakly dense in the space of distributions on $GL(3, F_{v_0})$ which transform on the left and on the right under the character $\theta$ of $N(F_{v_0})$ ([GK], principle of localization). Thus the distribution $B_{v_0}$ takes the same value on both functions, and the distribution $\mathcal{B}_v$ is well defined.

At a place $v_0 \in S$, we set

$$\mathcal{B}_{v_0}(f_1 \otimes f_2) = B_{v_0}(f'_{v_0}),$$

where $f'_{v_0}$ have matching orbital integrals with $f_1 \otimes f_2$, that is, $f'_{v_0} = f_{v_1} * f_{v_2}$.

Recall the decomposition

$$\mathcal{B}_v(f') = c(\pi) \prod_{v \in S} \mathcal{B}_v(f'_{v_0}).$$

It follows that we can write:

$$\mathcal{B}_\nu(f) = c(\pi) \prod_{v \in S} \mathcal{B}_v(f) \prod_{v \in S} \mathcal{B}_v(f_1 \otimes f_2).$$ \hfill (15)

The theorem will follow from (15) and a careful analysis of the distributions $\mathcal{B}_{v_0}$ (See the next two lemmas.).

**Lemma 1.** For every $v_0 \in S$, there is a unique element $\mathcal{P}_{v_0}$ of $\mathcal{H}_{v_0}$ such that

$$\mathcal{P}_{v_0}(f_v) = \sum_{u} \mathcal{U}(\Pi_v(f_v) u) \overline{\mathcal{P}_{v_0}(u)},$$

where the sum is over an orthonormal basis of $\mathcal{V}(\Pi_v)$.
Proof of the lemma. We first prove the uniqueness. Suppose $\mathcal{H}_0$ is any linear form such that

$$\sum_{u_i} \mathcal{H}_0(\Pi_i(f_v) u_i) \overline{\mathcal{H}_0(u_i)} = 0,$$

for any function $f_v$. Our task is to show that, for any vector $u_0$, $\mathcal{H}_0(u_0) = 0$.

We may as well assume that $u_0$ is a unit vector and even a member of the orthonormal basis $(u_i)$. We then choose a vector $u'$ such that $\mathcal{H}(u') \neq 0$ and a function $f_v$ such that $\Pi_i(f_v) u_0 = u'$ and $\Pi_i(f_v) u_i = 0$ for $i \neq 0$. We then obtain our conclusion by applying the hypothesis to $f_v$.

To prove the existence, we fix a place $w_0 \in S_i$ and let $w$ be the corresponding place of $E$. Let $\Pi^w$ be the restricted tensor product of the unitary representations $\Pi_v$ with $v \neq w$. We fix a unitary intertwining operator $A: \Pi^w \otimes \Pi_w \to \Pi$. In a precise way, the space of smooth vectors of $\Pi^w$ can be identified with the space $V^w$ spanned by the functions of the form $W^w(g)$ on the group $G^w$, the restricted product of the groups $G_v$ with $v \neq w$, where $W_v$ is in $\mathcal{H}(\Pi_v)$ and $W_v = W^K_v$ for $v \notin S$. Then $A$ has the form:

$$A(W^w \otimes W_w) = d \phi(g),$$

$$\phi(g) = \sum_{\gamma \in \mathbb{N}(2, E) \cap \mathbb{G}(2, E)} W(g) \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}.$$

The constant $d$ is chosen to make the map a unitary operator. Let $(u_v)$ be an orthonormal basis of $\mathcal{H}(\Pi_v)$ and $(m^w)$ be an orthonormal basis of $\mathcal{H}^w$. Then $A(m^w \otimes u_v)$ is an orthonormal basis of $\Pi$. If we set $\mathcal{H}(W^w(g)) = dW^w(e)$ then

$$\mathcal{H}(A(m^w \otimes u_v)) = \mathcal{H}_w(m^w) \mathcal{H}_w(u_v).$$

For every vector $m$ in $\mathcal{H}(\Pi^w)$, the linear form

$$u \mapsto \mathcal{H}(m^w \otimes u)$$

is invariant under $H_0(F_{w_0})$ thus belongs to $\mathcal{H}_{w_0}$. Thus there is a linear map $A_{H^w}: \mathcal{H}^w \to \mathcal{H}_{w_0}$ such that

$$\mathcal{H}(A(m^w \otimes u)) = A_{H}(m^w)(u).$$

The global distribution $\mathcal{H}_{H^w}$ can be written

$$\mathcal{H}_{H^w}(f) = \sum_{\mathfrak{p}} \sum_{m} \mathcal{H}_w(\Pi_w^w(f_m^w) m^w) \mathcal{H}_w(\Pi_w m^w m^w) A_{H^w}(m^w)(u_v).$$
Let $\mathcal{P}_k$ be a basis for the space $\mathcal{H}_w$. Note that, at this point, we do not know that the space is finite dimensional in general. We can write

$$A_H(m) = \sum_k A_k(m) \mathcal{P}_k,$$

where the $A_k$ are suitable linear forms on $\mathcal{H}_w$. Note that for a given $m$, $A_k(m) = 0$ for all but finitely many $k$'s. We get:

$$\mathcal{A}_H(f) = \sum_k \mathcal{A}_k(f) \mathcal{P}_k,$$

where we have set:

$$\mathcal{A}_k(f) = \sum \mathcal{A}_k(f) \mathcal{P}_k.$$

Now for each $v_0 \in S$, $v_0 \neq w_0$ we can choose a function $f_v$ such that $\mathcal{A}_k(f_v) \neq 0$ and, for $v_0 \in S$, functions $f_{v_1}, f_{v_2}$ such that $\mathcal{A}_k(f_{v_1} \otimes f_{v_2}) \neq 0$. Thus the distribution $\mathcal{R}_0$ is a linear combination of the distributions $\mathcal{A}_k$:

$$\mathcal{R}_0 = \sum_k \mathcal{A}_k.$$

Now we set

$$\mathcal{R}_0 = \sum_k \mathcal{A}_k,$$

and then $\mathcal{R}_0$ has the required form.

We need an analog of the previous lemma for a place $v_0 \in S$. It is in fact formal. We define an element $\mathcal{R}_0 \in \mathcal{H}_v$ as follows: we may identify the representations $\pi_{v_1}, \pi_{v_2}, \pi_{v_0}$; their common space is the space $\mathcal{H}(\pi_{v_0})$. Define an antilinear map $A$ from that space to itself by:

$$AW(g) = W(wg^{-1}).$$

Then $A\pi_{v_0}(g) = \pi_{v_0}(g^{-1}) A$ and $A^2 = 1$. Now consider the unitary representation attached to $\pi_{v_0}$ and let $\mathcal{H}$ be its Hilbert space. Of course, we use the same notation for the unitary representation attached to $\pi_{v_0}$. We regard the representation $\pi_{v_0}(g^{-1})$ has a representation on the conjugate Hilbert space. This new representation and the representation $\pi_{v_0}$ have the same character thus are equivalent. It follows that there is a unitary antilinear operator $U$ such that $U\pi_{v_0}(g) = \pi_{v_0}(g^{-1}) U$. Then $U^2$ is a unitary operator which commutes to $\pi_{v_0}$. It is therefore a scalar $\mu$ with
If $|\mu| = 1$, dividing $U$ by the square root of $\mu$ we may as well assume $U^2 = 1$. We have then (on the space of smooth vectors) $A = \lambda U$ with $\lambda \in \mathbb{C}$. Since $A^2 = 1$ we get $\lambda = \pm 1$ and so $A$ preserves the norm. This being so, for $u_1, u_2 \in \mathcal{W}(\pi_v)$, we set

$$\mathcal{P}_v(u_1 \otimes u_2) = (u_1, Au_2).$$

Then

$$\mathcal{P}_v(\pi_v(g) u_1 \otimes u_2) = \mathcal{P}_v(u_1 \otimes \pi_v(g) u_2).$$

We then define a distribution

$$\mathcal{H}_v(f_1 \otimes f_2) := \sum_{\alpha, \beta} \mathcal{H}_v(\pi_v(f_1) u_\alpha) \mathcal{H}_v(\pi_v(f_2) \mu^\beta) \mathcal{P}_v(u_\alpha \otimes \mu^\beta),$$

The sum is over orthonormal bases $(u_\alpha)$ and $(\mu^\beta)$ of $\pi_v$.

