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Automorphic forms on GL(3) I*

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Contents**

0. Introduction and notations

(0.1) Let F be an A-field. In [17], it was shown how to attach to any (unitary) irreducible representation π of $\mathrm{GL}(r, F_A)$ an infinite Euler-product $L(s, \pi)$, which is absolutely convergent in some right half-space. If π occurs as a discrete component of the space of cusp-forms or, as we shall say, is automorphic cuspidal, then $L(s, \pi)$ extends to a holomorphic function of s and satisfies a functional equation:

(0.1.1) $$L(s, \pi) = \varepsilon(s, \pi)L(1 - s, \tilde{\pi})$$

where $\tilde{\pi}$ is the representation contragredient to π. If $r = 1$, then π is a character of the idèle-class group $F_A^\times/F^\times$, i.e., a "grössencharacter" of the field F, $L(s, \pi)$ is nothing but the Dirichlet series attached to π times the appropriate gamma-factor and this result is due to Hecke. If $r = 2$ and say $F = \mathbf{Q}$, then giving π amounts to giving a new form (holomorphic or "à la Maass") and $L(s, \pi)$ is, apart from a translation, the Dirichlet series attached to that form, times the appropriate factor, so that the result is again classical.

In general, it is important to keep in mind how $L(s, \pi)$ is defined in the first place. The representation being written as an "infinite tensor product"

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** Sections 8 to 14 and the bibliography appear in the next issue.

$\pi = \bigotimes \pi_v$ where π_v is, for each place v, an irreducible representation of $G_v = \mathrm{GL}(r, F_v)$, we have:

(0.1.2)
$$L(s, \pi) = \prod_v L(s, \pi_v) ,$$
$$\varepsilon(s, \pi) = \prod_v \varepsilon(s, \pi_v, \psi_v) ,$$

(see notations below) where the factors $L(s, \pi_v)$ and $\varepsilon(s, \pi_v, \psi_v)$ are defined in terms of the local representation π_v (cf. [17]). This definition is repeated, for v non-archimedean in Section 1.

Let us recall also that, for v non-archimedean, $L(s, \pi_v) = P_v(q^{-s})^{-1}$ where $P_v \in \mathrm{C}[X]$ and $P_v(0) = 1$. In particular, if π_v contains the trivial representation of the maximal compact subgroup, which happens for almost all v, then π_v determines a semi-simple conjugacy class a_v in $\mathrm{GL}(r, \mathrm{C})$ and

$$P_v(X) = \det(1 - a_v X) .$$

For v archimedean, $L(s, \pi_v)$ is a product of gamma-factors. Finally, the factor $\varepsilon(s, \pi_v, \psi_v)$ is an exponential function of s equal to one for almost all v.

We will not recall the proof given in [17] of the properties of $L(s, \pi)$. It suffices to say that it generalizes in a straightforward manner Hecke's proof for $r = 1$.

This being so, it is natural to ask whether the converse theorem is true. So, assume $r \geqq 2$ and let π be a unitary irreducible representation of $\mathrm{GL}(r, \mathbf{A})$ which is trivial on the center of $\mathrm{GL}(r, F)$. Assume that $L(s, \pi)$ has the above properties. Is π automorphic cuspidal? Except in the most simple cases, this assumption is not enough. For $r = 2$, Weil was the first to propose the correct strengthening of the assumptions; in our language, we have to assume the above conditions not only for π but also for all the representations $\pi \otimes \chi$ where χ is a grössencharacter (see notations (0.5.7) below). One might say that the given Euler-product $L(s, \pi)$ is "twisted" by all grössencharacters χ. Indeed, if at a place v, $\chi_v(a) = |a|_v^t$ then

$$L(s, \pi_v \otimes \chi_v) = L(s + t, \pi_v) .$$

Under those strengthened assumptions (and a simple technical hypothesis), the representation π is indeed automorphic cuspidal ([23], [43]).

It is the purpose of this paper to show that the same is true for $r = 3$ (Theorem (13.6)). This result, sufficiently refined, can in fact be used to construct automorphic forms on the group GL(3). In other words, roughly speaking, all infinite Euler-products of degree 3 having a suitable analytic behavior are attached to automorphic representations of GL(3).

We hasten to add that this result cannot be true for $r \geqq 4$. Indeed, in order to have a characterization of automorphic representations, one must "twist" the given Euler-product $L(s, \pi)$ by all automorphic cuspidal representations of the groups $\mathrm{GL}(j)$ with $1 \leqq j \leqq r-2$. This, of course, depends on the theory of the corresponding Euler-products, a theory which is still in progress.

In order to prove the converse theorem, one needs, first of all, a proof of the direct theorem which, so to speak, can be "inverted," in contrast to the proof given in [17]. This new proof depends on the notion of Mellin transformation of a cusp-form φ. For $r = 2$ this is simply the integral

$$\int_{F_{\mathbf{A}}^{\times}/F^{\times}} \varphi \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^{\times}a . \tag{0.1.3}$$

For $r = 3$, this is the integral

$$\iiint \varphi \begin{pmatrix} a & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 0 \end{pmatrix} |a|^{s-1} \psi(-v)\, du dv d^{\times}a , \tag{0.1.4}$$

where $u, v, \in F_{\mathbf{A}}/F$ and $a \in F_{\mathbf{A}}^{\times}/F^{\times}$. Then one needs to relate integral (0.1.4) to the Euler-product $L(s, \pi)$ and this requires some lengthy local preparations, in fact, a new definition of the local L and ε factors (cf. §4). Once this is done, the converse theorem is easily established.

