Annals of Mathematics, **109** (1979), 213-258

Automorphic forms on GL(3) II

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8. Generic representations for archimedean fields

In Sections 8 to 11, the ground field F is **R** or **C**. We extend the results of the previous sections to that case; however we will limit ourselves to the minimum needed for the global applications.

In Sections 8 and 9, the integer r is arbitrary.

(8.1) Let π be a *unitary* representation of $G_r(F)$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^\infty$ be the space of C^∞-vectors. Denote by $\mathfrak{g}$ the Lie-algebra of the *real* Lie-group $G_r(F)$ and by $\mathfrak{U}$ the complex enveloping algebra of $\mathfrak{g}$. We identify $\mathfrak{U}$ with the convolution algebra of distributions on $G_r(F)$ with support contained in $\{e\}$. Both $G_r(F)$ and $\mathfrak{U}$ operate on $\mathcal{H}^\infty$. For instance, if X is in $\mathfrak{g}$ then, for $v \in \mathcal{H}^\infty$,

$$\pi(X)v = \frac{d}{dt}\pi(\exp tX)v\Big|_{t=0} .$$

We equip $\mathcal{H}^\infty$ with the topology defined by the semi-norms

$$||v||_D = ||\pi(D)v|| ,$$

where D is in $\mathfrak{U}$. Let θ be the character defined by (2.1.1) or more generally any generic character of $N(F)$. We will denote by $\mathcal{H}_\theta^*$ the space of all *continuous* linear forms λ on $\mathcal{H}^\infty$ such that

0003-486X/79/0109-2/0213/046 $ 02.30/1

* Sections 0 to 7 appeared in the previous issue.

$$\lambda[\pi(n)v] = \theta(n)\lambda(v)\ , \qquad \text{for } n \in N(F)\ , \quad v \in \mathcal{H}^\infty\ .$$

We shall say that π is *generic* if $\mathcal{H}_\theta^*$ is non-zero. Then by Theorem 3.1 of [34] it is one-dimensional if π is irreducible.

(8.2) Suppose π is a generic representation of $G_r(F)$ on a Hilbert space $\mathcal{H}$. Choose $\lambda \neq 0$ in $\mathcal{H}_\theta^*$. We shall denote by $\mathcal{W}(\pi; \psi)$ the space of all functions W on $G_r(F)$ of the form

$$W(g) = \lambda(\pi(g)v)\ , \qquad v \in \mathcal{H}^\infty\ .$$

We let K be the standard maximal compact subgroup of $G_r(F)$. We denote by $\mathcal{H}_0$ the space of K-finite vectors in $\mathcal{H}$ and by $\mathcal{W}_0(\pi; \psi)$ the subspace of functions W of the above form with v in $\mathcal{H}_0$. We note that since the character of π is a function, the representation $\tilde{\pi} = \bar{\pi}$ contragredient to π is equivalent to π^ι. In particular the statement analogous to (2.1.3) holds in the present case.

(8.3) We shall need some information on the behavior of the functions $W \in \mathcal{W}_0(\pi; \psi)$ at infinity. For $g \in G_r(F)$, set

$$||g|| = |\det g|^{-1/r}(\sum g_{ij}^2)^{1/2}$$

if $F = \mathbf{R}$, and

$$||g|| = (\det g\bar{g})^{-1/r} \sum g_{ij}\bar{g}_{ij}$$

if $F = \mathbf{C}$.

Clearly $||gh|| \leqq ||g||\,||h||$, $||g|| \geqq 1$ (if $r > 1$).

LEMMA (8.3.1). *There is a $t \geqq 0$ and there are $D_i \in \mathfrak{U}$ such that*

$$|W(g)| \leqq ||g||^t \sum_i ||\pi(D_i)W||$$

for all $g \in G(F)$ and $W \in \mathcal{W}_0(\pi; \psi)$.

($||W||$ is the norm of $W \in \mathcal{W}_0(\pi; \psi) \subseteq \mathcal{H}$).

Proof. Indeed, since λ is continuous, there are D_j in $\mathfrak{U}$ such that

$$|W(e)| \leqq \sum_j ||\pi(D_j)W||\ .$$

Apply this relation to $\pi(g)W$ to obtain

$$|W(g)| \leqq \sum_j ||\pi(D_j)\pi(g)W|| = \sum_j ||\pi(g^{-1}D_j g)W||\ .$$

Now one can find finitely many elements D_α of $\mathfrak{U}$ such that

$$g^{-1}D_j g = \sum \lambda_{j,\alpha}(g)D_\alpha\ ,$$

where the $\lambda_{j,\alpha}$ are coefficients of some finite dimensional representation of $G_r(F)/Z_r(F)$. Thus

$$|W(g)| \leqq \sum_{j,\alpha} |\lambda_{j,\alpha}(g)|\,||\pi(D_\alpha)W||\ .$$

There is a $t \geqq 0$ such that

$$|\lambda_{j,\alpha}(g)| \leqq ||g||^t$$

and we are done.

As in Section 2, we will introduce an ad hoc notion, the notion of a gauge. *A gauge* on $G(F)$ will be any function ξ such that

$$\xi(nak) = |a_1 a_2 \cdots a_{r-1}|_F^{-t} \phi(a_1, a_2, \cdots, a_{r-1}) , \tag{8.3.2}$$

for $n \in N(F)$, $k \in K$, and

$$a = \operatorname{diag}(a_1 a_2 \cdots a_r, a_2 \cdots a_r, \cdots, a_r) ,$$

where t is positive and $\phi \geqq 0$ is in $\mathcal{S}(F^{r-1})$.

We note that, if $t' > t$, there is a polynomial P (in the a_i if $F = \mathbf{R}$ and the a_i, $\bar{a}_i$ if $F = \mathbf{C}$) such that

$$|a_1 a_2 \cdots a_{r-1}|_F^{-t} \leqq |a_1 a_2 \cdots a_{r-1}|_F^{-t'} P(a_1, a_2, \cdots, a_{r-1}) .$$

It follows that for $t' > t$, any gauge ξ defined by t and ϕ is majorized by another gauge ξ' defined by t' and a suitable ϕ'. Similarly (2.3.4) and (2.3.5) are still true in the archimedean case. The result that we have in mind is

LEMMA (8.3.3). *If π is generic, for any $W \in \mathcal{W}_0(\pi; \psi)$, there is a gauge ξ which dominates W.*

Proof. It will suffice to show that given a compact set $\Omega \subset G_r(F)$, there are an $m \geqq 0$ and $\phi \geqq 0$ in $\mathcal{S}(F^{r-1})$ such that

$$|W(ag)| \leqq ||a||^m \phi(a_1, a_2, \cdots, a_{r-1}) ,$$

for $g \in \Omega$ and

$$a = \operatorname{diag}(a_1 a_2 \cdots a_{r-1}, \cdots, a_{r-1}, 1) .$$

There is an $f \in C_c^\infty(G(F))$ such that

$$W = W * \check{f} .$$

Since

$$W(ag) = W * \check{f} * \varepsilon_{\check{g}^{-1}}(a) ,$$

it suffices to obtain a majorization of $W * \check{f}(a)$ which is uniform for f in a bounded subset $\mathcal{B}$ of $C_c^\infty(G(F))$. Now

$$\begin{aligned} W * \check{f}(a) &= \int_{G(F)} W(ah) f(h) dh \\ &= \int_{N(F)\backslash G(F)} W(ah) dh \int_{N(F)} \theta(ana^{-1}) f(nh) dn . \end{aligned} \tag{8.3.4}$$

There is a compact subset Ω of $N(F)\backslash G(F) \simeq A(F)K$ such that for $f \in \mathcal{B}$, the inner integral vanishes unless $h \in \Omega$. Moreover, let V be the derived group of N. Let

$$\phi_h(x_1, x_2, \cdots, x_{r-1}) = \int_V f(vuh)dv ,$$

where

$$u = \begin{pmatrix} 1 & x_1 & & & 0 \\ & 1 & x_2 & & \\ & & \cdot & \cdot & \\ & & & \cdot & x_{r-1} \\ & & & & \cdot \\ 0 & & & & 1 \end{pmatrix} .$$

Then ϕ_h belongs to $\mathcal{S}(F^{r-1})$ and stays in a bounded set if h is in Ω and f in $\mathcal{B}$. The inner integral in (8.3.4) is nothing but the Fourier transform $\hat{\phi_h}$ evaluated at $(a_1, a_2, \cdots, a_{r-1})$. For $h \in \Omega$ and $f \in \mathcal{B}$, it stays in a bounded set of $\mathcal{S}(F^{r-1})$ so that there is $\phi \geqq 0$ in $\mathcal{S}(F^{r-1})$ such that

$$|\hat{\phi_h}| \leqq \phi .$$

The total integral is, by (8.3.1), dominated by a constant times

$$\int_\Omega ||ah||^t dh \phi(a_1, a_2, \cdots, a_{r-1}) \leqq ||a||^t \phi(a_1, a_2, \cdots, a_{r-1}) \int_\Omega ||h||^t dh$$

and the lemma follows.

We remark that a similar result was available to Harish-Chandra (private communication).

9. Some auxiliary integrals (archimedean fields)

In this section, the ground field is **R** or **C**.

(9.1) We establish the analogue of Theorem (3.1) for unitary representations and archimedean fields. However we prove only the results we need for the global theory.

We first review the results of [17]. Let π be an irreducible unitary representation of G. We will again consider the integrals

(9.1.1) $$Z(\Phi, s, f) = \int_G \Phi(x) f(x) |\det x|^s d^\times x ,$$

but now f is restricted to being a bi-K-finite coefficient of π and Φ will be in the subspace $\mathcal{S}(r \times r, F; \psi)$ of $\mathcal{S}(r \times r, F)$ as defined on page 115 of [17]. That subspace is dense in the entire Schwartz space, invariant under K acting on the right and left, invariant by the enveloping algebra of G and also by the Fourier transform. Each integral (9.1.1) converges in a half-space and extends to a meromorphic function of s. More precisely

$$Z(\Phi, s + (r-1)/2, f) = L(s, \pi) P(s) c^s ,$$

where P is a polynomial in s, $c > 0$ a constant depending on the choice of ψ, and $L(s, \pi)$ a function of the form

$$Q(s)\prod_i G_1(s + s_i)\prod_j G_2(s + s_j) ,$$

where Q is a fixed polynomial and

$$G_1(s) = \pi^{-s/2}\Gamma(s/2) , \qquad G_2(s) = (2\pi)^{1-s}\Gamma(s) .$$

Moreover when f and ϕ vary, the polynomials P span the ring $\mathbf{C}[s]$. Finally one has a functional equation

$$\begin{aligned} &Z(\Phi^\wedge, 1 - s + (r - 1)/2, f^\iota)/L(1 - s, \pi^\iota) \\ &\quad = \varepsilon(s, \pi, \psi)Z(\Phi, s + (r - 1)/2, f)/L(s, \pi) . \end{aligned} \tag{9.1.2}$$

Here $\Phi^\wedge$ and f^ι are as in (1.1) and $\varepsilon(s, \pi, \psi)$ has the form ac^{bs} for suitable constants a and b.

PROPOSITION (9.2). *Let π be an irreducible unitary generic representation of $G_r(F)$, $F = \mathbf{R}$ or $\mathbf{C}$.*

1) *For W in $\mathcal{W}_0(\pi; \psi)$ and Φ in $\mathcal{S}(r \times r, F)$, the integrals*

$$Z(\Phi, s, W) = \int\Phi(x) W(x)|\det x|^s d^\times x$$

converge absolutely in some right half-plane $\operatorname{Re} s > s_1$.

2) *They extend to the whole complex plane as meromorphic functions of s. If s_0 is large and P is a polynomial which cancels the poles of $L(s, \pi)$ in the strip $1 - s_0 \leqq \operatorname{Re}(s) \leqq s_0$, then the product*

$$Z(\Phi, s + (r - 1)/2, W)P(s)$$

is holomorphic and bounded in the same strip.

3) *The functional equation* (3.1.2) *is satisfied.*

Proof. Since each $W \in \mathcal{W}_0(\pi; \psi)$ is majorized by a gauge, the first assertion is easily proved.

Let us prove the second assertion at first for Φ in $\mathcal{S}(r \times r, F; \psi)$. Since Φ is right K-finite, there is a function ξ on K which is a sum of irreducible characters of K divided by their degree (an "elementary idempotent") such that

$$\Phi(x) = \int_K \Phi(kx)\xi(k)dk .$$

Thus for $\operatorname{Re}(s)$ large

$$Z(\Phi, s, W) = Z(\Phi, s, f) ,$$

where

$$f(g) = \int_K W(k^{-1}g)\xi(k)dk .$$

If $\lambda \neq 0$ is in π_0^*, then W has the form $W(g) = \lambda(\pi(g)v)$ for some v in $\mathcal{H}_0$ (Notations of (8.1) and (8.2).) Then:

$$f(g) = \mu(\pi(g)v)$$

where μ is the linear form on $\mathcal{H}_0$ (or $\mathcal{H}^\infty$) defined by

$$\mu(v) = \int_K \lambda(\pi(k)v)\xi(k^{-1})dk .$$

Clearly μ is K-finite. Thus f is a bi-K-finite coefficient of π. By (9.1),

$$Z(\Phi, s + (r - 1)/2, f) = L(s, \pi)R(s)c^s ,$$

where R is a polynomial. Thus, by Stirling's formula, if P cancels the poles of $L(s, \pi)$ in the strip $1 - s_0 \leqq \operatorname{Re}(s) \leqq s_0$, then $L(s, \pi)\ P(s)$ is holomorphic and rapidly decreasing in the same strip. Thus $P(s)\ Z(\Phi, s + (r - 1)/2, W)$ is also holomorphic and bounded in the same strip. We also obtain the third assertion (for Φ in $\mathcal{S}(r \times r, F; \psi)$) exactly as in Section 3.

Before extending these results to all of $\mathcal{S}(r \times r, F)$, we prove a lemma.

LEMMA (9.2.4). *There is an $s_2 > s_1$ with the following property. For any polynomial P the product*

$$\phi(s) = P(s)Z(\Phi, s, W)$$

is bounded in any vertical strip of the half-plane $\operatorname{Re}(s) > s_2$. *Moreover, if Φ approaches zero in $\mathcal{S}(r \times r, F)$, the function ϕ approaches zero uniformly in the same strip.*

Proof. Replacing π by $\pi \otimes \alpha^{i\sigma}$ if necessary, we may assume that π is trivial on the subgroup $\mathbf{R}_+^\times$ of the center of G. Then, for $\operatorname{Re}(s)$ large,

$$Z(\Phi, s, W) = \int_0^\infty t^{nrs} H(t)d^\times t ,$$

where $n = [F:\mathbf{R}]$, $d^\times t = dt/t$, and

$$H(t) = \int_{G_0} \Phi(tx)W(x)dx .$$

Here $G_0 = \{g \in G \mid |\det g| = 1\}$. Since W is dominated by a gauge, it is not hard to see that H is of slow increase for t small and rapid decrease for t large. Moreover

$$t\frac{d}{dt}\Phi(tx) = \Phi_1(tx) ,$$

where Φ_1 is again in the Schwartz space.

An integration by parts gives, for $\operatorname{Re}(s)$ sufficiently large,

$$Z(\Phi, s, W) = -\int_0^\infty \frac{t^{nrs}}{nrs} H_1(t)d^\times t ,$$

with

$$H_1(t) = \int_{G_0} \Phi_1(tx) W(x) dx \ .$$

By induction we get, for Re(s) large,

$$(nrs)^k Z(\Phi, s, W) = (-1)^k \int_0^\infty t^{nrs} H_k(t) d^\times t \ ,$$

where $H_k(t) = \int_{G_0} \Phi_k(tx) W(x) dx$ and Φ_k depends continuously on Φ. The first assertion follows. As for the second we need only remark that if $\Phi^\alpha \to 0$ then $|\Phi^\alpha| \leqq \Phi_0$ for some non-negative Schwartz function Φ_0.

The lemma being proved, we establish the second assertion of (9.2) for an arbitrary Φ. Choose a sequence Φ_i in $\mathcal{S}(r \times r, F; \psi)$ which approaches Φ. We take $s_0 + (r-1)/2$ larger than the s_2 of (9.2.4). Set

$$\phi_i(s) = P(s) Z\big(\Phi_i, s + (r-1)/2, W\big) \ ,$$

P a fixed polynomial.