**Lemma 2.** In fact:

$$\mathcal{H}_v = \mathcal{H}_v.$$

**Proof of the Lemma.** Indeed, let us take $\mu^\beta = A(u_\beta)$. Then

$$\mathcal{P}_v(u_\alpha \otimes \mu^\beta) = \langle u_\alpha, A^2 u_\beta \rangle = \langle u_\alpha, u_\beta \rangle = \delta_{\alpha, \beta}.$$

Then:

$$\mathcal{H}_v(f_1 \otimes f_2) = \sum_{\alpha} \mathcal{H}_v(\pi_v(f_1) u_\alpha) \mathcal{H}_v(\pi_v(f_2) Au_\alpha)
= \sum_{\alpha} \mathcal{H}_v(\pi_v(f_1) u_\alpha) \mathcal{H}_v(A\pi_v(f_2^*) u_\alpha)
= \sum_{\alpha} \mathcal{H}_v(\pi_v(f_1) u_\alpha) \mathcal{H}_v(\pi_v(f_2^*) \mu^\alpha u_\alpha).$$

But $\mathcal{H}_v(Au) = \mathcal{H}_v(\pi_v(w) u)$. Thus we get at last:

$$\mathcal{H}_v(f_1 \otimes f_2) = \sum_{\alpha} \mathcal{H}_v(\pi_v(f_1) u_\alpha) \mathcal{H}_v(\pi_v(w) u_\alpha),$$

which is indeed $\mathcal{H}_v(f_1 \otimes f_2)$. \[\square\]

Now we prove the theorem. Let $\mathcal{H}$ be the linear form on $\mathfrak{g}^*(II)$ defined by

$$\mathcal{H}(\phi) = \prod_{v_2 \in S_2} \mathcal{P}_{v_2}(W_{v_2}) \prod_{\alpha \in S_2} \mathcal{P}_{v_2}(W_{v_2} \otimes W_{v_2})$$

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when \( \phi \) is a pure tensor. Taking into account the computation of the norm of a pure tensor, we get:

\[
\sum_{\phi} \mathcal{N}(\Pi(f) \phi) \mathcal{T}(\phi) = c(\Pi) \prod_{n \in S} \mathcal{R}_{\pi}(f_n) \prod_{n \in S} \mathcal{R}_{\pi}(f_n) = c(\Pi) \mathcal{R}_{\pi}(f)
\]

Since \( \Pi \) is irreducible (See the proof of uniqueness in Lemma (1)) we get

\[
\mathcal{P} = \frac{c(\pi)}{c(\Pi)} \mathcal{P}
\]

and the theorem follows.

5. LOCAL RESULTS: SUPERCUSPIDAL CASE

Now we consider a local quadratic extension \( E/F \) of non-Archimedean local fields. We only consider irreducible unitary generic representations of \( GL(3, E) \). We say that such a representation \( \Pi \) is distinguished if the space \( \mathcal{H}(\Pi, H) \) of \( H \)-invariant linear forms is non-zero. Then the central character \( \Omega \) of \( \Pi \) is itself distinguished, that is, trivial on \( U_1 \). We then fix a character \( \omega \) of \( F^* \) such that \( \Omega(z) = \omega(zz) \).

**Theorem 2.** Suppose \( \Pi \) is supercuspidal. Then \( \Pi \) is distinguished if and only \( \Pi^* = \Pi \). The dimension of \( \mathcal{H}(\Pi, H) \) is then one. Let \( \Omega \) and \( \omega \) as above. Then \( \Pi \) is the base change of a unique cuspidal representation \( \pi \) of \( GL(3, F) \) with central character \( \omega \) and there exists a unique element \( \mathcal{P}_\pi \) of \( \mathcal{H}(\Pi, H) \) such that

\[
\sum_{\phi} \mathcal{N}(\Pi(f) \phi) \mathcal{T}_\pi(\phi) = \mathcal{B}_\pi(f^*),
\]

each time \( f \) and \( f^* \) have matching orbital integrals.

We first prove a result of density.

**Lemma 3.** If \( \Pi \) is supercuspidal and distinguished, let \( \mathcal{H}_\pi \) be the space spanned by the linear forms \( \mathcal{P} \) defined by:

\[
\mathcal{P}(u) = \int_{H(F)} (\Pi(h) u, \hat{u}) \, dh,
\]
where $\tilde{u}$ is a (smooth) vector. Then for any $\mathcal{P} \in \mathcal{H}(\Pi, H)$, the space $\text{Ker}(\mathcal{P})$ contains the intersection $\bigcap_{\rho \neq \rho_0} \text{Ker}(\mathcal{P})$.

Proof. We use Bernstein theory to write the space $\mathcal{C}$ of smooth functions on $GL(3, E_v)$ transforming under the character $\Omega$ and compactly supported modulo the center, as a direct sum of bi-invariant subspaces

$$\mathcal{C} = \mathcal{C}(\Pi) \oplus \mathcal{C}^H,$$

where $\mathcal{C}(\Pi)$ denotes the space spanned by the matrix coefficients of $\Pi_v$, that is, the functions of the form

$$g \mapsto (\Pi(g) u, \tilde{u}).$$

If we choose a linear basis $(u_i)$ of the space $\tau(\Pi)$ we can decompose further $\mathcal{C}(\Pi)$ as direct sum of the invariant spaces $\mathcal{C}_i$,

$$\mathcal{C}(\Pi) = \bigoplus \mathcal{C}_i,$$

where $\mathcal{C}_i$ is the space spanned by the functions of the form: $g \mapsto (\Pi(g) u, u_i)$. Choosing an index $i_0$, we may identify the space $\mathcal{C}_{i_0}$ to the space $\tau$ and view $\mathcal{P}$ as a linear form on $\mathcal{C}_{i_0}$. We may extend $\mathcal{P}$ to $\mathcal{C}$ by demanding that it be zero on $\mathcal{C}_i$ with $i \neq i_0$, and then zero on $\mathcal{C}^H$. Thus we now view the given linear form $\mathcal{P}_1$ as a distribution invariant on the right under $H(F)$. Its value on the function $f(g) = \langle H(g) u, u_{i_0} \rangle$ is equal to $\mathcal{P}(u)$. Since $u_{i_0}$ is invariant under a compact open subgroup $K'$ this distribution is invariant on the left under $K'$. It follows that there exists a distribution $\mu$ on $G(E)/H(F)$ such that, for any $f \in \mathcal{C}$,

$$\mathcal{P}_1(f) = \int \left( \int f(xh) \, dh \right) \, d\mu(x).$$

Moreover, this distribution is invariant on the left under $K'$. Thus, if $(x_j)$ is a set of representatives for the double cosets $Z(E) \backslash G(E)/H(F)$ we have, for suitable constants $\lambda_j$,

$$\mathcal{P}_1(f) = \sum_j \lambda_j \int \int f(k'x_jh) \, dh \, dk'.$$

For a given function $f$, there are only finitely many non-zero terms on the right. Coming back to the original linear form $\mathcal{P}_1$, we see that

$$\mathcal{P}_1(u) = \sum_j \lambda_j \int \langle \Pi_v(x_jh) u, u_{i_0} \rangle \, dh,$$
the sum on the right having only finitely many non-zero terms. If we set

$$u_j = \Pi(x_j^{-1})_j(u_0)$$

we get finally:

$$\mathcal{P}(u) = \sum_j \lambda_j \int (\Pi(h) u, u_j) \, dh.$$  

The lemma follows.