A detailed description of the contents of the paper follows. Sections 1 to 7 are published in this issue. There, the ground field F is local non-archimedean and the integer r is arbitrary in Sections 1 to 4. In (1.1) we review the definition of the local L and ε factors given in [17] and, in (1.2), add appropriate remarks of a rather technical nature. In (2.1) we introduce the notion of "generic representation" and attach to such a representation π a certain space of functions noted $W(\pi; \psi)$; this is an essential notion. In (2.2) to (2.5) we give or recall some technical properties of the functions in this space. In Section 3 we introduce some integrals which are, in a way, intermediate between the integrals of Sections 1 and 4; the results (Prop. (3.1) and Prop. (3.2)) are mere preparation for Section 4 and should not be taken too seriously. Then comes the crucial Section 4, where a new definition of L and ε is given for generic representations. The main results, Theorems (4.3) and (4.4), are, for the convenience of the reader, repeated for $r = 3$ in Section 7 (Theorem (7.4)). Section 5 is an exceptional section; we needed a technical result on the ε-factor. This result is stated without proof, and is only used in Sections 13 and 14. A proof appears in [24]. In Section 6 we are primarily interested in classifying the irreducible representations of $\mathrm{GL}(3, F)$. This is a rather technical matter and we had to content ourselves, most of the time, with a description of the results needed, even though the results are not all already in the literature. This section can be skipped to a large extent. Finally, in Section 7, we complete our results for $r = 3$. We show how to attach to every representation, generic or not, a space $\mathcal{W}(\pi; \psi)$, and how the results of Section 4 extend to this space. This is a highly technical refinement which can and should be ignored at first reading. In (7.5), we give some local results special to $r = 3$ (and 2). They are interesting in their own right. (7.6) contains a technical lemma needed for the global theory.

Sections 8 to 11 mimic Sections 1 to 7 for the archimedean case. More precisely, Section 8 mimics Section 2, Section 9 mimics Section 3, Section 10 mimics Section 6 and Section 11 mimics both Sections 4 and 7. Unfortunately, our results there are not as good as in the non-archimedean case, but they are sufficient for our purposes. In any case, the reader is advised to restrict himself to the function field case at first and therefore skip these sections—especially since no new ideas occur here.

Finally, the global theory is taken up in Sections 12, 13, 14. Section 12 contains preliminary material, namely the convergence of certain "Fourier series." It need not be read at first, but should be referred to whenever necessary. The main theorems are given in Section 13. The principal ideas are expounded in (13.1) to (13.6), the other numbers containing more technical material. Here again, the reader is advised to assume at first the ground field F to be a function field and all representations to be generic. Finally, applications are given in Section 14. The first application (Theorem (14.1)) is superceded by the more precise results of [10]. The most interesting one is (14.2): we show how to attach to every cubic extension of the ground field a family of automorphic representations.

Finally, a word about the organization of this paper. Each section is divided into subsections such as (1.1), (1.2), $\cdots$. In principle, every subsection presents a new idea. Generally speaking, theorems are to be regarded as more important than propositions and always carry the number of a subsection. Lemmas are merely auxiliary results.

We now give a list of our most frequently used notations.

(0.2) *Local fields*. In this paper, we denote by F a "ground field," which, depending on the section may be local or global. If F is local we denote by ψ_F or simply ψ a *non-trivial* additive character of F. Then the additive Haar-measure on F, denoted dx, dy, $\cdots$, is *always* assumed to be self-dual. The Haar-measure on the multiplicative group $F^\times$, denoted $d^\times x$, is normalized in various ways. The topological module of F is denoted α_F or α and we also write $\alpha_F(x) = |x|_F$ or $|x|$. Thus $d(mx) = |m|dx$ and $|z|_{\mathbf{C}} = z\bar{z}$.

If F is non-archimedean we denote by q_F or q, v_F or v the cardinality of the residual field and the normalized valuation respectively. Thus $|x| = q^{-v(x)}$. The ring of integers, the group of units, and the maximal ideal of F are denoted $\mathfrak{R}_F$, $\mathfrak{R}_F^\times$, $\mathfrak{P}_F$ or simply $\mathfrak{R}$, $\mathfrak{R}^\times$, and $\mathfrak{P}$. Often we assume that the character ψ_F has *exponent zero*, i.e., that the largest ideal on which ψ_F is trivial is $\mathfrak{R}$. We denote by $\tilde{\omega}$ a uniformizer for the field F.

If F is local, we denote by $\mathcal{S}(F^r)$ the space of Schwartz-Bruhat functions on F^r. We also denote by $\mathcal{S}(p\times q, F)$ the corresponding space on $M(p\times q, F)$ (matrices with p rows and q columns). The Fourier transform of a function Φ of this type is the function $\hat{\Phi}$ *on the same space* defined by

$$\hat{\Phi}(x) = \int \Phi(y)\psi(\mathrm{tr}({}^t yx))dy \ .$$

(In § 5 however a different convention is used.) Here again the Haar-measure is self-dual.