By (9.2.4) applied to a vertical line, given ε, there is an integer N_1 such that for $i, j \geqq N_1$ and Re$(s) = s_0$,

$$|\phi_i(s) - \phi_j(s)| \leqq \varepsilon \ . \tag{9.2.5}$$

Now fix P so that $L(s, \pi) P(s)$ is holomorphic in the strip $1 - s_0 \leqq \mathrm{Re}(s) \leqq s_0$. We have seen that each $\phi_i(s)$ is bounded in the strip $1 - s_0 \leqq \mathrm{Re}(s) \leqq s_0$. Assume at the moment that $|\phi_i(s) - \phi_j(s)| \leqq \varepsilon$ for $\mathrm{Re}(s) = 1 - s_0$ for $i, j \geqq N_2$. Let $N = \max(N_1, N_2)$. We contend then, that for $i, j \geqq N$, $|\phi_i(s) - \phi_j(s)| \leqq \varepsilon$ throughout the strip $1 - s_0 \leqq \mathrm{Re}(s) \leqq s_0$. In fact, fix $i, j \geqq N$. Each function $s\big(\phi_i(s) - \phi_j(s)\big)$ is bounded in the full strip. Thus if C_{ij} is large enough

$$|\phi_i(s) - \phi_j(s)| \leqq \varepsilon \quad \text{if} \quad t = |\mathrm{Im}(s)| \geqq C_{ij} \ .$$

Thus $|\phi_i(s) - \phi_j(s)| \leqq \varepsilon$ on each side of the rectangle bounded by $\mathrm{Re}(s) = s_0$, $\mathrm{Re}(s) = 1 - s_0$, $\mathrm{Im}(s) = C_{ij}$, $\mathrm{Im}(s) = -C_{ij}$. Hence thus is true throughout the rectangle. Letting C_{ij} increase, we obtain finally $|\phi_i(s) - \phi_j(s)| \leqq \varepsilon$ in the full strip.

Thus ϕ_i converges to a function ϕ in $1 - s_0 \leqq \mathrm{Re}(s) \leqq s_0$. The convergence being uniform, ϕ is holomorphic in the open strip, bounded in the closed strip.

We have, taking s_0 larger if necessary,

$$\phi_i(s) = P(s) \int \Phi_i(x) W(x) |\det x|^{s + (r-1)/2} d^\times x$$

if $s_3 \leqq \mathrm{Re}(s) \leqq s_0$. Taking limits we see that ϕ is an analytic continuation of $P(s)\ Z\big(\Phi, s + (r-1)/2, W\big)$ to $1 - s_0 < \mathrm{Re}(s) < s_0$. Finally, taking s_0 larger if necessary, we obtain the second assertion of (9.2).

It remains to show then that, for s_0 large enough, $\phi_i \to 0$ uniformly on the line $\operatorname{Re}(s) = 1 - s_0$. We have, with the above notation,

$$\phi_i(s) - \phi_j(s) = \varepsilon(s, \pi, \psi)^{-1} L(s, \pi)/L(1 - s, \tilde{\pi})$$
$$\times P(s)[Z(\Phi'_i, 1 - s + (r - 1)/2, \tilde{W}) - Z(\Phi'_j, 1 - s + (r - 1)/2, \tilde{W})] .$$

Suppose $\operatorname{Re}(s) = 1 - s_0$. The last factor tends to 0 uniformly. Write

$$L(s, \pi) = ac^{bs} Q(s) \prod_{i,j} \Gamma\left(\frac{1}{2}(s + a_i)\right)\Gamma(s + b_j) .$$

Since $\bar{\pi} = \tilde{\pi}$,

$$L(s, \tilde{\pi}) = \bar{a} c^{\bar{b}s} \overline{Q(\bar{s})} \prod_{i,j} \Gamma\left(\frac{1}{2}(s + \bar{a}_i)\right)\Gamma(s + \bar{b}_j) .$$

Here Q is a polynomial. Since we may enlarge s_0, we may assume that $Q(1 - \bar{s})$ in non-zero for $\operatorname{Re}(s) = 1 - s_0$. We may also assume that the gamma factors are holomorphic on this line. Write $s = \sigma + it$. Clearly $Q(\sigma + it)/Q(1 - \sigma + it)$ is bounded at infinity on any line. It remains to show say that $\Gamma(s + b)/\Gamma(1 - \bar{s} + b)$ is bounded on $\sigma = 1 - s_0$, if s_0 is large. In fact this follows immediately from the asymptotic expression

$$|\Gamma(\sigma + it)| \sim (2\pi)^{1/2} e^{-\pi t/2} t^{\sigma - 1/2} \quad \text{as} \quad |t| \longrightarrow \infty .$$

Finally the last assertion of (9.3) follows by continuity.

10. Problems of classifications: Archimedean case

We extend the results of Section 6 to the archimedean case. We consider almost exclusively *unitary* representations. Few proofs are given.

(10.1) The analogue of Proposition (6.1.1) is somewhat weaker.

PROPOSITION (10.1.1). *With the notation of* (6.1.1), *let σ_i be irreducible unitary and ξ be given by Mackey's construction. Then ξ admits at most one irreducible generic subrepresentation. Moreover if ξ contains such a subrepresentation then each σ_i is generic.*

As in the non-archimedean case we have the notion of a strongly generic unitary representation. Proposition (6.1.2) is still true as well as the ensuing remarks. Every strongly generic representation is also generic. Indeed the space of C^∞-vectors in π is contained in the space of C^∞-vectors in $\pi | P^1$. Since $\pi | P^1 \simeq I(P^1, N; \theta)$ we may regard the former space as being contained in the space of C^∞-functions on P^1. The inclusion being continuous, the proof is then the same as before.

The classification of irreducible admissible representations of $G_r(F)$ (or its "Hecke-algebra") is similar to (6.2) [28]: there is a bijection $\sigma \leftrightarrow \pi(\sigma)$ between these representations and the set of classes of semi-simple r-dimensional representations of the W-group W_F.

(10.2) Suppose $r = 2$. If τ is an irreducible representation (semi-simple) of $W_{\mathbf{R}}$ of degree two, then $\pi(\tau)$ is square-integrable (mod Z_2). If τ is a reducible representation of W_F of degree two, $\tau = \mu_1 \oplus \mu_2$ and $\pi(\tau) = \pi(\mu_1, \mu_2)$. This is unitary if and only if either both μ_1 and μ_2 are unitary or if $\mu_1 = \chi\alpha^s$, $\mu_2 = \chi\alpha^{-s}$ with χ unitary and $0 < s < 1/2$ (the complementary series). The unitary square-integrable representations and the unitary representations of the form $\pi(\mu_1, \mu_2)$ exhaust all of the unitary generic representations of $G_2(F)$. As in the non-archimedean case they are strongly generic.

(10.3) For $r = 3$, we content ourselves with pointing out that (6.7) is still true. Of course π cannot be square-integrable. With reference to (6.7), $\pi = I(G, P; \sigma, \nu)$ is generic if and only if σ is and is then strongly generic.

Finally the unitary generic representations of $G_3(F)$ correspond to the following three-dimensional representation of W_F: $\sigma = \tau \oplus \mu$ where τ is a two-dimensional irreducible unitary and μ a character, $\sigma = \mu_1 \oplus \mu_2 \oplus \mu_3$ where each μ_i is a character, and $\sigma = \chi\alpha^s \oplus \chi\alpha^{-s} \oplus \mu$ where χ and μ are characters and $0 < s < 1/2$. The corresponding representations $\pi(\sigma)$ of $G_3(F)$ are $I(G, P; \pi(\tau), \mu)$, $I(G, B; \mu_1, \mu_2, \mu_3)$ and $I(G, P; \pi(\chi\alpha^s, \chi\alpha^{-s}), \mu)$ respectively. For these representations we have (cf. [9]):

$$L(s, \pi(\sigma)) = L(s, \sigma),\ \varepsilon(s, \pi(\sigma), \psi) = \varepsilon(s, \sigma, \psi) \quad L(s, \tilde{\pi}(\sigma)) = L(s, \tilde{\sigma})\ .$$

11. The groups GL(3, R) and GL(3, C)

We partially extend the results of Section 4 to the case $r = 3$, $F = \mathbf{R}$ or $\mathbf{C}$.

(11.1) If Φ is in $\mathcal{S}(3 \times 3, F)$, we can still define the measure ρ_Φ on $SL(3, F)$ as in (4.3). In general ρ_Φ is not of compact support. However the following lemma which we will need for the global theory is true (cf. (13.6)).

LEMMA (11.1.1). *Let f be a continuous function on* SL(3, F). *Suppose $\rho_\Phi(f) = 0$ whenever ρ_Φ has compact support. Then $f(e) = 0$.*

Proof. We will need an auxiliary lemma which is better stated for arbitrary r. For $F = \mathbf{R}$, let $K^0 = SO(r, \mathbf{R})$, and, for $F = \mathbf{C}$, let $K^0 = SU(r)$. Let U be the open subset of $M(r-1 \times r, F)$ consisting of the matrices of rank $r-1$. Call B^+_{r-1} the group of $r-1$ by $r-1$ upper triangular matrices with positive diagonal entries. We may write an element Y in U in the form

$$Y = bH$$

where b is in B^+_{r-1} and the rows of H form an orthonormal system. There is exactly one matrix $k \in K^0$ whose last $r-1$ rows are the rows of H. Thus we may write

$$Y = [0, b]k\,, \qquad b \in B^+_{r-1}\,, \quad k \in K^0\,.$$

Moreover in this expression b and k are unique. One thus obtains a diffeomorphism of U with $B^+_{r-1} \times K^0$.

Similarly let U' be the set of r by r matrices X of the form

$$X = \begin{pmatrix} Z \\ Y \end{pmatrix}$$

where $Z \in M(1 \times r, F) = F^r$ and $Y \in U$. Every such element may be written uniquely in the form

$$X = \begin{pmatrix} Zk^{-1} \\ 0 \ \ b \end{pmatrix} \text{ with } k,\ b \in B^+_{r-1},\ k \in K^0\,. \tag{11.1.2}$$

With these notations we have

LEMMA (11.1.3). *Let $\phi_1 \in \mathcal{S}(F^r)$, $\phi_2 \in C^\infty_c(B^+_{r-1} \times K^0)$, and P be a polynomial on F^r. Let ϕ be the function on $M(r \times r, F)$ defined by*

$$\Phi(X) = \phi_1(Zk^{-1})\phi_2(b, k)P(Z)$$

for X in U' as in (11.1.2) *and $\Phi(X) = 0$ for $X \notin U'$. Then Φ is in $\mathcal{S}(r \times r, F)$.*

Proof. Let $\mathcal{V}$ be the space of functions on $F^r \times B^+_{r-1} \times K^0$ spanned by all functions Ψ of the form

$$\Psi(Z, b, k) = \phi_1(Zk^{-1})\phi_2(b, k)P(Z)$$

with ϕ_1, ϕ_2, and P as above. It is clear that $\mathcal{V}$ consists of smooth functions and that $\mathcal{V}$ is stable under the action of right invariant vector fields on the Lie-group $F^r \times B^+_{r-1} \times K^0$. Moreover any element of $\mathcal{V}$ has support contained in a set of the form

$$F^r \times \Omega\,,$$

where Ω is a compact set in $B^+_{r-1} \times K^0$, and is majorized by any negative power of $1 + ||Z||^2$.

Let $\mathcal{W}$ be the space of functions Φ on U' of the form

$$\Phi(X) = \Psi(Z, b, k)$$

as above. Each Φ is smooth. By transport of structure, $\mathcal{W}$ is stable by any differential operator in Z or Y with constant coefficients. Moreover each

Φ has support in a set $F^r \times \Omega$ where Ω is compact in U. Thus each Φ in $\mathcal{W}$ has a smooth extension to $M(r \times r, F)$ which is zero outside U'. Thus we may identify $\mathcal{W}$ with a space of smooth functions on $M(r \times r, F)$ stable under differential operators with constant coefficients. Since

$$1 + \|X\|^2 \geqq 1 + \|Z\|^2 , \quad \text{for} \quad X = \begin{pmatrix} Z \\ Y \end{pmatrix} ,$$

each Φ in $\mathcal{W}$ is majorized by any negative power of $1 + \|X\|^2$. Thus $\mathcal{W}$ is contained in $\mathcal{S}(r \times r, F)$. Q.E.D.

To prove (11.1.1), we specialize Φ as follows. We suppose $r = 3$. We set, for $X \in U'$,

$$\Phi(X) = \Phi\left[\begin{pmatrix} u & x_{12} & x_{13} \\ 0 & u_2 & x_{23} \\ 0 & 0 & u_3 \end{pmatrix} k\right] = \phi_1(u)\phi_{12}(x_{12})\phi_{13}(x_{13})\phi_2(u_2)\phi_{23}(x_{23})\phi_3(u_3)\phi_0(k) ,$$

where $\hat{\phi}_1, \phi_{12}, \phi_{13}, \phi_{23} \in C_c^\infty(F)$, $\phi_2, \phi_3 \in C_c^\infty(\mathbf{R}_+^\times)$, and $\phi_0 \in C^\infty(K^0)$. As before we set $\Phi(X) = 0$ for $X \notin U'$. Then by (11.1.3), $\Phi \in \mathcal{S}(3 \times 3, F)$; with the notations of (4.3), we readily find, for $k \in K^0$,

$$K_\Phi(x, u_2, v_2, u_3, v_3; k) = \hat{\phi}_1(-x)\phi_2(u_2)\phi_3(u_3)\hat{\phi}_{12}(v_2)\hat{\phi}_{23}(v_3)\phi_0(k)\hat{\phi}_{13}(0) .$$

Thus, with $\hat{\phi}_{13}(0) = 1$, we obtain for f continuous on SL(3, F),

$$\rho_\Phi(f) = \int f\left[\begin{pmatrix} b^{-1}c^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} k\right]$$
$$\times\ \hat{\phi}_1(-x)\phi_2(b)\hat{\phi}_{12}(b^{-1})\phi_3(c)\hat{\phi}_{23}(c^{-1})\phi_0(k)\,|b|\,|c|^2 d^\times b\, d^\times c\, dx\, dk .$$

Clearly ρ_Φ has compact support. It is clear that if $\rho_\Phi(f) = 0$ for all the Φ we are considering, then $f(e) = 0$. This concludes the proof of (11.1.1).

(11.2) The main theorem of this section is

THEOREM (11.2). *Let π be a unitary generic representation of $G_3(F)$. Denote by $\mathcal{W}'(\pi; \psi)$ the space of functions of the form $W * \mu^\vee$ where W is in $\mathcal{W}_0(\pi; \psi)$ and μ is a measure of compact support of the form $\mu = \rho_\Phi$ on* SL(3, F). *Suppose $W \in \mathcal{W}(\pi; \psi)$. Then:*

(1) *The integrals*

$$\Psi(s, W) = \int W\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^\times a ,$$

$$\tilde{\Psi}(s, W) = \int \tilde{W}\left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w'\right] |a|^{s-1} d^\times a\, dx$$

converge for $\mathrm{Re}(s)$ *sufficiently large.*

(2) *They extend to the complex plane as meromorphic functions of s and as such satisfy*

$$\widetilde{\Psi}(1-s,\, W)/L(1-s,\, \tilde{\pi}) = \varepsilon(s,\, \pi,\, \psi)\Psi(s,\, W)/L(s,\, \pi)\,.$$

(3) *If* P(resp. $\widetilde{P}$) *is a polynomial which cancels the poles of* $L(s,\pi)$ (resp. $L(s,\tilde{\pi})$) *in the strip* $1-s_0 \leqq \mathrm{Re}(s) \leqq s_0$, *and* s_0 *is large enough,*

$$P(s)\Psi(s,\, W) \qquad (\text{resp. } \widetilde{P}(s)\widetilde{\Psi}(s,\, W))$$

is holomorphic and bounded in the strip.

Proof. Recall that if ξ is a gauge and Ω a compact set in $G(F)$, then there is a gauge ξ_0 so that $\xi(g\omega) \leqq \xi_0(g)$ for $g \in G(F)$, $\omega \in \Omega$. Thus if $W = W_0 * \mu^{\vee}$, where W_0 is in $\mathcal{W}_0(\pi;\psi)$ and μ is of compact support, W is bounded by a gauge. The first assertion for $\Psi(s,\, W)$ follows immediately. We prove the first assertion for $\widetilde{\Psi}(s,\, W)$. Since $g \mapsto \widetilde{W}(gw')$ is also majorized by a gauge, we have only to see that

$$\int \xi\left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^{\times}a\, dx$$

is convergent for s large. Say $F = \mathbf{R}$. Using the Iwasawa decomposition this has the form

$$\int \phi\big(a,\, (1+x^2)^{1/2}\big)|a|^{s-1-t}(1+x^2)^{s-1-t/2} d^{\times}a\, dx$$

for suitable ϕ in $\mathcal{S}(F^2)$ and is clearly convergent for s large. The proof is similar for $F = \mathbf{C}$.

To proceed further we need an analogue of Lemma (4.3.1).