To finish the proof we write our local extension in the form $E_v/F_v$ where $E/F$ is a quadratic extension of number fields and $v_0$ a place of $F$ inert in $E$, $v$ the corresponding place of $E$. We assume that all the infinite places of $F$ split in $E$. We write the given supercuspidal representation as $\Pi_v$.

Suppose first that $\Pi_v = \Pi_v$. Then $\Pi_v$ is the base change of a supercuspidal representation $\pi_{v_0}$. In turn we may write $\pi_{v_0}$ as the local component of a cuspidal automorphic representation $\pi$. Let $\Pi$ be the base change of $\pi$. Then $\Pi_v$ is the local component of $\Pi$ at the place $v$. Since $\Pi$ is globally distinguished, it follows that $\Pi_v$ is (locally) distinguished.

Now we suppose that $\Pi_v$ is distinguished. Then its central character $\Omega_v$ is distinguished and we write, as before, $\Omega_v(z) = \omega_{v_0}(zz)$. We may further choose an idele class character $\omega$ of $F$ whose component at $v_0$ is $\omega_{v_0}$. We then set $\Omega(z) = \omega(zz)$. We first show that $\Pi_v$ is the local component at $v$ of a distinguished cuspidal automorphic representation $\Pi$ with central character $\Omega$. Since $\Pi_v = \Pi_v$ then, it will follow that, as claimed, $\Pi_v = \Pi_v$. From the previous density lemma, it follows there is a smooth vector $u_1$ in the space of $\Pi_v$ such that the linear form $\mathcal{P}_1$ defined by

$$\mathcal{P}_1(u) = \int_{H_v} (\Pi_v(h) u, u_1) \, dh$$

is non-zero. For any smooth vector $u$ in the space of $\Pi_v$ we set:

$$f^\pi(g_v) = (\Pi_v(g_v) u, u_1).$$

On the other hand, we let $f^\pi$ be a smooth function on the group $G^\pi$, transforming under the character $\Omega^\pi$ of $Z^\pi$, and compactly supported, modulo the center. We set $f(\gamma g) = f^\pi(\gamma) f^\pi(g_v)$. We define a function $\phi_{\pi, f^\pi}$ on $GL(3, E_A)$ as follows:

$$\phi_{\pi, f^\pi}(g) = \sum_{\gamma \in Z(E) \backslash G(E)} f(\gamma g).$$

The resulting function is invariant under $G(E)$ on the left, compactly supported modulo $Z(E_A) \backslash G(E)$ and cuspidal. Let $(\Pi_{v})$ be the family of cuspidal representations with central character $\Omega_v$, and for each $\pi$, let $f_{v}^\pi$ be
the space of smooth vectors of $\Pi_a$. Let $\phi_{a}^{\mu, f}$ be the orthogonal projection of $\phi_{a}^{\mu, f}$ on $\mathcal{F}_a$. Then

$$\phi_{a}^{\mu, f}(g) = \sum_{\pi} \phi_{a}^{\mu, f}(g).$$

The series converges in the space of rapidly decreasing functions on the quotient $G(E)/Z(E_{\Lambda}) \backslash G(E_{\Lambda})$. Thus we may write:

$$\int \phi_{a}^{\mu, f}(h) \, dh = \sum_{\pi} \int \phi_{a}^{\mu, f}(h) \, dh.$$

Moreover, Schur orthogonality relations imply the existence of a constant $d > 0$ such that

$$\int_{Z \backslash G_a} \phi_{a}^{\mu, f}(gg_v) u, u_0 \, dg_v = d(u, u)(\Pi_{\pi}(g_0) u_1, u_0).$$

This implies that

$$\int_{Z \backslash G_a} \phi_{a}^{\mu, f}(gg_v) u, u_0 \, dg_v = d(u, u) \phi_{a}^{\mu, f}(g).$$

Thus, for each $\pi$

$$\int_{Z \backslash G_a} \phi_{a}^{\mu, f}(gg_v) u, u_0 \, dg_v = d(u, u) \phi_{a}^{\mu, f}(g).$$

It follows that if the projection $\phi_{a}^{\mu, f}$ is not zero (for some choice of $u$ and $f$) then the representation $\Pi_a$ has the form $\Pi_a \otimes \Sigma$. We claim further that at least one of the representations $\Pi_a$ is distinguished. Indeed, suppose not. Thus, for all $\pi$

$$\int \phi_{a}^{\mu, f}(h) \, dh = 0.$$

It follows that for any function $f$

$$\int \phi_{a}^{\mu, f}(h) \, dh = 0.$$

Explicitly:

$$\sum_{\gamma \in G(E) \backslash H(F)} \int_{H_{E}} (\Pi_{\pi}(\gamma h) u, u_0) \, dh_{E_a} \int_{H_{E_a}} f^{\gamma}(h^{a}) \, dh^{a} = 0.$$
Now let $\Phi_{v_0}$ be the function on $\mathcal{Z}_{v_0}^+$ defined by

$$\Phi_{v_0}(g\tilde{g}) = \int_{H_{v_0}} (\Pi_v(gh_{v_0}) u, u_0) \, dh_{v_0}.$$  

Similarly, let $\Phi^o$ be the function on $\mathcal{Z}_v^+$ defined by:

$$\Phi^o(g\tilde{g}) = \int_{H^o} f^o(gh^o) \, dh^o.$$  

Finally let $\Phi$ be the product of $\Phi_{v_0}$ and $\Phi^o$. Thus $\Phi(sz) = \Phi(s) \omega(sz)$ for $z \in F^+$. Moreover the support of $\Phi$ is contained in a set of the form $MF^+$ where $M$ is a compact set. The above relation reads:

$$\sum_{\xi \in S(F)/F^+} \Phi(\xi) = 0.$$  

We can choose $\Phi^o$ in such a way that the above relation reduces to $\Phi_{v_0}(1) = 0$ or

$$\int_{H_{v_0}} (\Pi_v(h_{v_0}) u, u_0) \, dh_{v_0} = 0,$$

that is, $\Psi(u) = 0$, which is a contradiction. Thus one of the representations $\Pi_z$ is distinguished. As we have remarked before, this implies that $\Pi_z^o = \Pi_z$ and thus as claimed, $\Pi_z^o = \Pi_z$.