$\mathbf{R}_+^\times$ will denote the multiplicative group of positive real numbers.

If G is an algebraic F-group we often write G instead of $G(F)$.

(0.3) $\mathbf{A}$-*fields*. If F is an $\mathbf{A}$-field, i.e., a number field or a function field over a finite field, then we denote by $F_{\mathbf{A}}$ or simply $\mathbf{A}$ the ring of adèles of F and by $F_{\mathbf{A}}^\times$ or simply $\mathbf{I}$ the group of idèles. If v is a place of F then F_v denotes the corresponding (class of) local field. We then abbreviate $\alpha_{F_v}, |\ |_{F_v}, q_{F_v} \cdots$ by $\alpha_v, |\ |_v, q_v \cdots$. If x is an idèle, we denote by $\alpha_F(x)$ or $\alpha(x)$ or $|x|$ its module so that $\alpha_F(x) = \prod_v |x_v|_v$. We also write $|x|_v$ for $|x_v|_v$. We fix a non-trivial character ψ of $\mathbf{A}/F$ and denote by ψ_v its local component at the place v. Thus $\psi(x) = \prod_v \psi_v(x_v)$. If χ is a quasi-character of $F_{\mathbf{A}}^\times$, i.e., an homomorphism from $F_{\mathbf{A}}^\times$ to $\mathbf{C}^\times$ then we write similarly $\chi(x) = \prod_v \chi_v(x_v)$. Observe that for us a *character* has *module one*.

If G is an F-algebraic group then we denote $G(F_{\mathbf{A}})$ or simply $G(\mathbf{A})$ the group of points of G with values in $F_{\mathbf{A}}$. We set also:

(0.3.1) $$G^* = G(F)/G(\mathbf{A})$$

when this does not create confusion. For any place v we write $G_v = G(F_v)$. If S is a *finite* set of places, we set

(0.3.2) $$G_S = \prod_{v\in S} G_v \ ,$$

(0.3.3) $$G^S = \prod_{v\notin S} G_v \quad \text{(restricted product)} \ .$$

We can apply this to the set of infinite places, that we denote by the symbol ∞. Thus

(0.3.4) $$G_\infty = \prod G_v \quad (v \text{ infinite}) \,,$$

(0.3.5) $$G^\infty = \prod G_v \quad (v \text{ finite, restricted product}) \,.$$

(0.4) *Linear groups.* We denote by G_r the general linear group $\mathrm{GL}(r)$, regarded as an algebraic group over the ground field F. We denote by $Z_r \cong \mathrm{GL}(1)$ the center of G_r, by B_r the subgroup of upper-triangular matrices, by A_r the subgroup of diagonal matrices and by N_r the subgroup of matrices in B_r with unit diagonal. In general, by a parabolic subgroup R of G_r, we mean an F-parabolic subgroup. The unipotent radical of R is denoted U_R. If $R \neq G$ we say that U_R is a horicycle of R. Every parabolic subgroup is F-conjugate to an (upper) standard parabolic subgroup, i.e., to one which contains B_r. The type of such a parabolic is an s-tuple of integers $(n_1, n_2, \cdots, n_s)$ with $\Sigma n_i = r$. In particular we call P_r the parabolic of type $(r-1, 1)$. It consists of all matrices

(0.4.1) $$p = \begin{pmatrix} g & u \\ 0 & a \end{pmatrix}, \quad g \in G_{r-1}\,, \quad a \in G_1 \,.$$

We let U_r be its unipotent radical and P_r^1 the subgroup of $p \in P_r$ for which $a = 1$. We often drop the index r (equal to 3 in most of the paper) if this does not create confusion.

Similarly we identify G_{r-1} with the subgroup G' of $p \in P_r$ such that $a = 1$ and $u = 0$. Then the subgroups P_{r-1}, P_{r-1}^1, N_{r-1}, B_{r-1}, A_{r-1} are identified with the subgroups P', $P^{1\prime}$, N', B', A' of P_r.

The diagonal matrix with entries $(a_1, a_2, \cdots, a_r)$ is often denoted $\mathrm{diag}(a_1, a_2, \cdots, a_r)$. We also set

(0.4.2) $$w = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ (-1)^{r-1} & \cdots & 0 & 0 \end{pmatrix}, \quad w' = \begin{pmatrix} (-1)^r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & (-1)^{r-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & -1 \cdots & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

If g is a matrix then g_{ij} is the entry in the i-th row and j-th column. If g is a square invertible matrix, we also set

(0.4.3) $$g^\iota = {}^t(g)^{-1} = ({}^tg)^{-1} \,.$$

If R is a parabolic subgroup, then we set

(0.4.4) $$\bar{R} = {}^tR = R^\iota \,.$$

Thus if R is standard then $\bar{R}$ is a *lower* standard subgroup.

(0.5) *Representations of local groups.* We shall use various notions of representations. The reader must be able to pass unaided from one point of view to another. In particular if G is a locally compact group and H a closed subgroup, then every "representation" σ of H induces a representation π of G denoted

$$\pi = I(G, H; \sigma) . \tag{0.5.1}$$

Its space consists of functions f on G with values in the space of σ and transforming on the left according to

$$f(hg) = \sigma(h)\delta_H^{1/2}(h)\delta_G(h)^{-1/2}f(g) . \tag{0.5.2}$$

Here δ_G, δ_H are the modules of G and H respectively. (Thus if $d_l g$ is a left-Haar measure on G then $d_r g = \delta_G(g)d_l g$ is a right-Haar measure.)