LEMMA (11.2.4). *For* $W_0 \in \mathcal{W}_0(\pi;\psi)$, Φ *in* $\mathcal{S}(3 \times 3,\, F)$, *and* $\mathrm{Re}(s)$ *large,*

$$\int_{G_3(F)} W_0(g)\Phi(g)|\det g|^{s+1} d^{\times}g = \int_{\mathrm{SL}(3,F)} \int_{F^{\times}} W_0\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} h\right] |a|^{s-1} d^{\times}a\, d\rho_{\Phi}(h)\,.$$

Proof. We have seen that both integrals are convergent for $\mathrm{Re}(s)$ large. Proceeding as in (4.3.1) we need only see that

$$\int W_0\left[\begin{pmatrix} a & abv & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} k\right] |abc|^{s+1}|a|^{-2}|b|^{-1} K_{\Phi}(v,\, b,\, b^{-1},\, c,\, c^{-1};\, k)\, dv\, d^{\times}a\, d^{\times}b\, d^{\times}c\, dk$$

is absolutely convergent for $\mathrm{Re}(s)$ large. Since K_{Φ} depends continuously on

Φ and W is dominated by a gauge, we are reduced to the convergence of

$$\int \phi(a, b) |a|^{s-t-1} |b|^{2s-1-t} |c|^{3s} K(v, bc, (bc)^{-1}, c, c^{-1}) d^\times a \, d^\times b \, d^\times c \, dv$$

for $\operatorname{Re}(s)$ large. (Here $K \in \mathcal{S}(F^5)$.) This is easy.

Remark (11.2.5). Let Φ_1 be in $\mathcal{S}(3 \times 3, F)$. As in (4.5), set

$$H(g) = |\det g|^3 \int \Phi_1(ng) \bar{\theta}(n) dn \, .$$

As there, $\int H(g)\Phi(g) d^\times g$ is convergent for any Φ in $\mathcal{S}(3 \times 3, F)$. Moreover

$$\int H(g)\Phi(g) d^\times g = \int H\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} h\right] |a|^{-2} d^\times a \, d\rho_\Phi(h) \, .$$

As in (11.2.4), the proof reduces to the convergence of

$$\int \Phi_0\left[\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}\right] |a| |b|^2 |c|^3 K(v, b, b^{-1}, c, c^{-1}) \times d^\times a \, d^\times b \, d^\times c \, dv^\times dx \, dy \, dz \, ,$$

with $\Phi_0 \in \mathcal{S}(3 \times 3, F)$ and $K \in \mathcal{S}(F^5)$.

We return to the proof of (11.2). Let $\Phi \in \mathcal{S}(3 \times 3, F)$. Using (11.2.5) and proceeding exactly as in (4.5.2), we obtain for $\operatorname{Re}(s)$ large,

$$\text{(11.2.6)} \quad \int \widetilde{W}_0(g)\hat{\Phi}(wg) |\det g|^{s+1} d^\times g = \int \widetilde{W}_0\left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w' h^t\right] |a|^{s-1} d^\times a \, dx \, d\rho_\Phi(h) \, .$$

From (9.2.3), we have

$$Z(\Phi', 2 - s, \widetilde{W}_0) = \varepsilon'(s, \pi, \psi) Z(\Phi, s + 1, W_0)$$

for $W_0 \in \mathcal{W}_0(\pi; \psi)$. Set $W = W_0 * \rho_\Phi^\vee$. Then by (11.2.4),

$$Z(\Phi, s + 1, W_0) = \Psi(s, W) \, .$$

On the other hand, recalling that $\widetilde{W}(g) = W(wg^t)$, we obtain from (11.2.6)

$$Z(\Phi', s + 1, \widetilde{W}_0) = \widetilde{\Psi}(s, W) \, .$$

The second assertion now follows from (9.2.3). Similarly (3) follows from (9.2.2) and the two preceding identities.

12. Fourier expansions

In the remaining sections the ground field F is global. We first discuss the Fourier expansion of functions on $G(\mathbf{A})$, invariant on the left under $P(F)$ and cuspidal along any horicycle of G contained in N. It is best to do this for all r.

(12.1) A gauge ξ on $G(\mathbf{A})$ is a function invariant on the left under $N(\mathbf{A})$, on the right under the standard maximal compact subgroup K, and given on $A(\mathbf{A})$ by

$$\begin{aligned} \xi(a) &= |a_1a_2 \cdots a_{r-1}|^{-t}\phi(a_1, a_2, \cdots, a_{r-1}) , \\ a &= \operatorname{diag}(a_1a_2 \cdots a_r, a_2 \cdots a_r, \cdots, a_{r-1}a_r, a_r) . \end{aligned} \tag{12.1.1}$$

Here $\phi \geqq 0$ is in $\mathcal{S}(\mathbf{A}^{r-1})$ and t is positive. In particular, if ϕ has the form $\phi = \prod \phi_v$, then

$$\xi(g) = \prod_v \xi_v(g_v) , \tag{12.1.2}$$

where, for each v, the gauge ξ_v on G_v is defined by

$$\xi_v(a) = |a_1a_2 \cdots a_{r-1}|^{-t}\phi_v(a_1, a_2, \cdots, a_{r-1}) .$$

Note that in (12.1.2) almost all factors are equal to one.

LEMMA (12.1.3). *Suppose ξ is the gauge defined by* (12.1.1) *and $t' > t$ is given. Then there is $\phi' \in \mathcal{S}(\mathbf{A}^{r-1})$ such that the gauge ξ' defined by t' and ϕ' majorizes ξ.*

Proof. It suffices to show that, given $\phi \geqq 0$ in $\mathcal{S}(\mathbf{A}^{r-1})$ and $t > 0$, there is $\phi' \geqq 0$ in $\mathcal{S}(\mathbf{A}^{r-1})$ such that

$$\phi(x_1, x_2, \cdots, x_{r-1}) \leqq |x_1x_2 \cdots x_{r-1}|^{-t}\phi'(x_1, x_2, \cdots, x_{r-1}) .$$

We may assume $\phi = \prod \phi_v$. Then there is a finite set of places S, containing all archimedean places, such that, for $v \notin S$, the function ϕ_v is the characteristic function of $\mathfrak{R}_v^{r-1}$. Now it is clear that

$$\phi_v(x) \leqq |x_1x_2 \cdots x_{r-1}|^{-t}\phi_v(x)$$

for v not in S. On the other hand, for v in S, there is ϕ'_v such that

$$\phi_v(x) \leqq |x_1x_2 \cdots x_{r-1}|^{-t}\phi'_v(x) .$$

The function

$$\phi = \prod_{v \in S}\phi'_v \cdot \prod_{v \notin S}\phi_v$$

satisfies the above condition.

COROLLARY (12.1.4). *The sum of two gauges is majorized by a gauge.*

LEMMA (12.1.5). *For any compact subset Ω of $G(\mathbf{A})$ and any gauge ξ on $G(\mathbf{A})$, there is a gauge ξ' such that*

$$\xi(g\omega) \leqq \xi'(g) ,$$

for $g \in G(\mathbf{A})$ and $\omega \in \Omega$.

Proof. Enlarging Ω we may assume

$$\Omega = \prod_v \Omega_v ,$$

with

$$\Omega_v = K_v \text{ for all } v \text{ not in } S,$$

Ω_v being a compact subset of G_v for all v in S. Here again S is a finite set of places containing all the archimedean ones. We may assume that ξ is given by (12.1.1) with $\phi = \prod \phi_v$, ϕ_v being the characteristic function of $\mathfrak{R}_v^{r-1}$ for v not in S. Then ξ is given by (12.1.2). There is, for each v in S, a function ϕ'_v such that the gauge ξ'_v defined by t and ϕ'_v satisfies

$$\xi_v(g\omega) \leqq \xi'_v(g) \quad \text{for} \quad g \in G_v,\ \omega \in \Omega_v .$$

On the other hand,

$$\xi_v(g\omega) = \xi_v(g) \quad \text{for} \quad v \notin S,\ g_v \in G_v,\ \omega \in \Omega_v .$$

Hence the function

$$\xi' = \prod_{v \in S} \xi'_v \prod_{v \notin S} \xi_v$$

satisfies the conditions of the lemma. Arguing as in (12.1.3), we see that ξ' is majorized by a gauge.

(12.2) As in (0.4) we identify G_{r-1}, N_{r-1}, A_{r-1} with subgroups G', N', A' of $G_r = G$. We set $K' = K \cap G'(\mathbf{A})$.

PROPOSITION (12.2). *Let ξ be a gauge on $G(\mathbf{A})$. Then the series*

(12.2.1) $$\phi(g) = \sum_{\gamma \in N'(F)\backslash G'(F)} \xi(\gamma g)$$

converges uniformly on compact subsets of $G(\mathbf{A})$. Furthermore, if F is a number field and Ω is a compact subset of $G(\mathbf{A})$, c a positive constant, there is t_0 such that, if $t \geqq t_0$, then there is a constant c' with the property that:

(12.2.2) $$\phi(a\omega) \leqq c' \prod_{1 \leqq i \leqq r-1} |a_i|^{-ti+i(r-1-i)} ,$$

for ω in Ω and

$$a = \operatorname{diag}(a_1 a_2 \cdots a_{r-1} a_r,\ a_2 \cdots a_{r-1} a_r,\ \cdots,\ a_{r-1} a_r,\ a_r)$$

satisfying

$$|a_i| \geqq c , \quad \textit{for} \quad 1 \leqq i \leqq r-2 .$$

Note that there is no condition on a_{r-1}.

Proof. Let Ω be a compact subset of $G(\mathbf{A})$. There is a gauge ξ_1 such that

$$\xi(g\omega) \leqq \xi_1(g),\ g \in G(\mathbf{A}),\ \omega \in \Omega .$$

So for the first assertion, it suffices to establish the convergence for $g = e$.

Let V be a compact neighborhood of e in $G'(\mathbf{A})$ whose translates by $G'(F)$ do not meet it. Then there is a gauge ξ' such that

$$\xi(g) \leqq \xi'(gx) ,$$

for g in $G(\mathbf{A})$ and $x \in V$. Therefore

$$\sum_\gamma \xi(\gamma) \leqq \sum_\gamma \xi'(\gamma x)$$

and, for any real number s,

$$\begin{aligned} \sum_\gamma \xi(\gamma)\int_V |\det x|^s dx &\leqq \int_V \sum_\gamma \xi'(\gamma x)\,|\det x|^s dx \\ &\leqq \int_{G'(F)\backslash G'(\mathbf{A})} \sum_\gamma \xi'(\gamma x)\,|\det \gamma x|^s dx \\ &= \int_{N'(F)\backslash G'(\mathbf{A})} \xi'(x)\,|\det x|^s dx , \end{aligned}$$

the sum, as above, being extended over $N'(F)\backslash G'(F)$.

Using the Iwasawa decomposition, the last integral is found to be equal to

$$\int_{\mathbf{I}^{r-1}} \phi(a_1, a_2, \cdots, a_{r-1}) \prod_{1\leqq i\leqq r-1} |a_i|^{is-i(r-1-i)-t} d^\times a_1 d^\times a_2 \cdots d^\times a_{r-1} .$$

Since ϕ is in $\mathcal{S}(\mathbf{A}^{r-1})$, this last integral is finite for large s. Hence

$$\sum_{N'(F)\backslash G'(F)} \xi(\gamma) < +\infty .$$

Assume F is a number field. To prove the second assertion, we may take $\Omega = \{e\}$ and choose a in a fundamental domain for $A(F)$ in $A(\mathbf{A})$. So we may take a to be

$$a = \mathrm{diag}(a_1 a_2 \cdots a_{r-1}, a_2 \cdots a_{r-1}, \cdots, a_{r-1}, 1) , \quad |a_i| \geqq c \text{ for } 1 \leqq i \leqq r-2 ,$$

the a_i being of the following form:

$$(a_i)_v = 1 \text{ for } v \text{ non-archimedean,}$$
$$(a_i)_v = t_i,\ t_i > 0, \text{ for } v \text{ archimedean.}$$

Let V_1 and V_2 be compact neighborhoods of e in $N'(\mathbf{A})$ and $A'(\mathbf{A})$ respectively. Then the set V_a of elements x of the form

$$x = nbak,\ n \in V_1,\ b \in V_2,\ k \in K' ,$$

is a compact neighborhood of a in $G'(\mathbf{A})$. Furthermore

$$a^{-1}x = a^{-1}nabk$$

stays in a fixed compact set independent of a (but dependent on V_1, V_2, and c). So there is a gauge ξ' such that

$$\xi(ga) \leqq \xi'(gx)$$

for all $g \in G(\mathbf{A})$ and x in V_a. Reduction theory shows that the set of

$\gamma \in G'(F)$ such that

$$\gamma V_a \cap V_a \neq \varnothing$$

for at least one a (satisfying the above conditions) is finite. It follows there is a constant c' (independent of a) such that, for any function $f \geqq 0$ on $G'(F)\backslash G'(\mathbf{A})$,

$$\int_{V_a} f(x)dx \leqq c' \int_{G'(F)\backslash G'(\mathbf{A})} f(x)dx \; .$$

Hence

$$\begin{aligned}\sum\nolimits_{\gamma \in N'(F)\backslash G'(F)} \xi(\gamma a) \int_{V_a} |\det x|^s dx &\leqq \int_{V_a} \sum\nolimits_{\gamma \in N'(F)\backslash G'(F)} \xi'(\gamma x) |\det x|^s dx \\ &\leqq c' \int_{G'(F)\backslash G'(\mathbf{A})} \sum\nolimits_{\gamma \in N'(F)\backslash G'(F)} \xi'(\gamma x) |\det \gamma x|^s dx \\ &= c' \int_{N'(F)\backslash G'(\mathbf{A})} \xi'(x) |\det x|^s dx \; .\end{aligned}$$

As before if s is sufficiently large, the last integral is finite. On the other hand,

$$\int_{V_a} |\det x|^s dx = \prod\nolimits_{1 \leqq i \leqq r-1} |a_i|^{is - i(r-1-i)} \int_{V_1 \cdot V_2 \cdot K'} |\det x|^s dx \; .$$

So, if s is sufficiently large, we get a majorization

$$\phi(a) = \sum\nolimits_{\gamma \in N'(F)\backslash G'(F)} \xi(\gamma a) \leqq c'' \prod\nolimits_{1 \leqq i \leqq r-1} |a_i|^{-is + i(r-1-i)} \; . \qquad \text{Q.E.D.}$$

PROPOSITION (12.3). *Let ω be a character of $\mathbf{I}/F^\times$ and W a continuous function on $G(\mathbf{A})$ such that*

$$W(zng) = \theta(n) W(g) \omega(z)$$

for $n \in N(\mathbf{A})$, $z \in Z(\mathbf{A}) \simeq \mathbf{I}$. If the series

$$\phi(g) = \sum\nolimits_{N(F)\backslash P^1(F)} W(\gamma g)$$

converges absolutely, uniformly on compact subsets, its sum is continuous on $G(\mathbf{A})$, invariant under $P(F)$ on the left, and is cuspidal along any minimal horicycle of G contained in N.

Note that the series may be written as a sum on $N'(F)\backslash G'(F)$.

Proof. Only the last assertion needs to be proved. The case $r = 1$ being trivial, we may assume $r \geqq 2$ and our assertion proved for $r - 1$. We may write, identifying $P^1_{r-1} \subset G_{r-1}$ with a subgroup $P^{1\prime}$ of G':

(12.3.1)
$$\begin{aligned}\phi(g) &= \sum\nolimits_{\xi \in P^{1\prime}(F)\backslash G'(F)} w(\xi g) \; , \\ w(g) &= \sum\nolimits_{\gamma \in N'(F)\backslash P^{1\prime}(F)} W(\gamma g) \; ,\end{aligned}$$

both series being absolutely convergent, uniformly on compact sets.