Finally, it remains to prove that if $\Pi_z$ is distinguished then the dimension of $\mathcal{W}(\Pi_z, H_z)$ is one. We have just seen that $\Pi_z$ is the local component of a distinguished cuspidal representation $\Pi$, which is itself the base change of a cuspidal representation $\pi$. In particular, $\Pi_z$ is the local base change of the local component $\pi_{v_0}$; in fact $\pi_{v_0}$ is the unique irreducible representation with central character $\omega_{v_0}$ whose base change is $\Pi_z$. By lemma there is a unique element $\mathfrak{P}_0$ of $\mathcal{W}_{\Pi}$ such that \eqref{17} is satisfied. By the density result, it will suffice to prove that if $u$ is a vector in the kernel of $\mathfrak{P}_0$ then $\mathfrak{P}(u) = 0$ for every element $\mathfrak{P}_1$ of $\mathcal{W}_{\Pi_z}$. We may assume that $\mathfrak{P}_1$ has the form:

$$\mathfrak{P}_1(u) = \int_{H_z} (\Pi_z(h) u, u_1) \, dh.$$  

Now if $\Pi$ is any cuspidal automorphic representation of $GL(3, E_\Lambda)$ with local factor $\Pi_z$ and $\phi$ is any smooth vector of $\Pi$ which is a pure tensor of the form $u \otimes u^\vee$ then

$$\int \phi(h) \, dh = 0.$$
Indeed, this is clear if \( \Pi \) is not distinguished and follows from the factorization of the global period and the uniqueness of \( \pi_{\chi_0} \) otherwise. We now apply the previous construction. Each function \( \phi_{\alpha}^{\mu} \) corresponds to a pure tensor vector of the form \( u \otimes v^c \). Then

\[
\int \phi_{\alpha}^{\mu} (h) \, dh = 0.
\]

As we have seen this implies that in fact \( \mathcal{P}(u) = 0 \), as claimed. Thus the theorem is completely established.

### 6. SUPERCUSPIDAL REPRESENTATIONS FOR \( GL(2, E) \)

We briefly review the case of the group \( GL(2, E) \) (Cf. [HLR], [jH], [jHyF], [yF2], [P]). Let \( H_1 \) be a split-unitary group in two variables. Denote by \( H_1 \) the corresponding similitude group and by \( \ell \) the similitude ratio. Thus \( H_1, Z(E) \) has index two in \( \tilde{H}_1 \). Let \( h_1 \) be an element of \( \tilde{H}_1 - H_1, Z(E) \). We say that an irreducible admissible (unitary) representation \( \pi \) of \( GL(2, E) \) is distinguished by \( H_1 \) if the space \( \mathcal{H}(\pi, H_1) \) of linear forms invariant under \( H_1 \) is non-zero.

**Proposition 1.** Suppose that \( \Pi \) is supercuspidal. Then \( \Pi \) is distinguished by \( H_1 \) if and only if \( \Pi'' = \Pi \). Then \( \dim(\mathcal{H}(\Pi, H_1)) = 1 \). Moreover, let \( \pi \) be a supercuspidal representation whose base change is \( \Pi \) and let \( \omega \) be the central character of \( \pi \). Then in fact, for \( \mathcal{P} \in \mathcal{H}(\Pi, H_1) \) and \( h \in \tilde{H}_1 \):

\[
\mathcal{P}(\Pi(h) u) = \omega \mathcal{P}(\lambda(h)) \mathcal{P}(u).
\]

**Proof of the Proposition.** If \( \Pi \) is distinguished by \( H_1 \) then its central character \( \Omega \) is distinguished by \( U_1 \) and so has the form \( \Omega(z) = \omega(z) \) for a suitable \( \omega \). For \( \mathcal{P} \in \mathcal{H}(\Pi, H_1) \) set

\[
\mathcal{P}_1(u) = \frac{1}{2} (\mathcal{P}(\Pi(h_1) u) + \omega(\lambda(h_1)) \mathcal{P}(u)),
\]

\[
\mathcal{P}_{\text{wedge}}(u) = \frac{1}{2} (\mathcal{P}(\Pi(h_1) u) + \omega_{EF}(\lambda(h_1)) \mathcal{P}(u)).
\]

Then, for any \( h \in \tilde{H}_1 \) and any vector \( u \):

\[
\mathcal{P}_1(\Pi(h) u) = \omega(\lambda(h)) \mathcal{P}(u),
\]

\[
\mathcal{P}_{\text{wedge}}(\Pi(h) u) = \omega_{EF}(\lambda(h)) \mathcal{P}(u).
\]

We denote by \( \mathcal{H}(\Pi, H_1, \omega) \) (resp. \( \mathcal{H}(\Pi, H_1, \omega_{EF}) \)) the space of linear forms satisfying (20) (resp. (21)).
Lemma 4. The dimension of the space $\mathcal{H}(\Pi, H_1, \omega)$ is at most one.

Indeed, $\hat{H}_1$ is conjugate to $GL(2, F) \cdot Z(E)$ by an element of $GL(2, E)$ and
the conjugation takes the similitude ratio to $gz \mapsto \det gz$. Thus it suffices to prove
that the space $\mathcal{H}(\Pi, GL(2, F), \omega)$ of linear forms $\mathcal{I}$ such that
$$\mathcal{I}(\Pi(g) \cdot u) = \omega(\det g) \cdot \mathcal{I}(u)$$
for $g \in GL(2, F)$ has dimension at most one. We may extend $\omega$ to a character $\omega_1$ of $E^*$ and replace $\Pi$ by $\Pi \otimes \omega_1^{-1}$. We are then reduced to proving
that the space $\mathcal{H}(\Pi \otimes \omega_1^{-1}, GL(2, F))$ of linear forms invariant under
$GL(2, F)$ by the representation $\Pi \otimes \omega_1^{-1}$ has dimension at most one; this
is known (see [jH] and [yF2]). Moreover if that space is non-zero then (loc. cit.)
$$(\Pi \otimes \omega_1^{-1})^a = \Pi \otimes \omega_1^{-1} = \Pi \otimes \Omega_1^{-1}.$$ 
This relation is in fact equivalent to $\Pi^a = \Pi$. Thus if $\Pi$ is distinguished by
$H_1$ then it is invariant by $\sigma$.

Now we recall some global results. Let again $E/F$ be a global quadratic
extension of number fields. Let $\Pi$ be a cuspidal representation. Let $\omega$ be an
idèle class character of $F$. The two following conditions are equivalent
([HRL]): (i) the representation $\Pi$ is the base change of a cuspidal
representation $\pi$ with central character $\omega_{\pi_E/F}$; (ii) the restriction of the
central character $\Omega$ of $\Pi$ to $E^*$ is $\omega^2$ and there is $\phi$ in the space of $\Pi$ such that
$$\int_{Z(F_A) \cdot G(F)} \phi(g) \cdot \omega^{-1}(\det g) \cdot dg \neq 0.$$ 
Note that the central character $\Omega$ verifies then $\Omega(z) = \omega(zz)$. Thus, the
second condition amounts to: (iii) the central character $\Omega$ has the form
$\Omega = \omega \cdot \text{Norm}$ and there is $\phi$ in the space of $\Pi$ such that
$$\int_{Z(E_A) \cdot H(F_A)} \phi(g) \cdot \omega^{-1}(\lambda(h)) \cdot dh \neq 0.$$ 
Now we go back to the local problem and again write our local extension
in the form $E_v/F_v$ and write $\Pi_v$ instead of $\Pi$. If $\Pi_v$ is given and
$\mathcal{I}_v \neq 0$, then we can argue as before and find a cuspidal representation $\Pi$
of which $\Pi_v$ is the local component at the place $v$ and $\Pi$ satisfies (ii). Then
$\Pi$ is the base change of a representation $\pi$ with central character $\omega_{\pi_E/F}$.
Thus $\Pi_v$ is the base change of a representation $\pi_v$ with central character

The only other representation of which \( \Pi_v \) is the base change is the representation \( \pi_{\alpha v} \otimes \omega_{E_F, \alpha v} \) and it has the same central character. Likewise, if \( \mathcal{P}_{\alpha v} \neq 0 \) then \( \Pi_v \) is the base change of a representation \( \pi_{\alpha v} \) with central character \( \omega_{E_F, \alpha v} \).