The precise definition of the space of π depends on the context. In any case

$$\big(\pi(h)f\big)(g) = f(gh) . \tag{0.5.3}$$

In general if μ is a measure (or more generally a distribution) of compact support say, and f a function, then we set

$$\rho(\mu)f = f*\check{\mu} . \tag{0.5.4}$$

Here $\check{\mu}$ is the image of μ under the anti-automorphism $g \mapsto g^{-1}$. Thus

$$\rho(\mu)f(g) = \int f(gh)d\mu(h) . \tag{0.5.5}$$

If $\mu = \delta_h$ then we write $\rho(h)$ for $\rho(\delta_h)$.

If F is local, then we select a maximal compact K in the usual way:

$$K = 0(r, \mathbf{R}) \text{ if } F = \mathbf{R},\ K = \mathrm{U}(r) \text{ if } F = \mathbf{C},\ K = \mathrm{GL}(r, \mathfrak{R}) \text{ if } F \neq \mathbf{R}, \mathbf{C} . \tag{0.5.6}$$

Unless otherwise specified K_m denotes the subgroup of $k \in K$, $k \equiv 1 \bmod \mathfrak{P}^m$. Then the Hecke-algebra $\mathcal{H}_F$ or $\mathcal{H}$ of the group $G_r(F)$ is defined as in [23]. If F is non-archimedean, then admissible representations of the group $G_r(F)$ can be also interpreted as representations of $\mathcal{H}$. There is something similar for topological representations of $G_r(F)$ when F is archimedean; at any rate, in this case, we consider almost exclusively *unitary* representations of $G_r(F)$. Of course, even in the non-archimedean case we can consider unitary representations of $G_r(F)$; again a unitary irreducible representation determines an admissible irreducible representation.

If π is a representation of $G_r(F)$ on $\mathcal{V}$, then we can define the representation $\tilde{\pi}$ contragredient to π. If $\overline{\mathcal{V}}$ is the space imaginary conjugate to $\mathcal{V}$ then $\pi(g)$ defines also an operator $\bar{\pi}(g)$ on $\overline{\mathcal{V}}$ and so we get a representation $\bar{\pi}$ which is said to be conjugate to π. We can also define representations π^ι, and $\pi \otimes \chi$ if χ is a quasi-character of $F^\times$, on the same space $\mathcal{V}$ by:

(0.5.7) $$\pi^{\iota}(g) = \pi(g^{\iota}),\ \pi \otimes \chi(g) = \pi(g)\chi(\det g)\ .$$

If f is a function on $G_r(F)$ we shall also set

(0.5.8) $$f^{\iota}(g) = f(g^{\iota}),\ f^{\vee}(g) = f(g^{-1}),\ f \otimes \chi(g) = f(g)\chi(\det g)\ .$$

If R is a standard parabolic subgroup of type $(n_1, n_2, \cdots, n_s)$ in G_r then $R/U_R = M$ is isomorphic to $\prod \mathrm{GL}(n_i)$. Thus if σ_i is a representation of $\mathrm{GL}(n_i, F)$ on $\mathcal{V}_i$ we can form a representation $\sigma = \times\sigma_i$ of $M(F)$ on $\otimes\mathcal{V}_i$ and regard it as a representation of $R(F)$ trivial on $U_R(F)$. The representation of $G_r(F)$ it induces will be denoted

(0.5.9) $$I\big(G(F), R(F); \sigma_1, \sigma_2, \cdots, \sigma_r\big)\ .$$

Let π be a representation of $G_r(F)$. Suppose $\pi(z)$, $z \in Z_r$ is a scalar. Then there is a quasi-character ω of $F^\times$ such that

(0.5.10) $$\pi(z) = \omega(z)\cdot 1 \quad \text{if} \quad z \in F^\times \cong Z_r(F)\ .$$

We say that ω is the central quasi-character of π.

(0.6) *Representations of global groups.* Suppose F is an A-field. For each place v of F let π_v be a representation of G_v. If v is finite let us assume that π_v is admissible (but perhaps not irreducible) and contains the trivial representation of K_v, the standard maximal compact subgroup of G_v with multiplicity one for almost all v. If v is infinite let us assume that π_v is unitary. Then one can form an infinite tensor representation

(0.6.1) $$\pi = \otimes_v \pi_v$$

as in [23]. It can be interpreted as a representation of the global Hecke-algebra $\mathcal{H}$ which, in a sense, is the tensor product of the local Hecke-algebras $\mathcal{H}_v = \mathcal{H}_{F_v}$. If each π_v is unitary then one can think of π as being a unitary representation. Note that all unitary irreducible representations of $G_r(\mathbf{A})$ have the form (0.6.1) where the π_v are unitary irreducible.