Since $P^{1\prime}(\mathbf{A})$ is the stabilizer of the character $\theta \,|\, U(\mathbf{A})$, the function w

satisfies

$$w(ug) = \theta(u)w(g),\ u \in U(\mathbf{A})\ .$$

On the other hand, the characters

$$u \longmapsto \theta(\xi u \xi^{-1}),\ \xi \in P^{1'}(F)\backslash G'(F)\ ,$$

are all the non-trivial characters of $U(F)\backslash U(\mathbf{A})$. So (12.3.1) may be regarded as the Fourier expansion of ϕ on the group $U(F)\backslash U(\mathbf{A})$. Since this Fourier expansion has no constant term,

$$\int_{U(F)\backslash U(\mathbf{A})} \phi(ug)du = 0\ .$$

Furthermore

$$w(g) = \int_{U(F)\backslash U(\mathbf{A})} \phi(ug)\bar{\theta}(u)du\ . \tag{12.3.2}$$

Now let V be the unipotent radical of a standard parabolic subgroup Q of type (r_1, r_2) where $r_2 > 1$ and $r_1 + r_2 = r$. We want to show that

$$\int_{V(F)\backslash V(\mathbf{A})} \phi(vg)dv = 0\ .$$

The group V is commutative and a direct product

$$V = V' \cdot V_1\ ,$$

where

$$V_1 = V \cap U,\ V' = V \cap N' = V \cap G'\ .$$

Moreover U is also a direct product

$$U = V_1 \cdot V_2\ ,$$

where V_2 is contained in Q. Since V_2 is contained in Q it normalizes V. So the function

$$v_2 \longmapsto \int_{V(F)\backslash V(\mathbf{A})} \phi(vv_2 g)dv \tag{12.3.3}$$

on $V_2(\mathbf{A})$ is invariant under $V_2(F)$. To show—as we must—that this function vanishes, it suffices to show that all its Fourier coefficients vanish. For the constant Fourier coefficient, we obtain

$$\int_{V_2(F)\backslash V_2(\mathbf{A})} dv_2 \int_{V(F)\backslash V(\mathbf{A})} \phi(vv_2 g)dv = \int dv_2 \int \phi(v'v_1v_2g)dv'dv_1\ , \tag{12.3.4}$$

where we integrate with respect to

$$v_1 \in V_1(F)\backslash V_1(\mathbf{A}),\ v' \in V'(F)\backslash V'(\mathbf{A})\ .$$

Now

$$H = V'V_1V_2 = VV_2 = V'U = UV'$$

is a group. We claim that for any function $f \geqq 0$ on $H(F)\backslash H(\mathbf{A})$,

$$\int_{H(F)\backslash H(\mathbf{A})} f(h)dh = \int f(v'v_1v_2)dv'dv_1dv_2 ,$$

where the second integral is successively taken over

$$v' \in V'(F)\backslash V'(\mathbf{A}),\ v_1 \in V_1(F)\backslash V_1(\mathbf{A}),\ v_2 \in V_2(F)\backslash V_2(\mathbf{A}) .$$

Indeed one may assume f of the form

$$f(h) = \sum_{\xi \in H(F)} f_0(\xi h) .$$

Then

$$\int_{H(F)\backslash H(\mathbf{A})} f(h)dh = \int_{H(\mathbf{A})} f_0(h)dh .$$

On the other hand,

$$f(h) = \sum f_0(\xi_2\xi_1\xi'h) ,$$

the sum being for $\xi' \in V'(F)$, $\xi_1 \in V_1(F)$, $\xi_2 \in V_2(F)$. Thus

$$\int f(v'v_1v_2)dv'dv_1dv_2 = \int_{V_1(F)\backslash V_1(\mathbf{A}) \times V_2(F)\backslash V_2(\mathbf{A})} dv_1dv_2 \int_{V'(\mathbf{A})} \sum f_0(\xi_2\xi_1v'v_1v_2)dv' .$$

Since V_1 and V' commute, this is also

$$\int_{V_2(F)\backslash V_2(\mathbf{A})} dv_2 \int_{V_1(\mathbf{A}) \times V'(\mathbf{A})} \sum f_0(\xi_2v_1v'v_2)dv_1dv' = \int dv_2 \int_{V(\mathbf{A})} \sum f_0(\xi_2vv_2)dv .$$

Finally, since v_2 normalizes V, this is

$$\int_{V(\mathbf{A})} dv \int_{V_2(F)\backslash V_2(\mathbf{A})} \sum f_0(\xi_2v_2v)dv_2 = \int_{V(\mathbf{A}) \times V_2(\mathbf{A})} f_0(v_2v)dv_2dv = \int_{H(\mathbf{A})} f_0(h)dh ,$$

which proves our contention. Hence the integral (12.3.4) is nothing but

$$\int_{H(F)\backslash H(\mathbf{A})} \phi(hg)dh = \int_{V'(F)\backslash V'(\mathbf{A})} dv' \int_{U(F)\backslash U(\mathbf{A})} \phi(uv'g)du ,$$

this time because V' normalizes U. The inner integral vanishes, hence (12.3.4) vanishes as well.

In order to prove that the other Fourier-coefficients of (12.3.3) vanish, we note that we may apply the induction hypotheses to the function

$$h \longmapsto w(hg)$$

on $G'(\mathbf{A})$. In particular V' being a minimal horicycle of G' contained in N', we know that

$$\int_{V'(F)\backslash V'(\mathbf{A})} w(v'g)dv' = 0 .$$

Taking (12.3.2) into account we have

$$0 = \int dv' \int \phi(uv'g)\bar{\theta}(u)du$$
$$= \int dv' \int \phi(v_1v_2v'g)\bar{\theta}(v_2)dv_1dv_2 .$$

As before, noting that $\theta|U(\mathbf{A})$ extends to a representation of $H(\mathbf{A})$ trivial on $V(\mathbf{A})$ and $H(F)$, we may also write

(12.3.5) $$\int \bar{\theta}(v_2)dv_2 \int \phi(vv_2g)dv = 0 .$$

Any matrix γ of the form

$$\gamma = \begin{pmatrix} 1_{r_1} & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \zeta \in \mathrm{GL}(r_2 - 1, F) ,$$

will normalize V and V_2. Replace g by γg and note that ϕ is invariant under γ. Then (12.3.5) gives

$$\int \bar{\theta}(\gamma v_2 \gamma^{-1})dv_2 \int \phi(vv_2g)dv = 0 .$$

Since the non-trivial characters of $V_2(\mathbf{A})\backslash V_2(F)$ are of the form

$$v_2 \longmapsto \bar{\theta}(\gamma v_2 \gamma^{-1}) ,$$

we are done.

(12.4) Suppose F is a number field. Let R_F denote the ring of integers of F. Let $p = [F: Q]$.

Let v be an infinite place of F. For $g \in \mathrm{SL}(r, F_v)$ set

$$||g||_v = (\textstyle\sum g_{ij}^2)^{1/2}$$

if F_v is real, and

$$||g||_v = \textstyle\sum g_{ij}\bar{g}_{ij}$$

if F_v is complex (cf. (8.3)). Then $||g||_v \geqq 1$. For g in

$$\mathrm{SL}(r, F_\infty) = \textstyle\prod_{v\in\infty} \mathrm{SL}(r, F_v) ,$$

set

$$||g|| = \textstyle\prod_{v\in\infty} ||g_v||_v .$$

Again $||g|| \geqq 1$.

For $x \in M(r \times r, F_\infty)$, set

$$\tau(x) = (\textstyle\sum x_{vij}\bar{x}_{vij})^{1/2} ;$$

τ is a norm on $M(r \times r, F_\infty)$ regarded as a real vector space. We have $\tau(g) \leqq r^{1/2}||g||$ and $||g|| \leqq \tau(g)^p$ for all g in $\mathrm{SL}(r, F_\infty)$.

Let X be a finite subset of $\mathrm{SL}(r, F)$ and Ω a compact subset of $\mathrm{SL}(r, F_\infty)$.

Let c be a positive constant. A Siegel set in $\mathrm{SL}(r, F_\infty)$ is a set of elements of the form $g = \xi b \omega$, where ξ is in X, ω in Ω, and b varies over all elements of the form $b = \mathrm{diag}(b_1, b_2, \cdots, b_r)$, $b_i \in F_\infty^+$, $b_i/b_{i+1} \geqq c$, $b_1 b_2 \cdots b_r = 1$. By reduction theory, we can choose a Siegel set $\mathfrak{S}$ so that

$$\mathrm{SL}(r, F_\infty) = \mathrm{SL}(r, R_F)\cdot\mathfrak{S} .$$

The reduction functor provides an $\mathbf{R}$-algebra isomorphism ρ of $M(r \times r, F_\infty)$ into $M(pr \times pr, \mathbf{R})$, carrying $\mathrm{SL}(r, R_F)$ into $\mathrm{SL}(pr, \mathbf{Z})$ and Siegel sets in $\mathrm{SL}(r, F_\infty)$ into (standard) Siegel sets in $\mathrm{SL}(pr, \mathbf{Z})$ ([5]). By transport of structure, there is a norm σ on $M(pr \times pr, \mathbf{R})$ and constants $c_1 > 0$, $c_2 > 0$ so that

$$c_1\sigma(\rho(g)) \leqq \|g\| \leqq c_2\sigma(\rho(g))^p .$$

Then by 4.10 in [5], if $h = \gamma\xi b\omega$ with ξ, b, and ω as above and $\gamma \in \mathrm{SL}(r, R_F)$,

(12.4.1) $$\|h\|^p \geqq c'\|b\|$$

for a suitable constant $c' > 0$.

The following proposition will be applied in Section 13.

PROPOSITION (12.4.2). *Let F be a number field. Let ξ be a gauge on $G_r(\mathbf{A})$. Set*

$$\phi(g) = \sum\nolimits_{\gamma \in N'(F)\backslash G'(F)} \xi(\gamma g) .$$

Let Ω be a compact subset of $G_r(\mathbf{A})$. Then there is a constant c and an integer m so that

$$\phi(\omega_1 g \omega_2) \leqq c\|g\|^m$$

for all $\omega_1, \omega_2 \in \Omega$ and $g \in \mathrm{SL}(r, F_\infty)$.

Proof. Since G^∞ and G_∞ commute we may assume $\omega_1 = \{e\}$. We may also assume $\Omega = \Omega_0\Omega_\infty$ where $\Omega_0 \subset G^\infty$ and $\Omega_\infty \subset G_\infty$ are compact. By (12.1.5) we may assume $\Omega_0 = \{e\}$.

We may write $g \in \mathrm{SL}(r, F_\infty)$ in the form:

$$g = \begin{pmatrix} 1_{r-1} & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m & 1 \\ 0 & x \end{pmatrix} k ,$$

where $k \in K_\infty$, $x \in F_\infty^\times$, $m \in G_{r-1}(F_\infty)$. We may also assume

$$\det m_v > 0 , \qquad x_v > 0 .$$

Write $m = z \cdot h$ with $z \in F_\infty^\times$, $z_v > 0$, and $\det h_v = 1$. Note $z_v^{r-1} \cdot x_v = 1$. Then, for v real,

$$\|g_v\|_v^2 \geqq \sum m_{vij}^2 = |z_v|_v^2 \sum h_{vij}^2 \quad \text{or} \quad \|g_v\|_v \geqq |z_v|_v \|h_v\|_v .$$

The same holds for v complex so that

(12.4.3) $$\|g\| \geqq |z|_{\mathbf{A}} \|h\| .$$

Similarly

(12.4.4) $$||g|| \geqq |x|_{\mathbf{A}} .$$

Now, as above, we write

$$h = \gamma\xi b\omega ,$$

with γ in $\mathrm{SL}(r-1, R_F)$, ξ in a finite subset of $\mathrm{SL}(r-1, F)$, ω in a compact set, $b = \mathrm{diag}(b_1, b_2, \cdots, b_{r-1})$, $b_i \in F_\infty^+$, $b_i/b_{i+1} \geqq c$ and $\prod b_i = 1$. Then

$$\phi(g) = \phi\left[\begin{pmatrix} bz & 0 \\ 0 & x\end{pmatrix}\omega'\right]$$

where ω' is in a compact set in $G_r(\mathbf{A})$. By Proposition 12.2,

$$|\phi(g)| \leqq c_1 |z|_{\mathbf{A}}^{-(r-1)t} |x|_{\mathbf{A}}^{(r-1)t} |b_1^{r-1} b_2^{r-2} \cdots b_{r-1}|_{\mathbf{A}}^2 ,$$

t positive. Clearly $|b_i|_{\mathbf{A}} \leqq ||b||$. Thus by (12.4.1)

$$|\phi(g)| \leqq c_1 |z|_{\mathbf{A}}^{-(r-1)t} |x|_{\mathbf{A}}^{(r-1)t} ||b||^{r(r-1)} \leqq c_2 |z|_{\mathbf{A}}^{-(r-1)t} |x|_{\mathbf{A}}^{(r-1)t} ||h||^{r(r-1)p} .$$

This is also

$$c_2 |z|_{\mathbf{A}}^{-(r-1)t-r(r-1)p} |x|_{\mathbf{A}}^{(r(r-1)t} ||h||^{r(r-1)p} |z|_{\mathbf{A}}^{r(r-1)p}$$

or, since $|z|_{\mathbf{A}}^{r-1} x|_{\mathbf{A}} = 1$,

$$c_2 |x|_{\mathbf{A}}^{r(t+p)} (||h|| |z|_{\mathbf{A}})^{r(r-1)p} .$$

Our assertion then follows from (12.4.3) and (12.4.4).

13. The main theorem

In this section F is an **A**-field and $r = 3$.

(13.1) We want to establish a converse to Theorem (13.8) of [17]. We consider the following situation:

(13.1.1) ω is a character (of module one) of $F_{\mathbf{A}}^\times/F^\times$;

(13.1.2) For each infinite place v, π_v is an *irreducible, unitary, generic* representation of G_v or equivalently of the local Hecke algebra $\mathcal{H}_v$;

(13.1.3) For each finite place v, π_v is an irreducible admissible representation of G_v, generic or not.

Furthermore the following is assumed:

(13.1.4) For almost all v the representation π_v is unramified, thus of the form

$$\pi_v = \pi(\mu_{1v}, \mu_{2v}, \mu_{3v})$$

with $\mu_{i,v} = \alpha^{s_{i,v}}$. It is assumed that there is $t > 0$ so that $-t \leqq \mathrm{Re}(s_{i,v}) \leqq t$, for $i = 1, 2, 3$ and almost all v. This condition is automatically satisfied if we assume the representations π_v to be *unitary*.

Note also that if $\pi = \otimes_v \pi_v$ is an automorphic cuspidal representation then all the above conditions are satisfied. More precisely the representation π_v is then unitary generic for all v.

If v is finite, as in (6.6), let ξ_v be the induced representation of G_v associated to π_v. Recall that π_v is always a quotient of ξ_v (cf. § 6). As in (7.1) we may define the space $\mathcal{W}(\pi_v; \psi_v)$ which affords the representation ξ_v of G_v, noting again that, if π_v is generic, then $\xi_v = \pi_v$. For v infinite we set $\xi_v = \pi_v$. Let also $\pi = \otimes_v \pi_v$, $\xi = \otimes_v \xi_v$. We denote by $\mathcal{W}(\pi; \psi)$ the space spanned by the functions of the form

$$W(g) = \prod_v W_v(g_v) , \quad W_v \in \mathcal{W}(\pi_v; \psi_v) .$$

Here we assume that W_v is, for almost all v, the element in $\mathcal{W}(\pi_v; \psi_v)$ invariant under K_v and equal to one on K_v. If we also require that W_v be, for v infinite, in $\mathcal{W}_0(\pi_v; \psi_v)$, we obtain a subspace denoted by $\mathcal{W}_0(\pi; \psi)$. From (2.3.7) and (8.3.3) it is clear that every element W of $\mathcal{W}_0(\pi; \psi)$ is majorized by a gauge. Thus we may set (cf. (12.3))

(13.1.5) $$\phi(g) = \sum W(\gamma g) , \quad \gamma \in N(F)\backslash P^1(F) , \quad \text{for } W \in \mathcal{W}_0(\pi; \psi) .$$

Recall from (6.2.7) and (6.6) that if v is finite and ξ_v is the induced representation attached to π_v, the induced representation attached to $\tilde{\pi}_v$ is the image $\xi_v^{\vee}$ of ξ_v under the automorphism $g \mapsto wg^t w^{-1}$. As we know $\xi_v^{\vee}$ is equivalent to ξ_v^{ι} and the map $W_v \mapsto \tilde{W}_v$ (2.1.3) is a bijection of $\mathcal{W}(\pi_v; \psi_v)$ onto $\mathcal{W}(\tilde{\pi}_v; \psi_v)$. A similar remark applies to the space $\mathcal{W}_0(\pi_v; \psi_v)$ for v infinite. Thus in the above we may replace the triple $(\omega, (\pi_v), (\xi_v))$ by the triple $(\omega^{-1}, (\tilde{\pi}_v), (\xi_v^{\iota}))$. We obtain another space $\mathcal{W}_0(\tilde{\pi}; \psi)$ and the map $W \mapsto \tilde{W}$ defined by (2.1.3) is again a bijection of $\mathcal{W}_0(\pi; \psi)$ onto $\mathcal{W}_0(\tilde{\pi}; \psi)$. We set

(13.1.6) $$\tilde{\phi}(g) = \sum \tilde{W}(\gamma g) , \quad \gamma \in N(F)\backslash P^1(F) , \quad \text{for } W \in \mathcal{W}_0(\pi; \psi) .$$

Each ϕ in (13.1.5) is a continuous function on $G(\mathbf{A})$, invariant under $P(F)$ on the left and cuspidal along any proper horicycle of G contained in P((12.2), (12.3)). Since

$$W(g) = \int_{N^*} \phi(ng)\bar{\theta}(n)dn ,$$

the map $W \mapsto \phi$ is bijective. Hence the space spanned by the ϕ is invariant by convolution by elements of $\mathcal{H}$ and affords the representation ξ.

Thus it is clear that if ϕ is invariant under $G(F)$ on the left then ϕ is automorphic. More precisely by (12.4), ϕ is slowly increasing and is in fact a cusp form. Since the representation of $\mathcal{H}$ on the space of cusp forms is completely reducible each ξ_v must be completely reducible. Then each component of ξ_v must be generic. Since ξ_v has exactly one generic component, ξ_v is irreducible. Thus $\pi_v = \xi_v$ and $\pi = \otimes_v \pi_v$ is a component of the space of cusp forms.