We conclude that \( \mathcal{P}_{\alpha v} \) and \( \mathcal{P}_{\beta v} \) cannot be both non-zero. This already prove that if \( \mathcal{H}(\Pi_v, \mathcal{H}_1, \alpha v) \neq 0 \) then it has dimension one. Moreover \( \Pi_v \) is then a base change of a representation \( \pi_{\alpha v} \), \( \Pi_v = \Pi_{\alpha v}^* \) and the central character has the required properties.

If \( \Pi_v = \Pi_{\alpha v} \) then, as before, \( \Pi_v \) is the base change of a supercuspidal representation \( \pi_{\alpha v} \). We can find a cuspidal representation of which \( \pi_{\alpha v} \) is a component. We base change \( \Pi \) to \( \Pi_v \) and apply the previously recalled result to conclude that \( \Pi \) is distinguished by \( \Pi_1 \) and \( \Pi_v \) by \( \Pi_{\alpha v} \).

**Remark.** We could argue as in the previous section using the trace formula described in [JY] (Cf. [yF2]).

Now let \( H_2 \) be a unitary group which is not split. Let also \( \tilde{H}_2 \) be the corresponding similitude group and \( \lambda \) the similitude ratio. We define again \( \mathcal{H}(\Pi, \tilde{H}_2) \) as the space of linear forms on \( \mathscr{H}(\Pi) \) which are invariant under \( H_2 \). Then:

**Proposition 2.** Suppose that \( \Pi \) is supercuspidal. Then \( \Pi \) is distinguished by \( H_2 \) if and only if \( \Pi_v = \Pi. \) Then \( \dim(\mathcal{H}(\Pi, \tilde{H}_2)) = 1. \) Moreover, let \( \pi \) be a supercuspidal representation whose base change is \( \Pi \) and let \( \omega \) be the central character of \( \pi. \) Then in fact, for \( \mathcal{P} \in \mathcal{H}(\Pi, \tilde{H}_2) \) and \( h \in \tilde{H}_2: \)

\[
P(\mathcal{H}(h) u) = \omega(\mathcal{H}(h)) \mathcal{P}(u).
\]

Let \( G(F) \subset GL(2, E) \) be the multiplicative group of a quaternion algebra. It is known that the dimension of \( \mathcal{H}(\Pi, G(F)) \) is at most one. Moreover, \( \mathcal{H}(\Pi, G(F)) \neq 0 \) if and only if \( \mathcal{H}(\Pi, GL(2, F)) \neq 0 \) (see [jH] and [jHyF]). Arguing as before we can reduce this proposition to the previous one.

7. REPRESENTATIONS INDUCED FROM A CUSPIDAL REPRESENTATION

For the other unitary generic representations of \( GL(3, E) \) we propose the following conjecture:

**Conjecture 1.** Suppose \( \Pi \) is a unitary irreducible generic representation of \( GL(3, E) \). Then \( \Pi \) is distinguished by \( H \) if and only \( \Pi' = \Pi. \) Let \( \Omega \) and \( \omega \) as above. For each irreducible admissible representation \( \pi \) of \( GL(3, F) \)
with central character \( \omega \) whose base change is \( \Pi \), there exists a unique element \( \mathcal{P}_n \) of \( \mathcal{H}(\Pi, H) \) such that

\[
\sum_{\phi} \mathcal{H}(\Pi(f), \phi) \mathcal{P}_n(\phi) = \mathcal{B}(f'),
\]

(22)
each time \( f \) and \( f' \) have matching orbital integrals. Moreover, the linear forms \( \mathcal{P}_n \) form a basis of \( \mathcal{H}(\Pi, H) \).

We have established this conjecture when \( \Pi \) is supercuspidal. We prove it in another case. Suppose that \( \Pi \) is induced by a supercuspidal representation. In a precise way, let \( P = MU \) be the Levi decomposition of the parabolic subgroup \( P \) of type \( (2, 1) \) (upper generalized triangular matrices). Let \( \Pi_1 \) is a supercuspidal representation of \( GL(2, E) \) and \( \Pi_2 \) a character of \( E^\times \). Thus we may regard \( \Pi_1 \times \Pi_2 \) as a representation of \( M(E) \approx GL(2, E) \times E^\times \). We assume that \( \Pi \) is the corresponding normalized induced representation:

\[
\Pi = \text{Ind}(\Pi_1, \Pi_2).
\]

Thus \( \Pi \) operates by right shifts on the space of smooth maps \( \phi: GL(3, E) \rightarrow \mathcal{V}(\Pi_1) \) such that

\[
\phi(ph) = \delta_P(p)^{1/2} \Pi_1 \times \Pi_2(p) \phi(g)
\]
for every \( p \in P(E) \); here \( \delta_P \) is the module of \( P(E) \). We will content ourselves with proving the conjecture in this case. We begin with a lemma:

**Lemma 5.** The dimension of \( \mathcal{H}(\Pi, H) \) is at most two. Moreover, if \( \Pi \) is distinguished by \( H \) then \( \Pi_1 \) is distinguished by a split group in two variables, \( \Pi_2 \) is distinguished by \( \mathfrak{U}_1 \) and \( \Pi' = \Pi \).

**Proof of Lemma.** Let \( P_0 \) be the group of upper triangular matrices. We first study the orbits of \( P_0(E) \) on \( \mathfrak{S}(F) \); a system of representatives is given by the following matrices:

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & b
\end{pmatrix},
\begin{pmatrix}
b & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & b & 0 \\
0 & 0 & 0 \\
0 & 0 & b_3
\end{pmatrix},
\begin{pmatrix}
b_1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
b_2 & 0 & 0 \\
0 & b & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
where the elements $b$ and $b_i$ take their values in $F^*/F^+$. Next, a set of representatives for the orbits of $P(E)$ on $\mathcal{Z}(F)$ is given by the matrices:

$$
\begin{pmatrix}
0 & 0 & 1 \\
0 & b & 0 \\
1 & 0 & 0
\end{pmatrix}
\quad
\begin{pmatrix}
b_1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & b_3
\end{pmatrix}
$$

with $b, b_1, b_3$ as before. A system of representatives for the orbits of $P(E)$ on $\mathcal{Z}^+(F)$ is then given by the following matrices:

$$
\begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix}
\quad
\begin{pmatrix}
b & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -b^{-1}
\end{pmatrix}
$$

with $b \in F^* - F^+$. For each element $\sigma_i$ of the above type, let $\xi_i$ be such that $\xi_i^* \xi_i = \sigma_i$. Let $\delta_{\xi_i}$ be the module of the group $P_{\xi_i} := P(E) \cap \xi_i H(F) \xi_i^{-1}$. Then $\delta_{\xi_i}$ is the space of linear forms $\mu$ on the space $\mathcal{V}(\xi_i)$ such that

$$
\delta_{\xi_i}(p) \mu(\Pi_1 \times \Pi_2(p) v) = \delta_{\xi_i}(p) v,
$$

for every vector $u$ and every $p \in P_{\xi_i}$. Then

$$
\dim(\mathcal{V}(\xi_i)) \leq \sum_i \dim(\mathcal{V}(\xi_i)).
$$

Now $P_{\xi_0}$ contains the subgroup

$$
\begin{pmatrix}
1 & x & xx \\
0 & 1 & \hat{x} \\
0 & 0 & 1
\end{pmatrix}
$$

whose intersection with (or, more correctly, projection on) $M(E)$ contains the unipotent radical $U_1$ of the parabolic subgroup of type $(1, 1)$ of $M$. Thus for any $\mu \in \mathcal{V}(\xi_0)$, any $u \in U_1$ and any vector $v$:

$$
\mu(\Pi_1(u) v) = \mu(v)
$$

Because $\Pi_1$ is supercuspidal, this implies $\mu = 0$. Thus $\mathcal{V}(\xi_0) = 0$.