Those considerations may be extended to the group G_r^S, where S is a finite set of places. The corresponding Hecke-algebra is denoted $\mathcal{H}^S$, while $\mathcal{H}_S$ is the Hecke-algebra of G_S. When π has the form (0.6.1) we often set

(0.6.2) $$\pi_S = \otimes_{v \in S} \pi_v \quad \text{(finite tensor product)}\ ,$$

(0.6.3) $$\pi^S = \otimes_{v \notin S} \pi_v\ ,$$

which are representations of $G_{r,S}$ and G_r^S respectively (or $\mathcal{H}_S$ and $\mathcal{H}^S$).

Notations and definitions introduced in (0.5.2), (0.5.3), and (0.5.5) extend to global groups.

We also set

(0.6.4) $$K = \prod\nolimits_v K_v,\ K^S = \prod\nolimits_{v \notin S} K_v,\ K_S = \prod\nolimits_{v \in S} K_v\ .$$

Here for each place v, K_v denotes the standard maximal compact subgroup of G_v.

1. Local zeta-integrals

In Sections 1 to 7 the ground field F is local, non-archimedean. We first review the results of [17, Theorem (3.3)] adding the material required for our present purposes.

(1.1) Let π be an admissible irreducible representation of the group $G = \mathrm{GL}(r, F)$ on a complex vector space $\mathcal{V}$. In [17] we have defined the representation $\tilde{\pi}$ contragredient to π. It is admissible and operates on a space $\tilde{\mathcal{V}}$. On the product $\mathcal{V} \times \tilde{\mathcal{V}}$ there is a non-degenerate bilinear form denoted $\langle , \rangle$ which is invariant under G. Every function f of the form

$$f(g) = \langle \pi(g)v, \tilde{v} \rangle\,, \qquad v \in \mathcal{V},\ \tilde{v} \in \tilde{\mathcal{V}}\,,$$

is called a matrix coefficient of π. The function f defined by $f^{\vee}(g) = f(g^{-1})$ is a matrix coefficient of $\tilde{\pi}$. Since the representation π^{ι} of G_r on $\mathcal{V}$ defined by

$$\pi^{\iota}(g) = \pi(g^{\iota})$$

is actually equivalent to $\tilde{\pi}$ ([13]) we see that the function f^{ι} defined by $f^{\iota}(g) = f(g^{\iota})$ is a matrix coefficient of $\tilde{\pi}$.

For Φ in $\mathcal{S}(r \times r, F)$ and any matrix coefficient f of π, we set

$$Z(\Phi, s, f) = \int_G \Phi(x) f(x)\, |\det x|^s\, d^{\times}x\,. \tag{1.1.1}$$

Then the integral $Z(\Phi, s + (r-1)/2, f)$ converges in some half-space $\mathrm{Re}(s) > s_0$ and is, as a function of q^{-s}, a rational fraction of q^{-s}. When Φ and f vary, these fractions span a fractional ideal of the ring $\mathrm{C}[q^{-s}, q^{s}]$. It admits a unique generator of the form $1/Q(q^{-s})$ with $Q \in \mathrm{C}[X]$, $Q(0) = 1$. All those facts are expressed by saying that the integrals admit $1/Q(q^{-s})$ for "g.c.d." The same terminology will be used later for other integrals. We denote $1/Q(q^{-s})$ by $L(s, \pi)$, as in [17].

If we replace π by $\tilde{\pi} \simeq \pi^{\iota}$, then we get a factor $L(s, \tilde{\pi}) = L(s, \pi^{\iota})$ and a functional equation

$$\begin{aligned} &Z\big(\Phi^{\wedge}, 1 - s + (r-1)/2, f^{\iota}\big)/L(1-s, \pi^{\iota}) \\ &\qquad = \varepsilon(s, \pi, \psi) Z\big(\Phi, s + (r-1)/2, f\big)/L(s, \pi)\,. \end{aligned} \tag{1.1.2}$$

Here ψ is a non-trivial additive character and $\Phi^{\wedge}$ is obtained from the Fourier-transform of (3.3) in [17] by changing x into ${}^{t}x$; that is,

$$\Phi^{\wedge}(x) = \int \Phi(y) \psi\big(\mathrm{Tr}(y\,{}^{t}x)\big) dy\,.$$

This explains the presence of f^{ι} rather than the function $f^{\vee}$ of [17].

We also set:

$$\varepsilon'(s, \pi, \psi) = \frac{\varepsilon(s, \pi, \psi) L(1-s, \pi^{\iota})}{L(s, \pi)} . \tag{1.1.3}$$

(1.2) We shall need the fact that $L(s, \pi)$ is equal to *one* of the integrals $Z(\Phi, s, f)$. A priori it is only a sum of such integrals. This will require some results for which no convenient reference could be found.

Consider for one moment the following situation. Let G be a topological group with a fundamental system of neighborhoods of $\{e\}$ which are open compact subgroups. Let H be an open normal subgroup of finite index such that G/H is abelian. Finally let π be an admissible irreducible representation of G on $\mathcal{V}$. Call X_π the group of characters χ of G/H such that

$$\pi \cong \pi \otimes \chi .$$

Also let H' be the intersection of the kernel of the characters χ in X_π. Then the following lemma will easily be obtained by the reader:

LEMMA (1.2.1). *The representation $\pi|H$ is a finite direct sum of t irreducible representations σ_i of H. The commuting algebra of $\pi|H$ is identical with the commuting algebra of $\pi|H'$. If moreover G/H is cyclic, this algebra is commutative* (i.e., *the σ_i are pairwise inequivalent*) *and* $t = [G: H']$.