(13.2) Again suppose ϕ is invariant under $G(F)$. Then the function $g \mapsto \phi(wg^\iota)$ is also invariant under $G(F)$ and consequently ([29], [34]) has a Fourier expansion in terms of the function

$$\int_{N^*} \phi[wn^\iota g^\iota]\bar{\theta}(n)dn = \int_{N^*} \phi[nwg^\iota]\bar{\theta}(n)dn = \widetilde{W}(g) .$$

Hence $\phi(wg^\iota) = \tilde{\phi}(g)$ or $\phi(g) = \tilde{\phi}(wg^\iota)$.

Conversely suppose $\phi(g) = \tilde{\phi}(\gamma g^\iota)$ for some $\gamma \in G(F)$. Then ϕ is invariant under $P(F)$ and $\gamma^{-\iota} P^\iota \gamma^\iota$. Since for any $\gamma \in G(F)$, these two groups generate $G(F)$ we conclude that ϕ is $G(F)$-invariant.

In what follows we will have $\gamma = w'$; so we introduce

(13.2.1) $$\phi_1(g) = \tilde{\phi}(w'g^\iota) ,$$

where, as above, $\tilde{\phi}$ is defined by (13.1.6). Recall

$$w' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and that ϕ_1 is invariant on the left under $Q(F)$ where here $Q = w'P^\iota(w')^{-1}$. In particular ϕ_1 is still invariant under $U(F)$ and we may set

(13.2.2) $$V(g) = \int_{U^*} \phi(ug)\bar{\theta}(u)du ,$$

$$V_1(g) = \int_{U^*} \phi_1(ug)\bar{\theta}(u)du,\ \widetilde{V}(g) = \int_{U^*} \tilde{\phi}(ug)\bar{\theta}(u)du .$$

Of course if ϕ is $G(F)$-invariant then $\phi = \phi_1$ and $V = V_1$. The critical fact for us is the converse.

LEMMA (13.2.3). *If $V = V_1$ then $\phi = \phi_1$ and ϕ is $G(F)$-invariant.*

Proof. The assumption is that

$$\int_{U^*} (\phi - \phi_1)(ug)\bar{\theta}(\gamma u \gamma^{-1})du = 0$$

for $\gamma = 1$ and all $g \in G(\mathbf{A})$. Since $P(F) \cap Q(F)$ normalizes U and leaves $\phi - \phi_1$ invariant, we get the same relation for any γ in $P(F) \cap Q(F)$, in particular for

$$\gamma = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \alpha \in F, \beta \in F^\times .$$

Thus

$$\int_{(\mathbf{A}/F)^2} (\phi - \phi_1)\left[\begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} g\right] \psi(\alpha x + \beta y) dx dy = 0$$

for all α in F, β in $F^\times$. This implies that

$$\int_{\mathbf{A}/F} (\phi - \phi_1)\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} g\right] \psi(\beta y) dy = 0$$

for all $\beta \in F^\times$. Our conclusion follows from the following lemma which is true without any assumption on ϕ.

LEMMA (13.2.4). *With the above notations,*

$$\int_{\mathbf{A}/F} \phi\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} g\right] dx = \int_{\mathbf{A}/F} \phi_1\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} g\right] dx .$$

Proof. Let

$$w_1 = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Then

$$\phi(g) = \sum W\left[\begin{pmatrix}\alpha & 0 & 0\\ \gamma & \delta & 0\\ 0 & 0 & 1\end{pmatrix} g\right] + \sum W\left[w_1\begin{pmatrix}\alpha & 0 & 0\\ 0 & \delta & 0\\ 0 & 0 & 1\end{pmatrix} g\right], \quad \alpha \in F^\times, \delta \in F^\times, \gamma \in F .$$

Thus, in (13.2.4), the left side can be written as

$$\sum W\left[w_1\begin{pmatrix}\alpha & 0 & 0\\ 0 & \delta & 0\\ 0 & 0 & 1\end{pmatrix} g\right], \quad \alpha \in F^\times, \gamma \in F^\times . \tag{13.2.5}$$

On the other hand, the right side is

$$\int \tilde{\phi}\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} w' g^\iota\right] dx$$

which is, as above,

$$\sum_{\alpha, \delta \in F^\times} \tilde{W}\left[w_1\begin{pmatrix}\alpha & 0 & 0\\ 0 & \delta & 0\\ 0 & 0 & 1\end{pmatrix} w' g^\iota\right],$$

or, using the definition of $\tilde{W}$,

$$\sum_{\alpha,\delta\in F^\times} W\left[w_1\begin{pmatrix}\alpha & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & \delta\end{pmatrix}g\right].$$

Because of the invariance of W under $Z_3(F)$, this is the same as (13.2.5).

(13.3) In order to formulate the relation $V = V_1$ in terms of L-functions we need the following relation between V_1 and $\tilde{V}$:

$$V_1(g) = \int_{\mathbf{A}} \tilde{V}\left[\begin{pmatrix}1 & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}w'g^\iota\right]dx\,. \tag{13.3.1}$$

Again this is true without any assumption on ϕ. Indeed replace ϕ by $\tilde{\phi}$ to obtain the equivalent form

$$\int_{(\mathbf{A}/F)^2}\phi\left[\begin{pmatrix}1 & 0 & 0\\ x & 1 & y\\ 0 & 0 & 1\end{pmatrix}g\right]\psi(-y)dxdy = \int_{\mathbf{A}} V\left[\begin{pmatrix}1 & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}g\right]dx\,. \tag{13.3.2}$$

To show the right side of (13.3.2) is convergent, it suffices to show that the series

$$\sum_{\xi\in F} V\left[\begin{pmatrix}1 & 0 & 0\\ \xi & 1 & 0\\ 0 & 0 & 1\end{pmatrix}g\right] \tag{13.3.3}$$

is normally convergent for g in a compact set. Using the invariance of ϕ under $P(F)$ we see that this is also

$$\sum_{\xi\in F}\int_{(\mathbf{A}/F)^2}\phi\left[\begin{pmatrix}1 & 0 & u\\ 0 & 1 & v-u\xi\\ 0 & 0 & 1\end{pmatrix}g\right]\bar{\psi}(v)dvdu$$

or, with a change of variables,

$$\sum_{\xi\in F}\int_{(\mathbf{A}/F)^2}\phi\left[\begin{pmatrix}1 & 0 & u\\ 0 & 1 & v\\ 0 & 0 & 1\end{pmatrix}g\right]\bar{\psi}(v+u\xi)dudv\,.$$

This is a partial Fourier series for the smooth function $u \to \phi(ug)$ on $u \mapsto U(F)\backslash U(\mathbf{A})$. Thus the convergence is clear. Moreover the sum of the series (13.3.3) is

$$\int_{\mathbf{A}/F} \phi\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & v\\ 0 & 0 & 1\end{pmatrix} g\right] \bar{\psi}(v)dv .$$

Writing the integral on the right of (13.3.2) as a sum over F followed by an integral over $\mathbf{A}/F$, we obtain our conclusion.

(13.4) Let us express the identity of functions $V = V_1$ in terms of their Mellin transforms. Since ϕ is cuspidal along the radical of the parabolic subgroup of type (1, 2), we have (cf. (12.3.1) and (12.3.2))

$$V(g) = \sum_{\alpha \in F^\times} W\left[\begin{pmatrix}\alpha & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} g\right] . \tag{13.4.1}$$

Thus, at least formally,

$$\int_{\mathbf{I}/F^\times} V\left[\begin{pmatrix}a & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] |a|^{s-1} d^\times a = \int_{\mathbf{I}} W\left[\begin{pmatrix}a & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] |a|^{s-1} d^\times a . \tag{13.4.2}$$

The integral on the right we denote, in accordance with the local theory (Theorem 7.4), by $\Psi(s, W)$. Since W is majorized by a gauge, we find that the integral is dominated by one of the form

$$\int_{\mathbf{I}} \Phi(a, 1) |a|^{\mathrm{Re}(s)-t-1} d^\times a ,$$

with $\Phi \in \mathcal{S}(\mathbf{A}^2)$. Thus for $\mathrm{Re}(s)$ large, the integral $\Psi(s, W)$ is convergent and both sides of (13.4.2) are defined and equal. Before proceeding we prove:

LEMMA (13.4.3). *Let ξ be a gauge on $G(\mathbf{A})$. Then the integral*

$$\int_{\mathbf{A}}\int_{\mathbf{I}} \xi\left[\begin{pmatrix}a & 0 & 0\\ x & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right] |a|^{s-1} d^\times a dx$$

is convergent for s large.

Proof. Write $\xi = \Pi \xi_v$ as a product of local gauges ξ_v. By definition there is a finite set of places S, containing those at infinity such that, for $v \notin S$,

$$\xi_v(\mathrm{diag}(ab, b, 1)) = \Phi_v(a, b) |ab|^{-t} ,$$

where Φ_v is the characteristic function of $\mathfrak{R}_v^2$ and t is independent of v. Our integral (finite or not) is a product of local integrals of the form

$$\int_F \int_{F^\times} \xi_v \left[\begin{pmatrix} a_v & 0 & 0 \\ x_v & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] |a_v|_v^{s-1} d^\times a_v dx_v .$$

We have already observed that for a given v this integral is finite for s large. For $v \notin S$ the integrand vanishes unless $x_v \in \mathfrak{R}_v$ and then is independent of x_v. Thus as above we are reduced to the convergence of

$$\prod_{v \notin S} \int_{F_v^\times} \Phi_v(a_v, 1) |a_v|^{s-t-1} d^\times a_v$$

for s large, which is clear.

In accordance with the local theory (cf. (7.4)), we set

$$\widetilde{\Psi}(s, \widetilde{W}) = \int_{\mathbf{A}} \int_{\mathbf{I}} \widetilde{W} \left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w' \right] |a|^{s-1} d^\times a dx .$$

By the lemma just proved the integral is absolutely convergent for Re(s) large. Thus it may be written as

$$\int_{\mathbf{A}} \int_{\mathbf{I}/F^\times} \sum_{\alpha \in F^\times} \widetilde{W} \left[\begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w' \right] |a|^{s-1} d^\times a dx .$$

Since this is absolutely convergent we may interchange the integrals to obtain, after using (13.4.1) for $\widetilde{W}$:

$$\int_{\mathbf{I}/F^\times} |a|^{s-1} d^\times a \int_{\mathbf{A}} \widetilde{V} \left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w' \right] dx .$$

This is then convergent as an iterated integral, for Re(s) large. Changing x to ax in the inner integral and then a to a^{-1}, we obtain

$$\int_{\mathbf{I}/F^\times} |a|^{-s} d^\times a \int_{\mathbf{A}} \widetilde{V} \left[\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w' \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] dx$$

which by (13.3.1) is

$$\int_{\mathbf{I}/F^\times} |a|^{-s} d^\times a \, V_1 \left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] .$$

Thus, for Re(s) small,

$$\tilde{\Psi}(1-s, W) = \int_{\mathbf{I}/F^\times} V_1\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^\times a \,, \tag{13.4.4}$$

both sides being defined.

(13.5) Suppose that ϕ is invariant under $G(F)$. Let us indicate briefly how to obtain the functional equation for $L(s, \pi)$ with the present methods. We assume for simplicity that F is a function field. ϕ is then, as noted above, a cusp form thus a compactly supported function mod $Z(\mathbf{A})G(F)$. It can be shown then that

$$a \longmapsto V\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] = \int_{(\mathbf{A}/F)^2} \phi\left[\begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \psi(-v) du dv$$

is compactly supported on $\mathbf{I}/F^\times$. Thus the left side of (13.4.2) is a polynomial in Q^{-s}, Q^s and provides an analytic continuation of the right side, i.e. of $\Psi(s, W)$.

Since $V = V_1$, we have

$$\int_{\mathbf{I}/F^\times} V\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^\times a = \int_{\mathbf{I}/F^\times} V_1\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^\times a$$

for Re(s) small. Thus by (13.4.2) and (13.4.4)

$$\Psi(s, W) = \tilde{\Psi}(1-s, W) \,,$$

in the sense of analytic continuation. The functional equation

$$L(s, \pi) = \varepsilon(s, \pi) L(1-s, \tilde{\pi})$$

follows in the usual way from the local theory.

(13.6) We are now ready to formulate the main result of this section.

THEOREM (13.6). *Let π be as in* (13.1). *Assume that for any character χ of $\mathbf{I}/F^\times$ the products $L(s, \pi \otimes \chi)$ and $L(s, \tilde{\pi} \otimes \chi)$ extend to entire functions of s, bounded in vertical strips if F is a number field, and satisfy*

$$L(s, \pi \otimes \chi) = \varepsilon(s, \pi \otimes \chi) L(1-s, \tilde{\pi} \otimes \chi^{-1}) \,. \tag{13.6.1}$$

Then π is a component of the cusp forms and each π_v is unitary and generic.

Note that (13.6.1) is to be understood in the sense of analytic continuation.

Proof. Suppose first that F is a function field. For $\chi = 1$, the assumption is equivalent to the statement that

$$\Psi(s, W) = \tilde{\Psi}(1 - s, W) , \tag{13.6.2}$$

both sides being polynomials in Q^{-s}, Q^{s}. By (13.4.2) and (13.4.4), this gives

$$\int_{\mathbf{I}/F^\times} V\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1}\chi(a)d^\times a = \int_{\mathbf{I}/F^\times} V_1\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1}\chi(a)d^\times a \tag{13.6.3}$$

for $\chi = 1$. Actually, a priori, the integrals are defined in non intersecting half-spaces and it is only their analytic continuations (as polynomials in Q^{-s}, Q^{s}) which coincide. If we replace π by $\pi \otimes \chi$ we have to replace V by $V \otimes \chi$ and V_1 by $V_1 \otimes \chi$. Thus we get (13.6.3) for all χ—an equality between polynomials in Q^{-s}, Q^{s}. Comparing coefficients in these polynomials, we obtain

$$\int_{\mathbf{I}^1/F^\times} V\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \chi(a)d^\times a = \int_{\mathbf{I}^1/F^\times} V_1\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \chi(a)d^\times a ,$$

where $\mathbf{I}^1$ denotes the idèles of module one. Thus $V(e) = V_1(e)$. Since replacing W by a right translate corresponds to translating V and V_1 by the same element, we obtain $V = V_1$. Hence by (13.2.3) ϕ is a cusp form and, as we have seen, π is a component of the space of cusp forms.

Assume now that F is a number field. Referring to (11.2), we let μ be a measure of compact support on $G_\infty = \prod_{v \in \infty} G_v$ of the form $\mu = \otimes_v \mu_v$, where $\mu_v = \rho_{\Phi_v}$, $\Phi_v \in \mathcal{S}(3 \times 3, F_v)$. Let W' be of the form $W * \mu^\vee$, with $W \in \mathcal{W}_0(\pi; \psi)$. By (12.1.5) W' is majorized by a gauge; so the integrals defining $\Psi(s, W')$ and $\tilde{\Psi}(s, W')$ are convergent for $\mathrm{Re}(s)$ large and each is a product of the corresponding local integrals. Our assumptions on $L(s, \pi)$ and $L(s, \tilde{\pi})$ imply that both of these functions extend to entire functions and that

$$\Psi(s, W') = \tilde{\Psi}(1 - s, W') . \tag{13.6.4}$$

In the preceding computations replace W by $W * \mu^\vee$. Then we must replace ϕ by $\phi * \mu^\vee$, which is permissible because (13.1.5) converges uniformly on compact sets. Similarly we must replace $\tilde{W}$ by $\tilde{W} * \mu^t$ and $\tilde{\phi}$ by $\tilde{\phi} * \mu^t$. Then ϕ_1 is replaced by $\phi_1 * \mu^\vee$ and V, V_1, and $\tilde{V}$ respectively by $V * \mu^\vee$, $V_1 * \mu^\vee$ and $\tilde{V} * \mu^t$. Again since W' is dominated by a gauge, we find exactly as before,

$$\int_{\mathbf{I}/F^\times} V * \mu^\vee \left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1}\chi(a)d^\times a \tag{13.6.5}$$

$$= \int_{I/F^\times} V_1 * \mu^\vee \left[\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] |a|^{s-1} \chi(a) d^\times a \; ,$$

at first for $\chi = 1$, the identity being taken in the sense of analytic continuation of entire functions.

We note now that $\Psi(s, W')$ is bounded in vertical strips $a \leqq \mathrm{Re}(s) \leqq b$. From the integral expression this is obvious for a large. Similarly for $\tilde{\Psi}(s, W')$. Hence by (13.6.4) it is true for b small. On the other hand, from the expression of Ψ as a product, our assumption on $L(s, \pi)$, and (11.2.3), $\Psi(s, W')$ is moderately increasing in any strip of finite width. Our conclusion follows from the Phragmen-Lindelöf principle.

Replacing π by $\pi \otimes \chi$, we obtain (13.6.5) for all χ (recall that μ has support in $\mathrm{SL}(3, F_\infty)$). Since both sides are entire and bounded in vertical strips we may apply Lemma 11.3.1 of [23] to conclude that

$$V * \mu^\vee(e) = V_1 * \mu^\vee(e) \; .$$

By Lemma (11.1.1), we obtain $V(e) = V_1(e)$. Replacing W by its translates by elements of $Z(\mathbf{A})G^\infty$ and convolutes by elements of $\mathcal{H}_\infty$, we obtain $V = V_1$. Q.E.D.