Next, consider the case of the element $\sigma_i$. Then $P_{\xi_i}$ is the set of matrices

$$
\begin{pmatrix}
h_1 & 0 \\
0 & h_2
\end{pmatrix}
$$
with $h_1 \in H_1(F)$ and $h_2 \in U_1(F)$, where $H_1$ is the unitary group in $GL(2, E)$ for the Hermitian matrix
\[
\begin{pmatrix}
-1 & 0 \\
0 & 1
\end{pmatrix}.
\]
Thus $V(\xi_1) = 0$ unless $H_2$ is distinguished and then $V(\xi_1)$ may be identified with the space of linear forms $\mu$ on $\mathcal{Y}(H_1)$ such that
\[
\mu(H_1(h_1)v) = \mu(v)
\]
for any $h_1 \in H_1$ and any vector $v$. Thus $\dim(V(\xi_1)) \leq 1$ by the previous section. Moreover if $V(\xi_1) \neq 0$ then $H_1$ is distinguished by $H_1$ and $U_2$ by $U_1$.

In particular, it follows then that $H_1^* = H_1$ and $U_2^* = U_2$, and thus $\Pi'' = \Pi$ as well.

Now consider the case of the element $\sigma_2$. We let $H_2$ be the unitary group for the matrix
\[
\begin{pmatrix}
-b & 0 \\
0 & 1
\end{pmatrix}.
\]
Then $V(\xi_2) = 0$ unless $H_2$ is distinguished by $U_1$ and then $V(\xi_2)$ is isomorphic to the space of linear forms $\mu$ on $\mathcal{Y}(H_1)$ which are invariant under $H_2$. Thus $\dim(V(\xi_2)) \leq 1$ again. Furthermore, if $V(\xi_2) \neq 0$ then again $H_1$ and $H_2$ are invariant under $\sigma$ and so is $\Pi$.

Thus we do get $\dim(\mathcal{B}(\Pi, H)) \leq 2$. Moreover if $\Pi$ is distinguished by $H$ then $\Pi'' = \Pi$.

Now suppose that $\Pi'' = \Pi$. Then $\Pi_1'' = \Pi_1$ and $\Pi_2'' = \Pi_2$. The representation $\Pi_1$ is the base change of two supercuspidal representations, say $\pi_1$ and $\pi_1^\ast := \pi_1 \otimes \omega_E$. Let $\omega_1$ be their common central character and set $\omega_2 = \omega \omega_1$. Thus $\Pi$ is the base change of exactly two irreducible representations with central character $\omega$, the central character of $\Pi$ being $\Omega = \omega \cdot \text{Norm}$. The two representations in question are the induced representations
\[
\pi' = \text{Ind}(\pi_1, \pi_2), \quad i = 1, 2.
\]
It remains to show that there are two elements $\mathcal{A}_i$, $i = 1, 2$, of $\mathcal{B}(\Pi, H)$ with the following property. Define as before distributions $\mathcal{A}_i$, $i = 1, 2$, by
\[
\mathcal{A}_i(f) = \mathcal{A}_e(f'),
\]
for $(f, f')$ with matching orbital integrals. Then
\[
\mathcal{A}_i(f) = \sum \mathcal{A}_e(\Pi_i(f) u) \overline{\mathcal{F}_i(u)}.
\]
Note that $f'$ is an arbitrary smooth function of compact support on $G^+$ and the restrictions of the representations $\pi^i$ to $G^+$ inequivalent. It follows that the distributions $\mathcal{D}_i$ are linearly independent. The same is true of the distributions $\mathcal{D}_i$. It follows that the linear forms $\mathcal{D}_i$ are linearly independent (see Lemma (1)).

To apply the global theory, we again write our local extension in the form $E/F_{v_0}$, write $\Pi_i$ for $\Pi$ and so on. Our assertion follows from the global theory if there exist two cuspidal representations $\pi_{i_1}$, $i_2$ with components $\pi_{i_1} v_0$ at $v_0$. Indeed, the corresponding base change representations $\pi_i$ are then cuspidal and distinguished and we can argue as before. Of course, this will not be the case in general.

To remedy the situation, for every $z = q^{s_{v_0}}$ with $s \in \mathbb{C}$, we set:

$$\pi_{i_1}^{s_{v_0}} = \text{Ind}(\pi_{i_1} v_0 \otimes \pi_{i_2} v_0 \otimes \pi_{i_3} v_0 \otimes \pi_{i_4} v_0),$$

where $\pi_{i_0}$ denotes the module of $F_{v_0}$. We recall a standard lemma:

**Lemma 6.** Fix an index $i$. Let $X$ be the set of complex numbers $z$ of module 1 such that there is a cuspidal automorphic representation $\pi$ of $GL(3, F_{v_0})$ whose component at $v_0$ is $\pi_{i_0}$. The set $X$ is infinite.

**Proof of the Lemma.** For the convenience of the reader we provide a proof. For the proof of the lemma we fix the index $i$ and drop it from the notations and consider the representation

$$\pi_{i_0}^{s_{v_0}} := \text{Ind}(\pi_{i_1} v_0 \otimes \pi_{i_2} v_0 \otimes \pi_{i_3} v_0 \otimes \pi_{i_4} v_0).$$

Let $\mathcal{C}(G_{v_0}, \omega_{v_0}^{-1})$ be the space of smooth functions transforming under the character $\omega_{v_0}^{-1}$ and compactly supported modulo the center. We recall that as a consequence of Bernstein’s theory, there is a bi-invariant subspace $\mathcal{C}_0$ of the space $\mathcal{C}(G_{v_0}, \omega_{v_0}^{-1})$ with the following properties. There is a direct sum decomposition

$$\mathcal{C}(G_{v_0}, \omega_{v_0}^{-1}) = \mathcal{C}_0 \oplus \mathcal{C}_0^0$$

where $\mathcal{C}_0$ is also bi-invariant. Let $f \in \mathcal{C}_0$. For any irreducible admissible representation $\pi_{i_0}$ of $GL(3, F_{v_0})$ with central character $\omega_{v_0}$ we have $\pi_{i_0}(f) = 0$ unless $\pi = \pi_{i_0}$ for some $i$. If $f$ is given and $f_0$ is its projection on $\mathcal{C}_0$ in the above decomposition then $\pi_{i_1}(f) = \pi_{i_2}(f_0)$ for any $i$. Given $m + 1$ complex numbers $z_0, z_1, z_2, ..., z_m$ one can find $f \in \mathcal{C}(G_{v_0}, \omega_{v_0}^{-1})$ such that $\pi_{i_0}(f) \neq 0$ but $\pi_{i_1}(f) = 0$, $1 \leq i \leq m$. This follows from the fact the representations are irreducible and inequivalent. Thus there is an element of $\mathcal{C}_0$ with the same property, namely $f_0$. In particular, suppose $Y$ is a set of complex numbers of module 1 with the following property of density: for $f \in \mathcal{C}_0$ the
relations $\pi_z(f) = 0$ for all $y \in Y$ imply $f = 0$. Then the set $Y$ must be infinite. Thus it suffices to show $X$ has precisely this property of density.