Coming back to our main course, for any integer d, we let G^d be:

$$G^d = \{g \in G(F) \,|\, v(\det g) \in \mathbf{Z}\, d\} . \tag{1.2.2}$$

In particular

$$G^1 = G(F),\ G^0 = \{g \,|\, |\det g| = 1\},\ G^r = G^0 Z(F) .$$

We shall apply the previous lemma to the group $G = G^1$ and the subgroup $H = G^r$. A character of G^1/H has the form

$$g \longmapsto \chi(\det g) ,$$

where χ is an unramified character of $F^\times$ whose order divides r. On the other hand, if χ is an unramified character such that

$$\pi \cong \pi \otimes \chi ,$$

where π is irreducible and admissible, then the two representations must agree on the center and therefore $\chi^r = 1$.

For any integer d, let X_d denote the group of unramified characters χ of $F^\times$ such that $\chi^d = 1$. Then:

LEMMA (1.2.3). *Let π be an admissible irreducible representation of*

$G(F)$ on $\mathcal{V}$. Let d be the integer such that

$$\chi \in X_d \Longleftrightarrow \pi \cong \pi \otimes \chi, \ (\chi \text{ unramified}) .$$

Then

$$\mathcal{V} = \bigoplus \mathcal{V}_i \quad (0 \leqq i \leqq d-1) ,$$

where each $\mathcal{V}_i$ is invariant under G^d and irreducible under G^0 (or equivalently G^r). Moreover:

$$\mathcal{V}_i = \pi(a^i)\mathcal{V}_0$$

if $v(\det a) = 1$. Finally the representations of G^0 on the $\mathcal{V}_i$ are inequivalent.

Proof. Since X_r is cyclic, any subgroup of it is of the form X_d with $d|r$. So Lemma (1.2.3) is just a reformulation of (1.2.1) in the present case.

Now let σ be the representation of G^d on $\mathcal{V}_0$. Then one can also express (1.2.3) as

$$\pi = I(G, G^d; \sigma) .$$

In particular if we call $\mathcal{A}_0$ (resp. $\mathcal{A}$) the space spanned by the coefficients of σ (resp. π), we see that the elements of $\mathcal{A}_0$ may be regarded as functions on G which vanish outside G^d. As such, they are matrix coefficients of π. Now by the previous lemma every $f \in \mathcal{A}$ can be written as

$$f(g) = \sum f_{i,j}(a^i g a^j) , \quad f_{i,j} \in \mathcal{A}_0 , \quad 0 \leqq i, j \leqq d-1 . \tag{1.2.4}$$

Moreover, $\mathcal{A}_0$ is invariant under right and left translations by G^d or G^0. Since $\mathcal{V}_0$ is irreducible under G^0, the space $\mathcal{A}_0$ is irreducible under $G^0 \times G^0$. Then:

PROPOSITION (1.2.5). *With the above notations,*

$$L(s, \pi) = 1/Q(q^{-s}) , \quad Q \in \mathbf{C}[X^d] ,$$

and given $f \neq 0$ in $\mathcal{A}_0$, there is $\Phi \in \mathcal{S}(r \times r, F)$ such that

$$Z\big(\Phi, s + (r-1)/2, f\big) = L(s, \pi) .$$

Proof. We recall that the functions $f \in \mathcal{A}_0$ have support in G^d. It follows that the integrals

$$Z\big(\Phi, s + (r-1)/2, f\big) , \quad f \in \mathcal{A}_0 ,$$

are fractions in q^{-ds} and in fact span an ideal I_0 of the ring

$$\mathbf{C}[q^{-ds}, q^{ds}] .$$

From relation (1.2.4) it follows that, if I is the ideal spanned by the integrals

$$Z(\Phi, s + (r - 1)/2, f)\,, \quad f \in \mathcal{A}\,,$$

we have

$$I = I_0\mathbf{C}[q^{-s}, q^s]\,.$$

Now I_0 contains 1 and admits a generator of the form

$$1/R(q^{-ds})\,, \quad R \in \mathbf{C}[X]\,, \quad R(0) = 1\,.$$

It is also a generator of I, that is, identical to $L(s, \pi)$. Therefore there are $f_i \in \mathcal{A}_0$ and Φ_i such that

$$\sum Z(\Phi_i, s + (r - 1)/2, f_i) = L(s, \pi)\,.$$

On the other hand, there are $h_{i,j}$ and $k_{i,l}$ in G^0 such that

$$f_i(g) = \sum\nolimits_{j,l} \lambda_{i,j,l} f(h_{i,j} g k_{i,l})\,.$$

It suffices to take for Φ the function

$$\Phi(g) = \sum\nolimits_{i,j,l} \lambda_{i,j,l} \Phi(h_{i,j}^{-1} g k_{i,l}^{-1})$$

to get the required results. Q.E.D.

2. Generic representations

In this section, the ground field F is local and non-archimedean.