(13.7) For applications we require a somewhat stronger theorem.

THEOREM (13.7). *Let π be as in* (13.1) *and S a finite set of finite places. Assume that for any character of $F_\mathbf{A}^\times/F^\times$, unramified at each place of S, the function $L(s, \pi \otimes \chi)$ is entire, bounded in vertical strips if F is a number field, and satisfies*

$$L(s, \pi \otimes \chi) = c\varepsilon(s, \pi \otimes \chi) L(1 - s, \tilde{\pi} \otimes \chi^{-1})$$

where $c \neq 0$ is a constant independent of χ. Then there is a space $\mathcal{V}$ of smooth functions on $G(F)\backslash G(\mathbf{A})$ transforming under $Z(\mathbf{A})$ according to ω and affording the representation $\xi^S = \otimes_{v \notin S} \xi_v$ of $\mathcal{H}^S$. Moreover, if F is a number field the elements of $\mathcal{V}$ are slowly increasing.

Proof. We may choose ψ so that ψ_v is of exponent zero for each $v \in S$. Next for each v in S, we choose once for all $a_v \geqq 1$ and $W_v^0 \neq 0$ in $\mathcal{W}(\pi_v; \psi_v)$ so that the conditions of Lemma (7.6) are satisfied. We let $K_v^{a_v}$ be the open subgroup of G_v defined in that lemma and set

$$K'_S = \prod_{v \in S} K_v^{a_v} \; , \quad G' = K'_S G^S$$

with $G^S = \prod_{v \notin S} G_v$. We denote by $\mathcal{W}'_0(\pi; \psi)$ the subspace of $\mathcal{W}_0(\pi; \psi)$ spanned by the functions of the form

$$W(g) = \prod_{v \notin S} W_v(g_v) \prod_{v \in S} W_v^0(g_v) \; .$$

Note that W is determined by its restriction to G^S. From now on we restrict ourselves to the functions W in that space. Note that they are all K-finite. Clearly that space transforms like ξ^S under $\mathcal{H}^S$.

Let f be any of the functions W, ϕ, V or V_1. Then f depends linearly on W and if we replace W by a right translate we must replace f by the corresponding translate. Thus we find

$$f(gh) = \omega(h_{33})f(g) \quad \text{for} \quad h \in K'_S\,, \quad g \in G(\mathbf{A})\,, \tag{13.7.1}$$

$$\int_{\mathfrak{p}_v^{-1}} f\left[g\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix}\right]dx = 0 \quad \text{for} \quad v \in S\,, \quad g \in G(\mathbf{A})\,. \tag{13.7.2}$$

For the sake of clarity, let us assume F is a number field. We leave the function field case to the reader. From the hypothesis, we get, instead of (13.6.5) the relation

$$\begin{aligned}&\int_{\mathbf{I}/F^\times} |a|^{s-1}\chi(a)d^\times a \int V\left[\begin{pmatrix}a & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}h\right]d\mu(h)\\ &= c\int_{\mathbf{I}/F^\times} |a|^{s-1}\chi(a)d^\times a \int V_1\left[\begin{pmatrix}a & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}h\right]d\mu(h)\,,\end{aligned} \tag{13.7.3}$$

this time for those χ such that χ_v is unramified for v in S. But from (13.7.1) we see that for any b in $\prod_{v\in S}\mathfrak{R}_v^\times$, a in $\mathbf{I}$, $h \in \mathrm{SL}(3, F_\infty)$,

$$\begin{aligned}V\left[\begin{pmatrix}ab & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}h\right] &= V\left[\begin{pmatrix}a & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}h\begin{pmatrix}b & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\\ &= V\left[\begin{pmatrix}a & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}h\right].\end{aligned}$$

The same identity is true for V_1.

Thus both sides of (13.7.3) vanish if χ_v is ramified for some v in S. Hence (13.7.3) is satisfied for all χ and we obtain as before $V(h) = cV_1(h)$ for $h = e$. We may replace W by a convolute by an element of $\mathcal{H}^S$ and a translate by an element of $Z(\mathbf{A})$ to obtain the same identity for h in $Z(\mathbf{A})G^S$. Finally, by (13.7.1) the same identity is true on $Z(\mathbf{A})G'$. Otherwise said,

$$\int_{U^*} (\phi - c\phi_1)(ug)\bar{\theta}(\gamma u\gamma^{-1})du = 0$$

for $g \in Z(\mathbf{A})G'$ and $\gamma = 1$. As in (13.2), we obtain the same identity for any

$\gamma \in G' \cap P(F) \cap Q(F)$ and, in particular,

$$\int_{(\mathbf{A}/F)^2} (\phi - c\phi_1)\left[\begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} g\right]\psi(\alpha x + \beta y)dxdy = 0 \tag{13.7.4}$$

for all g in G', $\alpha \in F$, $\beta \in F^\times$ satisfying

$$|\alpha|_v \leqq 1 , \quad |\beta|_v = 1$$

for $v \in S$. By (13.7.1) the function of x

$$\int_{\mathbf{A}/F} (\phi - c\phi_1)\left[\begin{pmatrix}1 & 0 & x\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix} g\right]\psi(\beta y)dy$$

is, for $g \in G'$, invariant under the subgroup $\prod_{v \in S} \mathfrak{R}_v$. Thus (13.7.4) is trivially true if, for at least one v in S, $|\alpha|_v > 1$. Thus it is true for all α in F and we get

$$\int_{\mathbf{A}/F} (\phi - c\phi_1)\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} g\right]\psi(\beta x)dx = 0 \tag{13.7.5}$$

for $g \in G'$, $\beta \in F^\times$ with $|\beta|_v = 1$ for all $v \in S$. Again by (13.7.1), (13.7.5) is trivially true if for at least one v in S, $|\beta|_v > 1$. Now if $|\beta|_v < 1$ for some v in S, including the case $\beta = 0$, then

$$\int_{\mathfrak{P}_v^{-1}} dy = \int_{\mathfrak{P}_v^{-1}} \psi(-\beta y)dy \neq 0 ,$$

and

$$\int_{\mathfrak{P}_v^{-1}} \psi(-\beta y)dy \int_{\mathbf{A}/F} (\phi - c\phi_1)\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} g\right]\psi(\beta x)dx$$

$$= \int_{\mathfrak{P}_v^{-1}}\int_{\mathbf{A}/F} (\phi - c\phi_1)\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x+y\\ 0 & 0 & 1\end{pmatrix} g\right]\psi(\beta x)dxdy .$$

Now this vanishes for all $g \in G'$. For we may assume that $g \in G^S$ and then this is

$$\int_{\mathbf{A}/F} \psi(\beta x)dx \int_{\mathfrak{P}_v^{-1}} (\phi - c\phi_1)\left[\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & x\\ 0 & 0 & 1\end{pmatrix} g \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\right] dy$$

which vanishes by (13.7.2). Thus (13.7.5) is true for all $\beta \in F$ and we conclude that on G' or $Z(\mathbf{A})G'$, $\phi = c\phi_1$. To continue we will use the following lemma.

LEMMA (13.7.6). *The group* $G(F) \cap G'$ *is generated by* $P(F) \cap G'$ *and* $Q(F) \cap G'$.

Proof. Every matrix in $G(F) \cap G'$ can be written as a product

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \beta & 1 \end{pmatrix}\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix},$$

the first matrix being in $Q(F) \cap G'$, the last one in $P(F) \cap G'$. The middle one is in $G' \cap G(F)$. But if we select $\gamma \in F^\times$ with $v(\gamma) = 1$ at each v in S, the middle matrix can be written as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \beta\gamma^{-1} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\beta\gamma^{-1} & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and any matrix in this expression is in $P(F) \cap G'$ or $Q(F) \cap G'$.

The lemma being proved, we see that $\phi | G'$ is invariant on the left under $G(F) \cap G'$. Since $G(\mathbf{A}) = G(F)G'$, there is a unique function ϕ_2 on $G(F)\backslash G(\mathbf{A})$ which coincides with ϕ on G'. Because both ϕ and ϕ_2 are invariant under $P(F)$ and

$$P(\mathbf{A}) = P(F)\big(P(\mathbf{A}) \cap G'\big),$$

in fact ϕ and ϕ_2 coincide on the larger set $P(\mathbf{A})G'$.

It is easy to see that ϕ_2 is smooth and that $W \mapsto \phi_2$ is a map commuting with the action of $\mathcal{H}^S$. Since $N(\mathbf{A}) = N(F)N'$ with $N' = N(\mathbf{A}) \cap G'$ we get

$$W(g) = \int_{N' \cap N(F) \backslash N'} \phi(ng)\theta(n)dn , \quad g \in G' ,$$

and the map is injective. Thus the space $\mathcal{V}$ of all the functions ϕ_2 obtained in this way affords the representation ξ^S of $\mathcal{H}^S$. It remains to see that such functions ϕ_2 are slowly increasing.

Let us derive a majorization of

$$\phi_2\left[\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} g\right], \tag{13.7.7}$$

where g is in a compact Ω of $G(\mathbf{A})$, $t_i \in F_\infty^+$, and $t_1/t_2 \geqq c_1$, $t_2/t_3 \geqq c_2$, $t_1t_2t_3 = 1$. Clearly, Ω is contained in a finite union $\bigcup \gamma\Omega'$, where $\gamma \in X$ a (finite) set in $G(F)$, Ω' a compact set in G'. Write accordingly $g = \gamma h$, $\gamma \in X$, $h \in \Omega'$. Then

$$\phi_2\left[\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} g\right] = \phi_2(uh) ,$$

where

$$u = \gamma^{-1}\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}\gamma = \gamma_\infty^{-1}\begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix}\gamma_\infty .$$

But u is in SL(3, F_∞), so $\phi_2(uh) = \phi(uh)$ and, by (12.4),

$$|\phi_2(uh)| \leqq c\,||u||^m \leqq cc_\gamma(t_1^2 + t_2^2 + t_3^2)^{m/2} ,$$

where c_γ is a constant (depending on γ). Thus we have a majorization of (13.7.7) by $c'(t_1^2 + t_2^2 + t_3^2)^{m/2}$ and we see that ϕ_2 is slowly increasing. Q.E.D.

(13.8) Let us examine in more detail the conclusion of (13.7). In particular we want to prove the following complement. As before, S is a given finite set of finite places, and $\pi^S = \bigotimes_{v \notin S} \pi_v$.

THEOREM (13.8). *Under the assumptions of* (13.7) *there is for each* $v \in S$ *a generic representation* π'_v *of* $\mathcal{H}_v$ *of central character* ω_v *with the following property. Set* $\pi' = \bigotimes_{v \in S} \pi'_v \otimes \pi^S$ (*so that* $\pi'_v \cong \pi_v$ *for* $v \notin S$). *Then for any character* χ *of* $F_{\mathbb{A}}^\times/F^\times$ *the function* $L(s, \pi' \otimes \chi)$ *is meromorphic and satisfies*

$$L(s, \pi' \otimes \chi) = \varepsilon(s, \pi' \otimes \chi)L(1 - s, \tilde{\pi}' \otimes \chi^{-1}) .$$

Moreover π'_v *is uniquely determined by this condition.*

Proof. Let us show first the uniqueness of the π'_v, $v \in S$. Suppose π''_v, $v \in S$, is another choice. Let w be a place in S. By (7.1.6) it is enough to show that, for any character η of $F_w^\times$,

(13.8.1) $$\varepsilon'(s, \pi'_w \otimes \eta, \psi_w) = \varepsilon'(s, \pi''_w \otimes \eta, \psi_w) .$$

Now

$$L(s, \pi')/L(s, \pi'') = \prod\nolimits_{v \in S} L(s, \pi'_v)/L(s, \pi''_v) ,$$
$$L(s, \tilde{\pi}')/L(s, \tilde{\pi}'') = \prod\nolimits_{v \in S} L(s, \tilde{\pi}'_v)/L(s, \tilde{\pi}''_v) ,$$

and

$$\varepsilon(s, \pi')/\varepsilon(s, \pi'') = \prod\nolimits_{v \in S} \varepsilon(s, \pi'_v, \psi_v)/\varepsilon(s, \pi''_v, \psi_v) .$$

Comparing functional equations, we get

$$\prod\nolimits_{v \in S} \varepsilon'(s, \pi_v, \psi_v) = \prod\nolimits_{v \in S} \varepsilon'(s, \pi'_v, \psi_v)$$

or, replacing π' and π'' by $\pi' \otimes \chi$ and $\pi'' \otimes \chi$,

$$\prod\nolimits_{v \in S} \varepsilon'(s, \pi'_v \otimes \chi_v, \psi_v) = \prod\nolimits_{v \in S} \varepsilon'(s, \pi''_v \otimes \chi_v, \psi_v)$$

for all characters χ of $F_{\mathbf{A}}/F^{\times}$. We can select a χ such that $\chi_w = \eta$ and χ_v is as ramified as we wish for v in S, $v \neq w$. Then, by (5.6),

$$\varepsilon'(s, \pi'_v \otimes \chi_v, \psi_v) = \varepsilon'(s, \pi''_v \otimes \chi_v, \psi_v) ,$$

for $v \in S$, $v \neq w$; and then (13.8.1) follows.

To prove the existence of the π'_v we will need the following proposition. It is best to state it for all r so that $G = \mathrm{GL}(r)$. Since the proof is essentially given in [23, § 10] we give only an outline here. Set $\mathcal{H}^S = \otimes_{v \notin S} \mathcal{H}_v$, as before.

PROPOSITION (13.8.2). *Let F be an* **A***-field and S a finite set of finite places. Let ϕ be a continuous function on $G(\mathbf{A})$, R an F-parabolic subgroup of G, $R = MU$ a Levi-decomposition of R, Z_M the center of M. We assume that ϕ is invariant under $M(F)U(\mathbf{A})$ on the left, K-finite on the right and that the representation of $\mathcal{H}^S$ on $\rho(\mathcal{H}^S)\phi$ is admissible. Then ϕ is $Z_M(\mathbf{A})$-finite on the left.*

We shall use the following lemma. Let T be a finite set of places and ϕ a continuous function on G_T. We will say that ϕ is $\mathcal{H}_T$-admissible if the representation of $\mathcal{H}_T$ on $\rho(\mathcal{H}_T)\phi$ is admissible.

LEMMA (13.8.3). *Let T be a finite set of places. Suppose R is a parabolic subgroup of G, $R = MU$, and Z_M is the center of M. Let ϕ be a continuous function on G_T, K_T-finite on the right, invariant under $U_T = \prod_{v \in T} U_v$ on the left. Suppose ϕ is $\mathcal{H}_T$-admissible. Then ϕ is $(Z_M)_T$-finite on the left.*

We take the lemma for granted. To prove the proposition we introduce for each finite set T of places the set $\Omega(T)$ of all ideles x whose component x_v, $v \notin T$, has module one. We can choose T, containing the archimedean places, so that $T \cap S = \varnothing$ and $\mathbf{I} = F^{\times}\Omega(T)$. Let K' be an open compact subgroup of G_0 such that ϕ is invariant by K' on the right. Enlarging T if necessary, we can write $G(\mathbf{A})$ as a finite union

$$G(\mathbf{A}) = \bigcup M(F)U(\mathbf{A})G_T g_j K'$$

where the g_j are in G_S.

Now set $Z_T = \prod_{v \in T} (Z_M)_v$ and let Z' denote the set of $a \in Z_M(\mathbf{A})$ such that $a_v = 1$ for $v \in T$ and $a_v \in K_v$ for $v \notin T$. Then

$$Z_M(\mathbf{A}) = Z_M(F)Z_T Z' .$$

Since $G(\mathbf{A}) = R(\mathbf{A})K$ it is clear that ϕ is Z'-finite on the left. Let $\{\phi_i\}$ be a finite basis for the space of left-translates of ϕ under Z'. It suffices to show that each ϕ_i is Z_T-finite. Since each ϕ_i is $U(\mathbf{A})$-invariant on the left and K' invariant on the right it will suffice to show that each function ψ of the form $\psi(g) = \phi_i(gg_j)(g \in G(\mathbf{A}))$ has a Z_T-finite restriction to G_T. Now each ψ is clearly $\mathcal{H}_T$-admissible, and since a quotient of an admissible representation is admissible, each of their restrictions to G_T is also $\mathcal{H}_T$-admissible. Our conclusion follows from Lemma (13.8.2).

We return to the proof of the existence of the π'_v. Let ${}^0L_2(G(F)\backslash G(\mathbf{A}), \omega)$ denote the Hilbert space of functions on $G(\mathbf{A})$ transforming like ω under $Z(\mathbf{A})$, square integrable mod $G(F)Z(\mathbf{A})$, and cuspidal.