To show this is the case, consider an element $f_0 \in \mathcal{G}_0$. Define a function

$$f_{v_0}(g) = \int_{G_{v_0}} f_0(gx) f_0(x) \, dx$$

on $G_{v_0}$. Choose another place $v_1$ and a cuspidal element $f_{v_1}$ of $\mathcal{E}(G_{v_1}, \omega_{v_1})$ with $f_{v_1}(e) \neq 0$. Choose also an element $f^{v_0v_1}$ of $\mathcal{E}(G^{v_0v_1}, \omega^{v_0v_1})$ and set

$$f = f_{v_0} f_{v_1} f^{v_0v_1}, \quad f^* = f_{v_0} f_{v_1} f^{v_0v_1}.$$

Consider the sums

$$\phi(g) = \sum_{\gamma \in Z(F) \backslash G(F)} f(\gamma g), \quad \phi'(g) = \sum_{\gamma \in Z(F) \backslash G(F)} f'(\gamma g).$$

Both functions are cuspidal. Let again $\tilde{\pi}_x$ denote the set of cuspidal representations with central character $\omega$ and let $\phi_x, \phi'_x$ denote the corresponding orthogonal projections. Then

$$\phi_x = \pi_{v_0v_1}(f_0) \phi'_x.$$

Thus if $\phi_x \neq 0$ then $\pi_{v_0v_1} = \pi_z$ for some $z \in X$. Now suppose that $f_0$ is such that $\pi_z(f_0) = 0$ for all $z \in X$. Then $\phi_x = 0$ for all $x$ and so $\psi = 0$. In particular

$$\sum f(\gamma) = 0,$$

for all choices of $f^{v_0v_1}$. This implies $f_{v_1}(e) = 0$ and so $f_0 = 0$. Thus $X$ is infinite, as claimed.

For any $z$ of module 1, let $\Pi_z$ be the base change of the representation $\pi_z$. Let $\mathcal{B}^{v_0v_1}$ be the distribution corresponding to $\mathcal{B}_{v_0v_1}$. For $z \in X$, there is thus a linear form in $\mathcal{H}(\Pi_z^*, H)$ with the required property. More precisely, let $\mathcal{W}^z$ be any non-zero Whittaker linear form on $\mathcal{H}(\Pi_z^*)$, then there is a unique $\mathcal{B}_z \in \mathcal{H}(\Pi_z^*)$ such that

$$\mathcal{B}_z^z(f) = \sum_{w} \mathcal{W}^z(\Pi_z^*(f) \, W_w) \mathcal{B}_z(W_w).$$

Our task is to show that for every $z$ (of module 1) there is such a linear form.

At this point, we may as well revert to a local notation, writing our extension as $E/F$ and writing simply $H^\tau$ rather than $H^\tau_z$ and so on. We may
regard the representations $\pi^{h,z}$ as a fiber bundle of representations. In a precise way, we set $K' = GL(3, \mathbb{C})$ and let $V'$ be the space of smooth functions $\phi: K' \to \pi(\pi_1)$ such that $\phi(mk) = \pi_1 \times \pi_2(m)$ for $m \in M(F) \cap K'$. Then we may regard all the representations $\pi^{h,z}$ as operating on $V'$. For every $u \in V'$ the map $z \mapsto \pi^{h,z}(u)$ takes its values in a fixed finite dimensional vector space and is a polynomial function of $z$. Similarly, there is a holomorphic family of Whittaker linear forms $W^z$ on $V'$. More precisely, for every $u \in V'$ the map $z \mapsto W^z(u)$ is a polynomial in $z$. Now suppose that $f'$ is bi-invariant under the compact open subgroup $K'_1$. Then the Bessel distribution corresponding to $W^z$ is given by:

$$B^z_i(f') = \sum_n \pi(n) \pi(n)^z(u_n) \overline{W^z(u)}$$

where $u_n$ is a fixed orthonormal basis of the space of $K'_1$-invariant vectors in $V$. It follows that

$$z \mapsto B^z_i(f')$$

is a polynomial in $z$. The same is therefore true of the map $z \mapsto \overline{B^z_i(f')}$.

We introduce the notation of generalized vector and write

$$W^z(W) = (W_1, W^z), \mathcal{P}^z(W) = (W_1, \mathcal{P}^z).$$

In terms of generalized vectors, we see that for $z \in X$:

$$(\Pi^z(f) \mathcal{P}^z, W^z) = \overline{B^z_i(f)}.$$
where $W_z$ is a function such that
\[
W_z(ngk_1) = \theta(n\tilde{m}) W_z(g),
\]
for $n \in N(E)$ and $k_1 \in K_1$. For every $g$ the map $z \mapsto W_z(g)$ is a polynomial in $z$.

We claim that $W_z$ belongs to the Whittaker model of $\Pi^z$. This is true for $z \in X$. Indeed, for $z \in X$, we have:
\[
W_z(g) = (\Pi^z(g) P_{K_1} \phi_z^\dagger, \phi^z).
\]

Again the representations $\Pi^z$ form a fiber bundle of representations all operating on the same space $\gamma'$ smooth of functions on $K = GL(3, \mathbb{C})$ with values in $\gamma'(\Pi_1)$. There is also an analytic family of Whittaker linear forms $\phi^z$ on $\gamma'$. Let $e_\mu$, $1 \leq \mu \leq M$, be a basis of the space of vectors in $\gamma'$ invariant under $K_1$. The functions $W^z_\mu$ defined by:
\[
W^z_\mu(g) = \phi^z(\Pi^z(g) e_\mu)
\]
form a basis of the space of $K_1$ invariant elements in the Whittaker model of $\Pi^z_\mu$. Thus for $z \in X$ we have a unique decomposition:
\[
W_z(g) = \sum_{\mu \leq M} \lambda_\mu(z) W^z_\mu(g).
\]

Let $z_0$ be a point not in $X$. Now choose $M$ elements $(g')$ such that
\[
D(z) := \det(W^z_\mu(g'))
\]
is non-zero at $z_0$. Then $D(z) \neq 0$ on a subset $X_0$ of $X$ which is also infinite. For $z \in X_0$ the scalars $(\lambda_\mu(z))$ are solutions of the Cramer system
\[
W_z(g') = \sum_{\mu \leq M} \lambda_\mu(z) W^z_\mu(g').
\]
Thus there exist polynomials $(P_\mu)$ such that
\[
\lambda_\mu(z) = \frac{P_\mu(z)}{D(z)}
\]
for $z \in X_0$. Then
\[
W_z(g) = \sum_{\mu \leq M} \frac{P_\mu(z)}{D(z)} W^z_\mu(g)
\]
for $z \in X_0$. Thus the same relation is true at $z_0$. Our assertion follows.
This being so the above result for \( z = 0 \) amounts to saying that for every \( K \) there is a unique vector \( u_{K_1} \) invariant under \( K_1 \) such that, for any function \( f \),

\[
\mathcal{R}'(f^{K_1}) = (\Pi(f) u_{K_1}, \mathcal{W}).
\]

Now if \( K_2 \supseteq K_1 \) then \( (f^{K_2})^{K_1} = f^{K_2} \). Thus

\[
\mathcal{R}'(f^{K_2}) = (\Pi(f^{K_2}) u_{K_2}, \mathcal{W}) = (\Pi(f) P_{K_1} u_{K_1}, \mathcal{W}).
\]

It follows that \( P_{K_1} u_{K_1} = u_{K_2} \). It follows there is a generalized vector \( \mathcal{P} \) such that

\[
\mathcal{R}'(f) = (\Pi(f) \mathcal{P}, \mathcal{W}).
\]

By definition the distribution \( \mathcal{R}' \) is invariant under \( H \) on the right. Thus the generalized vector \( \mathcal{P} \) is also invariant and we are done.

### 8. CONCLUDING REMARKS

The same technics can be used to prove the conjecture for other representations. As a matter of fact, this is done in [LR]. However, it is difficult to prove the conjecture for all representations thus we prefer to limit ourself to the above cases.