(2.1) Let θ be the character of $N(F)$ defined by

(2.1.1) $$\theta(n) = \prod \psi(n_{i,i+1})\,, \quad 1 \leqq i \leqq r - 1\,.$$

If π is any admissible representation of G on a complex vector space $\mathcal{V}$, we denote by $\mathcal{V}_\theta^*$ or π_θ^* the space of linear forms λ on $\mathcal{V}$ such that

$$\lambda[\pi(n)v] = \theta(n)v \qquad \text{for} \qquad v \in \mathcal{V}\,, \quad n \in N\,.$$

If π is irreducible, we say that it is generic if π_θ^* is non-trivial. In that case the dimension of that space is actually one ([13], [34]). Suppose π is irreducible and generic; select $\lambda \neq 0$ in $\mathcal{V}_\theta^*$. We denote by $\mathcal{W}(\pi; \psi)$ the space spanned by the functions

(2.1.2) $$W(g) = \lambda[\pi(g)v]\,, \quad v \in \mathcal{V}\,.$$

Clearly $\mathcal{W}(\pi; \psi)$ is invariant on the right under G and the representation of G on that space equivalent to π. Each W in $\mathcal{W}(\pi; \psi)$ satisfies

$$W(ng) = \theta(n)W(g)\,.$$

These properties characterize the space $\mathcal{W}(\pi; \psi)$.

If π is irreducible and generic, then the same is true of the representation $\tilde{\pi}$ contragredient to π. For the automorphism $g \mapsto wg^t w^{-1}$ transforms π into a representation equivalent to $\tilde{\pi}$ ([13]) and, on the other hand, fixes N and θ. In a precise way if W is in $\mathcal{W}(\pi; \psi)$ then the function $\widetilde{W}$ defined by

(2.1.3) $$\widetilde{W}(g) = W(wg^{\iota})$$

is in the space $\mathcal{W}(\tilde{\pi}; \psi)(=\mathcal{W}(\pi^{\iota}; \psi))$. If π is irreducible and generic, then for any quasi-character χ of $F^{\times}$ the representation $\pi \otimes \chi$ is still generic. More precisely if W is in $\mathcal{W}(\pi; \psi)$ then the function $W \otimes \chi$ defined by

$$W \otimes \chi(g) = W(g)\chi(\det g)$$

is in $\mathcal{W}(\pi \otimes \chi; \psi)$.

Note that we could replace θ by any *generic* character of N, that is, by any character of the form

$$\xi(n) = \prod \psi(a_i n_{i,i+1}) \,, \quad a_i \neq 0 \,.$$

For such a, ξ has the form

$$\xi(n) = \theta(ana^{-1})$$

with some $a \in A$. In particular the property of being generic does not depend on the choice of ψ although the space $\mathcal{W}(\pi; \psi)$ does.

Finally observe that for $r = 1$ the condition of being generic is empty and all quasi-characters of $F^{\times}$ are "generic representations." The space $\mathcal{W}(\pi; \psi)$ consists of all scalar multiples of π, regarded as a function on $F^{\times}$.

(2.2) The functions in $\mathcal{W}(\pi; \psi)$ have a very simple form. Recall that a finite function on a locally compact abelian group is a continuous function whose translates span a finite dimensional vector space (cf. [23, § 8]).

PROPOSITION (2.2). *Let π be a generic representation of $G(F)$. Then one can select finite-functions λ_i on $(F^{\times})^{r-1}$, $1 \leqq i \leqq t$, with the following property: for any W in $\mathcal{W}(\pi; \psi)$, there are t functions ϕ_i in $\mathcal{S}(F^{r-1})$ such that:*

(2.2.1) $$W(a) = \sum_{1 \leqq i \leqq t} \lambda_i(a_1, a_2, \cdots, a_{r-1})\phi_i(a_1, a_2, \cdots, a_{r-1})$$

for

(2.2.2) $$a = \operatorname{diag}(a_1 a_2 \cdots a_{r-1}, a_2 \cdots a_{r-1}, \cdots, a_{r-1}, 1) \,.$$

Proof. We begin with some simple remarks on representations. Suppose σ is a smooth representation of $F^{\times}$ on a complex vector space $\mathcal{V}$; suppose that the algebra $\mathcal{A}$ spanned by the operators $\sigma(a)$, with $a \in F^{\times}$, is finite dimensional. Then there are a finite set X of finite functions on $F^{\times}$ and for each $\chi \in X$ an element A_{χ} of $\mathcal{A}$ such that

$$\sigma(a) = \sum_{\chi \in X} \chi(a)A_{\chi} \quad \text{for every} \quad a \in F^{\times} \,.$$

The set X depends only on the isomorphism class of $\mathcal{A}$. Moreover every element of $\mathcal{A}$ satisfies a polynomial equation. Thus $\mathcal{A}$ is spanned, as a vector space, by the operators $\sigma(a)$, $|a| \leqq 1$. Next we prove the following lemma:

LEMMA (2.2.1). *Let $\mathcal{V}$ be a space of smooth functions on the group*

$$H = \prod_{i=1}^{n} H_i , \quad H_i \cong F^\times .$$

Assume $\mathcal{V}$ invariant under translations. Suppose each ϕ in $\mathcal{V}$ vanishes on a when $|a_i|$ is sufficiently large for each i. On the other hand let $\mathcal{V}_i$ be the space of $\phi \in \mathcal{V}$ which vanish for $|a_i|$ small enough and σ_i the representation of H_i on $\mathcal{V}/\mathcal{V}_i$. Suppose the algebra $\mathfrak{A}_i$ spanned by the operators $\sigma_i(a)$, $a \in H_i$, is finite dimensional for each i. Then there is a finite set X of finite functions on H such that every $\phi \in \mathcal{V}$ can be written

$$\phi(a) = \sum \chi(a)\phi_\chi(a) , \quad \phi_\chi \in \mathcal{S}(F^n) , \quad \chi \in X .$$