(13.8.4) Let $\mathcal{V}$ be as in (13.7). Suppose $\mathcal{V}$ is contained in ${}^0L_2(G(F)\backslash G(\mathbf{A}),\omega)$. Since the representation of $G(\mathbf{A})$ on the latter space is a direct sum of irreducible representations, each representation ξ_v is the direct sum of its components. These components are generic and since ξ_v has a unique generic component we find that ξ_v is irreducible, or $\xi_v = \pi_v$ for all $v \notin S$. Moreover there is an invariant irreducible subspace $\mathcal{V}$ of ${}^0L_2$ and a $\mathcal{H}^S$-morphism from $\mathcal{V}$ to $\mathcal{V}'$. Call π' the representation of $\mathcal{H}$ on $\mathcal{V}$. Then the representation of $\mathcal{H}_v$ on $\mathcal{V}'$ is a multiple of π'_v for all v. Thus $\pi_v \cong \pi'_v$ for $v \notin S$. Since π_v is generic for all v, our contention is obvious.

(13.8.5) Now suppose ϕ is a non-cuspidal element of $\mathcal{V}$. Let R be a horicycle in G with the property that

$$\phi_R(g) = \int_{R^*} \phi(rg)dr$$

is non-zero, and maximal with this property. Essentially three cases remain. Suppose first that $R = N$. By Proposition (13.8.2), ϕ_N is $A(\mathbf{A})$-finite on the left. In particular there is a non-zero function f_0 on $G(\mathbf{A})$ which is a linear combination of left-translates of ϕ_N under $A(\mathbf{A})$ and which transforms on the left under $A(\mathbf{A})$ according to a quasi-character of $A(\mathbf{A})$ trivial on $A(F)$. Of course f_0 is $N(\mathbf{A})$-invariant on the left, K-finite on the right, i.e., f_0 belongs to the space of an induced representation $\eta = I(G(\mathbf{A}), B(\mathbf{A}); \sigma)$. Of course σ is trivial on $N(\mathbf{A})A(F)$.

Let ξ' be the representation of $\mathcal{H}^S$ on $\rho(\mathcal{H}^S)\phi$. There is a map of $\mathcal{H}^S$-modules $\xi' \to \eta$ sending ϕ to f. Choosing ϕ to correspond to an element e of the form $e = \otimes_{v\notin S} e_v$ in the factorable representation $\xi^S = \otimes_{v\notin S}\xi_v$, we get $\xi' = \otimes_{v\notin S}\xi'_v$, where ξ'_v is the representation of $\mathcal{H}_v$ on $\xi_v(\mathcal{H}_v)e_v$ and, for almost all v, e_v is K_v-fixed. Set $\eta_v = I(G_v, B_v; \sigma_v)$ and note that $\eta = \otimes_v \eta_v$. There is then, for each $v \notin S$, an $\mathcal{H}_v$ morphism $\xi'_v \to \eta_v$ where in each case e_v has a non-zero image. Thus, for $v \notin S$, ξ_v and η_v have a common component ζ_v which contains the trivial representation of K_v for almost all v. We let, for $v \in S$, π'_v be the generic component of η_v (cf. (6.1)). Then for all χ,

$$\varepsilon'(s, \pi'_v \otimes \chi_v, \psi_v) = \varepsilon'(s, \eta_v \otimes \chi_v, \psi_v)$$

for $v \in S$. For $v \notin S$ let $\pi'_v = \pi_v$; then π'_v is a component of ξ_v; so the same is true for $v \notin S$. Moreover for almost all v, π'_v and ζ_v contain the trivial representation of K_v so that in fact $\pi'_v = \zeta_v$. Hence, given χ, for almost all v

$$L(s, \pi'_v \otimes \chi_v) = L(s, \eta_v \otimes \chi_v) .$$

It follows that $L(s, \pi' \otimes \chi)$ is meromorphic. On the other hand by [17, (Theorem 3.4)] the functions

$$L(s, \eta) = \prod_v L(s, \eta_v) , \quad L(s, \tilde{\eta}) = \prod_v L(s, \tilde{\eta}_v) , \quad \varepsilon(s, \eta) = \prod_v \varepsilon(s, \eta_v, \psi_v) ,$$

satisfy

$$L(s, \eta) = \varepsilon(s, \eta)L(1 - s, \tilde{\eta}) .$$

It follows that $L(s, \pi')$ and more generally $L(s, \pi' \otimes \chi)$ satisfy the required equation.

(13.8.6) Suppose $\phi_N = 0$ but that $\phi_U = 0$. Again by Proposition (13.8.2), ϕ_U is $Z_M(\mathbf{A})$-finite. Here $P = MU$ is the parabolic of type (2, 1). As before, there is a non-zero function f_0 on $G(\mathbf{A})$, which is a linear combination of left translates of ϕ_U by elements of $Z_M(\mathbf{A})$, transforming on the left under $Z_M(\mathbf{A})$ according to a quasi-character μ trivial on $Z_M(F)$.

Again let ξ' denote the representation of $\mathcal{H}^S$ on $\rho(\mathcal{H}^S)\phi$. The representation of $\mathcal{H}^S$ on $\rho(\mathcal{H}^S)f$ is a quotient of ξ'.

Now it is easy to see that, for each $g \in K$, each function $m \mapsto f(mg)$, $f \in \rho(\mathcal{H}^S)f_0$ is slowly increasing on $M(\mathbf{A})$. Moreover there is $u \in \mathbf{R}$ so that the functions $m \mapsto f(mg)\,|\det m|^u$ actually belong to ${}^0L_2(M(F)/M(\mathbf{A}), \mu)$. Projecting onto an appropriate component of the latter space, we obtain a non-zero $\mathcal{H}^S$-morphism from $\rho(\mathcal{H}^S)f_0$ to the space of an induced representation $\eta = I(G(\mathbf{A}), P(\mathbf{A}); \sigma)$, σ being a representation of $M(\mathbf{A})$ which is automorphic and cuspidal. Set $\eta_v = I(G_v, P_v; \sigma_v)$. For $v \in S$, we take π'_v to be the generic component of η_v and proceed exactly as in (13.8.5).

Finally we must consider the case when $\phi_N = 0$ but $\phi_V \neq 0$, V being the radical of a parabolic of type (1.2). The proof is essentially as before. This concludes the proof of (13.8).

In passing, note the following result:

PROPOSITION (13.8.7). *Suppose π and π' are two automorphic cuspidal representations of* GL(3, **A**). *Suppose also that $\pi_v \cong \pi'_v$ for all v outside a finite set of finite places. Then in fact $\pi_v \cong \pi'_v$ for all v.*

Proof. First if ω (resp. ω') is the central character of π (resp. π'), then $\omega_v = \omega'_v$ for almost all v. This implies that $\omega = \omega'$. Then, since π_v and π'_v are generic, the proof is exactly the same as the proof of the uniqueness assertion of (13.8).

(13.9) In the applications we will often use the above theorems in the following form:

PROPOSITION (13.9.1). *Let π be given as in* (13.1). *Let S be a finite set of finite places. Assume that, for any character χ of $F_{\mathbf{A}}^{\times}/F^{\times}$ whose ramification at each place v in S is sufficiently high, $L(s, \pi \otimes \chi)$ is entire, bounded in vertical strips if F is a number field, and satisfies*

$$L(s, \pi \otimes \chi) = c\varepsilon(s, \pi \otimes \chi)L(1 - s, \tilde{\pi} \otimes \chi^{-1}) ,$$

where c is independent of χ. Then $c = 1$ and the conclusions of (13.7) *and* (13.8) *apply.*

Proof. We may apply (13.8) to the representation $\pi \otimes \eta$, where η_v is sufficiently ramified for $v \in S$. We obtain then a representation $\pi' \otimes \eta$ such that $\pi'_v = \pi_v$ for $v \notin S$ and for which

$$L(s, \pi' \otimes \chi) = \varepsilon(s, \pi' \otimes \chi)L(1 - s, \tilde{\pi}' \otimes \chi^{-1})$$

for all characters χ of $F_{\mathbf{A}}^{\times}/F^{\times}$. Comparing this with the corresponding statement for π, we obtain

$$c \prod_{v \in S} \varepsilon'(s, \pi_v \otimes \chi_v, \psi_v) = \prod_{v \in S} \varepsilon'(s, \pi'_v \otimes \chi_v, \psi_v) ,$$

whenever χ_v is sufficiently ramified for $v \in S$. But by taking this ramification to be high enough and applying (5.6), we obtain $c = 1$. Q.E.D.

We also have:

PROPOSITION (13.9.2). *Suppose the conditions of* (13.6) *are satisfied except that the functional equation reads*

$$L(s, \pi \otimes \chi) = c\, \varepsilon(s, \pi \otimes \chi)L(1 - s, \tilde{\pi} \otimes \chi^{-1}) ,$$

where c is a constant independent of χ. Then $c = 1$ and the conclusion of (13.6) *applies.*

Proof. Take $S = \{v\}$, v being any finite place and apply (13.9.1) to get $c = 1$. Then apply (13.6). Q.E.D.

In (13.9.1) it should be noted that the representations π_v, $v \in S$, which appear in the hypothesis really play an artificial role. More precisely, (13.9.1) should be stated in the following way:

PROPOSITION (13.9.3). *Let ω be a character of $F_{\mathbf{A}}^{\times}/F^{\times}$ and S a finite set of finite places. For $v \notin S$ let π_v be an irreducible representation of $\mathcal{H}_v$ of central character ω_v. Assume the relevant conditions of* (13.1) *are satisfied. Suppose that, for any character χ of sufficient ramification at each v in S, the functions*

$$L(s, \pi^S, \chi) = \prod_{v \notin S} L(s, \pi_v \otimes \chi_v) , \quad L(s, \tilde{\pi}^S, \chi) = \prod_{v \notin S} L(s, \tilde{\pi}_v \otimes \chi_v) ,$$

are entire, bounded in vertical strips if F is a number field and satisfy

$$L(s, \pi^S, \chi)$$
$$= c \prod_{v \notin S} \varepsilon(s, \pi_v \otimes \chi_v, \psi_v) \prod_{v \in S} \varepsilon(s, \chi_v \omega_v, \psi_v) \varepsilon(s, \chi_v, \psi_v)^2 L(1-s, \tilde{\pi}^S, \chi^{-1}) ,$$

where c is a constant independent of χ. Then $c = 1$ and the conclusions of (13.7) *and* (13.8) *apply.*

Proof. For $v \in S$, let π_v be any irreducible admissible representation of G_v with central character ω_v. If χ is sufficiently ramified at each $v \in S$,

$$L(s, \pi_v \otimes \chi_v) = L(s, \tilde{\pi}_v \otimes \chi_v^{-1}) = 1$$

and

$$\varepsilon(s, \pi_v \otimes \chi_v, \psi_v) = \varepsilon(s, \chi_v \omega_v, \psi_v) \varepsilon(s, \chi_v, \psi_v)^2 .$$

Then the hypotheses of (13.9.1) are satisfied and our conclusion follows.

14. Applications

We give some applications to division algebra of degree 9 and 3-dimensional representations of the W-group.

THEOREM (14.1). *Let H be a division algebra of center F and degree 9, σ an automorphic unitary irreducible representation of $H_{\mathbf{A}}^{\times}$ which is not one-dimensional. Call ω its central character.*

(1) *Let S be the finite set of (finite) places v where H does not split. For each v in S, there is a unique unitary irreducible representation π_v of G_v whose central character is ω_v such that, for all characters χ of $F_v^{\times}$,*

$$L(s, \pi_v \otimes \chi) = L(s, \sigma_v \otimes \chi) , \quad L(s, \tilde{\pi}_v \otimes \chi) = L(s, \tilde{\sigma}_v \otimes \chi) ,$$
$$\varepsilon(s, \pi_v \otimes \chi, \psi_v) = \varepsilon(s, \sigma_v \otimes \chi, \psi_v) .$$

Moreover π_v is generic.

(2) *For $v \notin S$, let π_v be the representation of G_v obtained from σ_v by transport of structure ($G_v \cong H_v^{\times}$). Then π_v is generic.*

(3) *The representation $\pi = \otimes \pi_v$ of $G(\mathbf{A})$ is automorphic cuspidal.*

Proof. Let at first π_v be, for $v \in S$, any irreducible representation of central character ω_v. If χ is any character of $F_{\mathbf{A}}^{\times}/F^{\times}$ highly ramified at each $v \in S$, we have (cf. (5.1)), for $v \in S$,

$$L(s, \pi_v \otimes \chi_v) = L(s, \sigma_v \otimes \chi_v) = 1 , \quad L(s, \tilde{\pi}_v \otimes \chi_v^{-1}) = L(s, \tilde{\sigma}_v \otimes \chi_v^{-1}) = 1$$

and

$$\varepsilon(s, \pi_v \otimes \chi_v, \psi_v) = \varepsilon(s, \sigma_v \otimes \chi_v, \psi_v) .$$

From the functional equation for $L(s, \sigma \otimes \chi)$, we have

$$L(s, \pi \otimes \chi) = \varepsilon(s, \pi \otimes \chi)L(1 - s, \tilde{\pi} \otimes \chi^{-1}) ,$$

whenever χ_v is highly ramified for each $v \in S$. We may apply (13.9.1) (with $c = 1$), and in fact select π_v for $v \in S$, generic with central character ω_v, so that the functional equation above is true *for all characters* χ of $F_{\mathbf{A}}^{\times}/F^{\times}$, both sides being only meromorphic at first.

Since the ε and L factors are the same for $\pi \otimes \chi$ and $\sigma \otimes \chi$ outside S, we get, by comparing the functional equations for π and σ,

$$\prod_{w \in S} \varepsilon'(s, \pi_w \otimes \chi_w, \psi_v) = \prod_{w \in S} \varepsilon'(s, \sigma_w \otimes \chi_w, \psi_w) ,$$

for all characters χ of $F_{\mathbf{A}}^{\times}/F^{\times}$. Taking χ_w highly ramified for all but one place v in S, we obtain

$$\varepsilon'(s, \pi_v \otimes \eta_v, \psi_v) = \varepsilon'(s, \sigma_v \otimes \eta, \psi_v) ,$$

for all characters η of $F_v^{\times}$.

If σ_v is not one-dimensional, by (4.4) in [17], the right side is a monomial in q^{-s}. Hence by (7.5.5), π_v is supercuspidal. Therefore, the relations (14.1.1) are satisfied in this case, the L-factors being equal to one.

If σ_v is one-dimensional, then $\sigma_v = \zeta \circ \nu$, ν being the reduced norm and ζ a character of $F_v^{\times}$. Let σ_3 be the special representation. Recall that it is the generic component of $I(G_v, B_v; \eta_1, \eta_2, \eta_3)$ where $\eta_i = \alpha_v^{3/2-i}$. Then

$$\varepsilon'(s, \sigma_3 \otimes \zeta\eta, \psi_v) = \prod_{i=1}^{3} \varepsilon'(s, \eta_i\zeta\eta, \psi_v) .$$

But by (5.6.3), (4.7) and (7.11) of [17], the right side is also $\varepsilon'(s, \sigma_v \otimes \zeta\eta, \psi_v)$. Since σ_3 is generic, comparing with the previous equality, we must have $\pi_v = \sigma_3 \otimes \zeta$. Then, by (4.4) and (7.11) of [17], it follows that

$$L(s, \pi_v \otimes \eta) = L(s, \sigma_v \otimes \eta) ,$$

for all characters η of $F_v^{\times}$. Hence (14.1.1) is completely established.

Taking π_v, for $v \notin S$, as in (14.1.2) we may apply (13.6) directly to conclude that π is automorphic cuspidal.

(14.2) We give an application to Artin-Hecke L-functions. Let W_F be the W-group attached to F and W_v the W-group attached to F_v. Let σ be an irreducible unitary representation of W_F and call σ_v the composite of σ with the natural homomorphism $W_v \to W_F$.

THEOREM (14.2). *Assume that σ is of degree* 3 *and that, for any character χ of $F_{\mathbf{A}}^{\times}/F^{\times}$, the function $L(s, \sigma \otimes \chi)$ is entire, bounded in vertical strips if F is a number field. Then*:

(1) *For each finite place v, there is a unique irreducible unitary representation π_v of central character $\omega_v = \det \sigma_v$ such that, for any character η of $F_v^{\times}$,*

$$L(s, \pi_v \otimes \eta) = L(s, \sigma_v \otimes \eta), \quad L(s, \tilde{\pi}_v \otimes \eta) = L(s, \tilde{\sigma}_v \otimes \eta)$$

and

$$\varepsilon(s, \pi_v \otimes \eta, \psi_v) = \varepsilon(s, \sigma_v \otimes \eta, \psi_v) .$$

Moreover π_v *is generic. For* v *infinite, set* $\pi_v = \pi(\sigma_v)$ ((10.3)).

(2) *The representation* $\pi = \otimes_v \pi_v$ *of* $G(\mathbf{A})$ *is automorphic cuspidal.*

Proof. We have to use the following facts.