One expects the above results to generalize in a straightforward way to the groups \( GL(n) \) with \( n \) odd. For \( n \) even, the situation is more complicated. Even if we assume that the infinite places of \( F \) split in \( E \), we have to deal with more than one unitary group. Thus it is reasonable to conjecture that a cuspidal representation which is a base change is distinguished by some unitary group; it is then a separate issue to show that it is in fact distinguished with respect to the quasi-split group \( H \), that is the unitary group for the Hermitian matrix \( w \) with entries \( w_{i,j} = \delta_{i+j, n+1} \). Assuming that it is the case, it is best to introduce the similitude group \( \tilde{H} \) and global period integrals of the form:

\[
\int_{Z(E_\mathbb{A}) \backslash \tilde{H}(E_\mathbb{A})} \phi(h) \omega^{-1}(\lambda(h)) \, dh.
\]

Such an integral should be non-identically zero if and only if \( II \) is the base change of a cuspidal representation \( \pi \) with central character \( \omega \sigma_{E/F}^2 \). It should factor as product of local invariant linear forms.
Now we discuss the local situation for \( n \) even. So let \( E/F \) be a local non-Archimedean quadratic extension. For every generic irreducible representation \( \Pi \), we should introduce, for a character \( \omega \) of \( F^\times \), the space \( \mathcal{H}(\Pi, \hat{\Pi}, \omega) \) of linear forms \( \mathcal{P} \) such that

\[
\mathcal{P}(\Pi h) u = \omega(\lambda(h)) \mathcal{P}(u).
\]

Then it should be that \( \mathcal{H}(\Pi, \hat{\Pi}, \omega) \neq 0 \) if and only if \( \Pi \) is the base change of at least one representation \( \pi \) with central character \( \omega \circ \phi_{E/F} \). This conjecture is motivated by the following property of the local transfer factor \( \gamma(\cdot, \cdot_0) \), at an inert place \( v_0 \): the transfer factor is a function defined on the group \( A(F_{v_0}) \) of diagonal matrices with entries in \( F_{v_0} \). If \( n \) is even, for any scalar matrix \( z \in F_{v_0}^\times \),

\[
\gamma(az, \psi) = \gamma(a, \psi) \omega_{E/F}(z)^{n/2}.
\]

Going back to a local situation, for global purposes, we deal with functions \( \Phi \) supported on the set of Hermitian matrices in the orbit of \( w \), that is, whose determinant is in \( \det w F^+ \). The matching functions \( f' \) are supported on the set \( w G^+ \). It is more convenient to consider the symmetric space \( \Xi_\omega \) of matrices \( s \) such that \( s = s^* \) where we have set \( g^* := w g^* w \). The group \( GL(n, E) \) operates on \( \Xi_\omega \) by \( s \mapsto g s^* g^{-1} \). We consider the orbit of \( w \), that is, the set \( \Xi_\omega^+ \) of matrices with \( \det s \in F^+ \). Then the condition of matching reads

\[
\Phi(g g^*) = \int f(g h) \, dh,
\]

\[
\int \Phi(n a w n^*) \theta(n_1) \, dn = \gamma(a, \psi) \int f'(n_1 a w n_2) \theta(n_1) \theta(n_2) \, dn_1 dn_2.
\]

The relative Bessel distribution is defined in the same way as before but the Bessel distribution is now defined by:

\[
\mathcal{B}_\pi(f') := \sum_x \#(\pi(f')) W_x \overline{\#(W_x)}.
\]

The distributions \( \mathcal{B}_\pi \) and \( \mathcal{B}_{\pi \otimes \phi_{E/F}} \) have the same restriction to \( G^+ \). We have the following lemma:

**Lemma 7.** The restriction of \( \mathcal{B}_\pi \) to \( G^+ \) is non-zero. Let \( \pi_i \), \( 1 \leq i \leq m \) be a family of irreducible generic representations of \( GL(n, F) \) such that for any pair \((i, j)\), \( i \neq j \), the representations \( \pi_i \) and \( \pi_j \) (resp. \( \pi_i \) and \( \pi_i \otimes \phi_{E/F} \)) are
inequivalent. Then the restrictions of the Bessel distributions \( \mathcal{B}_n \) to \( G^+ \) are linearly independent.

**Proof of the Lemma.** Indeed, the first assertion is clear if the restriction of \( \pi \) to \( G^+ \) is irreducible (see Lemma (1)). Suppose it is not. Then \( \pi \) is induced by an irreducible representation \( \pi_0 \) of \( G^+ \) and \( \pi = \pi \otimes \omega_{EF} \). Suppose that \( W \in \mathcal{W}(\pi) \) is such that \( \mathcal{W}(W) \neq 0 \). Then the function \( W_i \) defined by:

\[
W_i(g) = \frac{1}{2} (W(g) + W(g) \omega_{EF}(\det g))
\]

is a non-zero element of \( \mathcal{W} \) supported on \( G^+ \). It follows that

\[
\mathcal{W}(\pi) = \mathcal{W}^+ \oplus \pi(r) \mathcal{W}^+,
\]

where \( \mathcal{W}^+ \) is the space of elements of \( \mathcal{W}(\pi) \) supported on \( G^+ \) and\( \det r \neq F^+ \). It follows that the representation \( \pi^+ \) on \( \mathcal{W}^+ \) by right shifts is equivalent to \( \pi_0 \) or \( \pi_0^+ \) and is irreducible. Since \( \mathcal{W} \) vanishes on \( \pi(r) \mathcal{W}^+ \), for \( f' \) supported on \( G^+ \), we may take the sum defining the Bessel distribution over a basis of \( \mathcal{W}^+ \). The first assertion follows.

To prove the second assertion, for each \( i \), denote by \( \pi_{i,0} \) the restriction of \( \pi_i \) to \( G^+ \), if this restriction is irreducible, or the irreducible representation of \( G^+ \) which induces \( \pi_i \) if not. Then the representations \( \pi_{i,0} \) are irreducible and pairwise inequivalent. By the first part of the proof, the restriction of the Bessel distribution \( \mathcal{B}_n \) to \( G^+ \) may be viewed as a (generalized) matrix coefficient of \( \pi_{i,0} \). The second assertion follows.

The lemma implies that in the (still conjectural) relative trace formula, for a global extension \( E/F \) of number fields, where the infinite places of \( F \) split in \( E \), a cuspidal automorphic representation of \( GL(n, F_{\infty}) \) cannot give a zero contribution, even if we consider only functions \( f' \) supported on \( G^+ \). This will prove that any base change representation of \( GL(n, E_{\infty}) \) is distinguished by a quasi-split unitary group. Note that this argument is insufficient if some real place of \( F \) is inert in \( E \).

The lemma also suggests the following conjecture: suppose that \( \mathcal{H}(\Pi, \tilde{H}, \omega) \) is not zero. Let \( \pi_1, \pi_2, \pi_3, \ldots, \pi_m \) be representations of \( GL(n, F) \) with central character \( \omega \omega_{EF}^{\mathbb{Z}} \) which base change to \( \Pi \). We assume that for any other representation \( \pi \) which base change to \( \Pi \) there is exactly one index \( i \) such that either \( \pi = \pi_i \) or \( \pi = \pi_i \otimes \omega_{EF} \). Then, for each index \( i \), there is a unique element \( \mathcal{P}_i \) of \( \mathcal{H}(\Pi, \tilde{H}, \omega) \) such that

\[
\mathcal{H}_i(f) = \mathcal{B}_n(f')
\]

if \( f \) and \( f' \) have matching orbital integrals. Moreover, the vectors \( \mathcal{P}_i \), \( 1 \leq i \leq m \), form a basis of \( \mathcal{H}(\Pi, \tilde{H}, \omega) \).
REFERENCES


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