(14.2.3) If v is finite and σ_v is irreducible, then $L(s, \sigma_v) = 1$. If $\omega_v = \det \sigma_v$ and χ_1, χ_2, and χ_3 are quasi-characters of $F_v^\times$ whose product is ω_v, then as soon as η is sufficiently ramified,

$$L(s, \sigma_v \otimes \eta) = 1 , \quad \varepsilon(s, \sigma_v \otimes \eta, \psi_v) = \prod_{i=1}^3 \varepsilon(s, \chi_i \eta, \psi_v) .$$

(See [9].)

Given σ_v a unitary representation of W_v, let $\pi(\sigma_v)$, when it exists, be the unique irreducible representation of G_v satisfying the relations of (14.2.1).

There are a number of cases when the existence is easy to establish.

If σ_v has the form

$$\sigma_v = \mu_{1v} \oplus \mu_{2v} \oplus \mu_{3v}, \; \omega_v = \mu_{1v}\mu_{2v}\mu_{3v} , \tag{14.2.4}$$

where μ_{iv} is a character of W_v (or $F_v^\times$), we may take for $\pi(\sigma_v)$ the induced representation $I(G_v, B_v; \mu_{1v}, \mu_{2v}, \mu_{3v})$. Note that $\pi(\sigma_v)$ is then generic.

Suppose σ_v has the form

$$\sigma_v = \tau_v \oplus \mu_v , \tag{14.2.5}$$

where τ_v is irreducible of degree two, μ_v is a character and $\omega_v = \mu_v \det \tau_v$. There are a number of cases when the representation of GL(2, F_v), denoted by $\rho_v = \pi_v(\tau_v)$ is known to exist. Recall that it is generic, its central character is $\det \tau_v$, and that, for all characters η of $F_v^\times$,

$$L(s, \rho_v \otimes \eta) = L(s, \tau_v \otimes \eta) , \quad L(s, \tilde{\rho}_v \otimes \eta) = L(s, \tilde{\tau}_v \otimes \eta) ,$$
$$\varepsilon(s, \rho_v \otimes \eta, \psi_v) = \varepsilon(s, \tau_v \otimes \eta, \psi_v) .$$

In that case we may take for $\pi(\sigma_v)$ the generic representation $I(G_v, P_v; \rho_v, \mu_v)$. Note that either (14.2.4) or (14.2.5) applies to each infinite place (cf. (10.3)).

If we knew the existence of $\pi(\sigma_v)$ for all v, the hypotheses of (13.1) being then automatically satisfied, we could apply (13.6) directly to conclude that $\pi = \otimes \pi(\sigma_v)$ is automorphic cuspidal.

In fact we proceed instead as in (14.1). Let S then be any finite set of finite places such that for $v \notin S$, $\pi_v = \pi(\sigma_v)$ exists. First choose in any way the representations π_v, $v \in S$, with central character ω_v. Set $\pi = \otimes_v \pi_v$. Then, whenever χ is highly ramified at v in S,

$$L(s, \pi \otimes \chi) = \varepsilon(s, \pi \otimes \chi) L(1 - s, \tilde{\pi} \otimes \chi^{-1}) .$$

Here of course we use (14.2.3). Thus, by (13.9.1), we can choose the π_v anew, for $v \in S$, so that this functional equation is satisfied for all χ.

As before we obtain

$$\varepsilon'(s, \pi_v \otimes \eta, \psi_v) = \varepsilon'(s, \sigma_v \otimes \eta, \psi_v)$$

for any $v \in S$, any character η of $F_v^\times$. If σ_v is irreducible, the right side is a monomial in q^{-s}. Thus π_v is supercuspidal. Since the L-factor for σ_v or π_v is one, we see that $\pi_v = \pi(\sigma_v)$.

If σ_v has the form (14.2.4) we already know the existence of $\pi(\sigma_v)$. Finally suppose σ_v has the form (14.2.5). We have, for all characters η of $F_v^\times$,

$$L(1 - s, \tilde{\sigma}_v \otimes \eta^{-1})/L(s, \sigma_v \otimes \eta) = L(1 - s, \mu_v^{-1}\eta^{-1})/L(s, \mu_v \eta) .$$

The factor $L(1 - s, \tilde{\pi}_v \otimes \eta^{-1})/L(s, \pi_v \otimes \eta)$ differs from this by a unit in $C[q^{-s}, q^s]$. Checking the complete list of irreducible representations of G_v shows that π_v is necessarily of the form $\pi_v = I(G_v; P_v, \rho_v, \mu_v)$ where ρ_v is a supercuspidal representation of $G_2(F_v)$ with central character $\omega_v \mu_v^{-1}$. Thus ρ_v is unitary and so is π_v. Finally

$$L(s, \rho_v \otimes \eta) = L(s, \tau_v \otimes \eta) = L(s, \tilde{\rho}_v \otimes \eta) = L(s, \tilde{\tau}_v \otimes \eta) = 1$$

for all η. Hence:

$$L(s, \pi_v \otimes \eta) = L(s, \sigma_v \otimes \eta) \quad \text{and} \quad L(s, \tilde{\pi}_v \otimes \eta) = L(s, \tilde{\sigma}_v \otimes \eta) .$$

Thus again $\pi_v = \pi(\sigma_v)$.

Indeed $\pi(\sigma_v)$ exists for all v and we may apply (13.6) to obtain (14.2.2).

The theorem of course applies to monomial representations. More precisely let K be a separable cubic extension of F. A character ρ of $K_A^\times$ may be regarded as a one-dimensional representation of W_K. The theorem applies then to the representation σ of W_F induced by ρ,

$$\sigma = I(W_F, W_K; \rho) ,$$

provided σ is irreducible. We remark that if K/F is normal σ is either irreducible or a direct sum of three one-dimensional representations. If K/F is not normal it is easy to see that either σ is irreducible or a sum of a character and a monomial representation of degree two.

(14.3) Using the global theory one can derive purely local results.

PROPOSITION (14.3.1). *Let F be a local field, H a division algebra of center F and degree 9, σ an irreducible unitary representation of $H^\times$ of central character ω. Then there is a unique representation π of* GL(3, F)

with central character ω *such that*

$$L(s, \pi \otimes \chi) = L(s, \sigma \otimes \chi) , \quad L(s, \tilde{\pi} \otimes \chi) = L(s, \tilde{\sigma} \otimes \chi) ,$$
$$\varepsilon(s, \pi \otimes \chi, \psi) = \varepsilon(s, \sigma \otimes \chi, \psi)$$

for all characters χ *of* $F^\times$. *Moreover* π *is unitary and square integrable.*

Proof. The uniqueness of π is clear (7.5.3). If $\sigma = \eta \circ \nu$ then $\pi = \sigma_3 \otimes \eta$ is square-integrable. If σ is not one-dimensional then, by the argument used in the proof of (14.1), π is supercuspidal and thus, square-integrable.

For the existence we may, after changing notation, assume that the given local field is F_w, where F is global, and the given local division algebra is H_w, where H is a global division algebra. We have an irreducible unitary representation σ_w of $H_w^\times$. Let f_w be a matrix coefficient of the admissible representation σ_w such that $f_w(e) \neq 0$. Extend in any way the central character ω_w of σ_w to a character ω of $\mathbf{I}/F^\times$. For each $v \neq w$, let f_v be a smooth function on $H_v^\times$ which transforms under the center Z_v of $H_v^\times$ according to ω_v and is compactly supported mod Z_v. Let K_v' be a maximal compact subgroup of H_v chosen as on page 305 of [23]. We assume that, for almost all v, f_v has support in $Z_v K_v'$ and is invariant under K_v'.

Set $f(g) = \prod_v f_v(g_v)$ and let

$$\phi(g) = \sum f(\xi g) , \quad \xi \in Z(F)\backslash H^\times(F) .$$

For g in a compact set the series has only finitely many terms. By shrinking the support of f at some place other than w we may assume $\phi(e) = f(e)$. Thus we may choose f so that $\phi \neq 0$. Thus ϕ is a smooth function on $H^\times(F)\backslash H^\times(\mathbf{A})$ transforming under $Z(\mathbf{A})$ according to ω and thus has a non-zero projection on some irreducible component of $L^2(H^\times(F)\backslash H^\times(\mathbf{A}), \omega)$. It follows that there is an automorphic representation σ of $H^\times(\mathbf{A})$ whose component at w is σ_w and it suffices to apply (14.1). Q.E.D.

PROPOSITION (14.3.2). *Let* F *be a local field,* K *a separable cubic extension,* ρ *a character of* $K^\times$ *and* σ *the representation*

$$I(W_F, W_K; \rho) .$$

Then $\pi(\sigma)$ *exists and is supercuspidal if* σ *is irreducible.*

Proof. Suppose first σ is not irreducible. If $\sigma = \mu_1 \oplus \mu_2 \oplus \mu_3$ where each μ_i is a character of W_F or $F^\times$, then $\pi(\sigma) = I(G, B; \mu_1, \mu_2, \mu_3)$. If $\sigma = \tau \oplus \mu$ where μ is a character and τ a two-dimensional irreducible representation, then τ is monomial, the representation $\pi(\tau)$ of $G_2(\mathbf{A})$ is defined and $\pi(\sigma) = I(G, P; \pi(\tau), \mu)$. Assume σ is irreducible. Changing notations we may assume that the given fields are F_w and K_w where F and K are global, K is a cubic extension of F, w a place of F which does not split in K. Moreover we may assume that the given character of $K_w^\times$ has the form ρ_w where ρ is a character of $K_A^\times/K^\times$. Then if

$$\sigma = I(W_F, W_K; \rho) ,$$

we see that σ_w is the given representation of W_w. Certainly σ is irreducible and thus by (14.2) the representation $\pi(\sigma_w)$ exists. Q.E.D.

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REFERENCES

[1] B. BLAIKADAR, Thesis, University of California, Berkeley, 1900.

[2] I. N. BERNSTEIN and A. V. ZELEVINSKY, Representations of the group GL(n, F) where F is a non-archimedean local field, Uspekhi Mat. Nauk **31**: 3 (1976), 5–70 (Russian Math. Surveys **31**: 3 (1976), 1–68).

[3] ———, Induced representations of the group GL(n) over a p-adic field, Funkcional Anal. i Prilǒzen, **10**: 3 (1976), 74–75 (=Functional Anal. Appl. **10**: 3 (1976), 225–227).

[4] ———, Induced representations of reductive p-adic groups I, preprint.

[5] A. BOREL, Introduction aux Groupes Arithmétiques, Hermann, (1969).

[6] ———, Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup, Inven. Math. **35** (1969), 233–259.

[7] W. CASSELMAN, The Steinberg character as a true character, in *Harmonic Analysis on Homogeneous Spaces*, Proc. Sympos. Pure Math. **26** (1972), 413–418.

[8] ———, Some general results in the theory of admissible representations of p-adic reductive groups, preprint.

[9] P. DELIGNE, Les constantes des équations fonctionelles des fonctions L, in *Modular Functions of One Variable* II, Springer Verlag, Lecture Notes **343** (1973), 501–597.

[10] D. E. FLATH, A comparison of the automorphic representations of GL(3) and its twisted form, Thesis, Harvard University, 1977.

[11] A.I. FOMIN, Quasi-simple irreducible representations of SL($3, R$), Funkcional Anal. i Prilǒzen, **9**: 3 (1975), 67–74 (=Functional Anal. Appl. **9**: 3 (1975), 237–243).

[12] I. M. GELFAND and M. I. GRAEV, Unitary representations of the real unimodular group, Izv. Akad Nauk. SSR, Ser. Math. **17** (1953), 189–248 (=A. M. S. Trans. Ser. 2, **2**, 147–205).

[13] I. M. GELFAND and D. A. KAZDAN, Representations of GL(n, K) where K is a local field, in *Lie Groups and their Representations*, John Wiley and Sons (1975), 95–118.

[14] I. M. GELFAND and M. A. NAIMARK, Unitary representations of the complex unimodular group, Math. Sbornik 21: 63 (1947), 405–434 (=A. M. S. Trans. Ser. 1, **9**, 1–41).

[15] ———, *Unitary Representations of the Classical Groups*, Trudy Mat. Inst. Steklov. **36** (1950) (= *Unitäre Darstellungen der klassichen Gruppen*, Akademie Verlag, Berlin, 1957).

[16] I. M. GELFAND, M. I. GRAEV and I. I. PYATETSKII-SHAPIRO, *Generalized Functions*, Vol. 6, Nauka Press, 1966 (= *Representation Theory and Automorphic Functions*, W. B. Saunders, 1969).

[17] R. GODEMENT and H. JACQUET, *Zeta Functions of Simple Algebras*, Springer-Verlag, Lecture Notes, **260** (1972).

[18] A. D. GVISHANI, The problem of integral geometry on the group $P_n(k)$ and its application to the theory of representations, Funkcional Anal. *i* Prilǒzen, **10**: 3 (1976), 76–78 (=Functional Anal. Appl. **10**: 3 (1976), 227–229).

[19] R. HOWE and A. SILBERGER, Why any unitary principal series representation of SL_n over a p-adic field decomposes simply, Bull. A. M. S. **81**: 3 (1975), 599-601.

[20] ———, Any unitary principal series representation of GL_n over a p-adic field is irreducible, Proc. A. M. S. **54** (1976), 376-378.

[21] H. JACQUET, Generic representations in non-commutative harmonic analysis, Marseille-Luminy 1976, Springer-Verlag Lecture Notes **587** (1976), 91-100.

[22] ———, Fonctions L automorphes du groupe linéaire, in *Automorphic Forms, Representations, and L-functions*, A. M. S. Summer Institute, 1977.

[23] H. JACQUET and R. P. LANGLANDS, *Automorphic Forms on* GL(2), Springer-Verlag Lecture Notes **114** (1970).

[24] H. JACQUET, I. I. PIATETSKII-SHAPIRO and J. A. SHALIKA, Construction de formes automorphes pour le groupe GL(3). C. R. Acad. Paris, **282** (1976), 91-93.

[25] ——— , Construction of cusp forms on GL(3), University of Maryland, Lecture Notes **16** (1975).

[26] A. W. KNAPP and G. ZUCKERMAN, Classification theorem for representations of semi-simple Lie groups, in *Non-Commutative Harmonic Analysis*, Marseille-Luminy 1976, Springer-Verlag Lecture Notes **587** (1976), 139-159.

[27] R. P. LANGLANDS, Problems in the theory of automorphic forms, in *Lectures on Modern Analysis and Applications*, Springer-Verlag Lecture Notes **170** (1970), 18-86.

[28] ———, On the classification of irreducible representations of real algebraic groups, preprint.

[29] I. I. PIATETSKII-SHAPIRO, Euler subgroups, in *Lie Groups and Their Representations*, John Wiley and Sons (1975), 597-620.

[30] ———, Zeta functions of GL(n), mimeographed notes, University of Maryland, 1976.

[31] F. RODIER, Whittaker models for admissible representations of real algebraic groups, in *Harmonic Analysis on Homogeneous Spaces*, Proc. Symp. Pure Math. **26**, (1976), 425-430.

[32] ———, Modèles de Whittaker des représentations admissibles des groupes réductifs p-adiques déployés, preprint.

[33] ———, Décomposition spectrale des représentations lisses, in *Non-Commutative Harmonic Analysis*, Marseille-Luminy 1976, Springer-Verlag, Lecture Notes **587** (1976), 177-195.

[34] J. A. SHALIKA, The multiplicity one theorem for GL(n), Annals of Math. **100** (1974), 171-193.

[35] T. SHINTANI, On an explicit formula for class-1 "Whittaker functions" on GL_n over p-adic fields, Proc. Japan Acad. **52** (1976), 180-182.

[36] A. SILBERGER, Classification of representations of p-adic reductive groups, preprint.

[37] B. E. M. SPEH, Some results on principal series for GL(n, **R**), Thesis, M. I. T. 1977.

[38] M. TSUCHIKAWA, On the representations of SL(3, C), III, Proc. Japan Acad. **44** (1968), 130-132.

[39] I. VAKUTINSKII, Unitary irreducible representations of the group GL(3, **R**) consisting of matrices of the third order, Math. Sbornik **75**: 117 (1968), 303-320 (in Russian).

[40] N. WALLACH, Representations of reductive Lie groups, in *Automorphic Forms, Representations, and L-Functions*, A. M. S. Summer Institute, 1977.

[41] G. WARNER, *Harmonic Analysis on Semi-simple Lie Groups* I, Springer Verlag, 1972.

[42] A. WEIL, *Basic Number Theory*, Springer Verlag, 1967.

[43] ———, *Dirichlet Series and Automorphic Forms*, Springer Verlag, Lecture Notes **189**, 1971.

[44] A. V. ZELEVINSKII, Classification of irreducible non-cuspidal representations of the groups GL(n) over p-adic fields, Funkcional Anal. i Prilǒzen, **11**: 1 (1977), 67.

[45] ———, Classification of representations of the group GL(n) over p-adic fields, preprint (in Russian).

(Received October 25, 1977)
(Revised May 8, 1978)