Proof of Lemma (2.2.1). The assertion is trivial for $n = 0$; assume $n > 0$ and the assertion true for $n - 1$. We may then apply the above remarks to the representation σ_n and obtain the existence of a finite set Y of finite functions on $H_n \cong F^\times$, and for each $\eta \in Y$ an operator A_η belonging to the linear span of the set $\{\rho(b) | b \in H_n \cong F^\times , |b| \leqq 1\}$ so that for any $\phi \in \mathcal{V}$

$$\rho(b)\phi \equiv \sum_{\eta \in Y} \eta(b)A_\eta\phi \mod \mathcal{V}_n .$$

Thus for a given $\phi \in \mathcal{V}$ we may find c so small that

$$\phi(a_1, a_2, \cdots, a_{n-1}, bc) = \sum \eta(b) \ (A_\eta\phi)(a_1, a_2, \cdots, a_{n-1}, c)$$

for any b with $|b| \leqq 1$, any a_i's. From there we conclude at once that given ϕ there are b_j's on $F^\times$ and ϕ_j in $\mathcal{S}(F)$ so that

$$\phi(a_1, a_2, \cdots, a_{n-1}, a_n) = \sum_{\eta, j} \eta(b)\phi_j(a_n)\phi(a_1, a_2, \cdots, a_{n-1}, b_j) .$$

Now fix $b \in F^\times$. Let $\mathcal{V}'$ be the space of functions $a \mapsto \phi(a, b)$ on the group $H' \cong \prod_{i=1}^{n-1} H_i$. It is invariant under translations; for $1 \leqq i \leqq n - 1$ let $\mathcal{V}'_i$ and σ'_i be the analogues of $\mathcal{V}_i$ and σ_i. If ϕ is in $\mathcal{V}_i$ then the function $a \mapsto \phi(a, b)$ on H' is in $\mathcal{V}'_i$. Thus σ'_i is a quotient of σ_i. Therefore the space $\mathcal{V}'$ satisfies the assumptions of the lemma; applying the induction hypothesis we find a set X' of finite functions on H'. But that set depends only on the equivalence classes of the σ'_i thus also of the σ_i; in particular it may be chosen independently of b. The lemma follows then at once.

Going back to the proof of (2.2) we let P_i be the standard parabolic subgroup of type $(i, r - i)$, V_i its unipotent radical. We shall apply the lemma with $n = r - 1$, the group H being identified with the group of matrices of the form

$$\text{diag}\,(a_1a_2 \cdots a_{r-1}, a_2 \cdots a_{r-1}, \cdots, a_{r-1}, 1) .$$

Then H_i is contained in the center of P_i. Let $\mathcal{W}_i$ be the space spanned by the vectors $\pi(u)W - W$ with $u \in V_i$. By [7, (3.3.1)] or [2, (3.14)] the representation τ_i of H_i on $\mathcal{W}(\pi;\psi)/\mathcal{W}_i$ satisfies the above condition. Now let $\mathcal{V}$ be the space spanned by the restrictions of the functions in $\mathcal{W}(\pi;\psi)$ to H. Introduce, as in the lemma, $\mathcal{V}_i$ and σ_i. If W is in $\mathcal{W}_i$ its restriction is in $\mathcal{V}_i$. Thus σ_i is a quotent of τ_i and all the assumptions of Lemma (2.2.1) are satisfied. This concludes the proof of (2.2).

Remark (2.2.5). Later (see after 7.2) we shall encounter the following situation. Let $\mathcal{W}$ be a space of functions W on G, smooth on the right and satisfying

$$W(ng) = \theta(n)W(g) \qquad (n \in N) .$$

Suppose $\mathcal{W}$ is stable by right translations. Let π denote the corresponding smooth representation of G. Suppose π is finitely generated and admissible. Then with $\mathcal{W}$ replacing $\mathcal{W}(\pi;\psi)$ the conclusion of Proposition (2.2) is still valid.

(2.3) Finally it will be necessary to obtain a majorization of the elements of $\mathcal{W}(\pi;\psi)$ in a form suitable for the global theory. We define a *gauge* on $G(F)$ to be a function ξ invariant on the left under $N(F)$, on the right under K and which on $A(F)$ has the form:

$$\xi(a) = |a_1a_2\cdots a_{r-1}|^{-t}\phi(a_1, a_2, \cdots, a_{r-1}) \tag{2.3.1}$$

if

$$a = \operatorname{diag}(a_1a_2\cdots a_r, a_2\cdots a_r, \cdots, a_{r-1}a_r, a_r) , \tag{2.3.2}$$

where t is real $\geqq 0$ and $\phi \geqq 0$ is a Schwartz-Bruhat function on F^{r-1}. In particular ξ is invariant under $Z(F)$. We prove a few elementary properties of gauges:

LEMMA (2.3.3). *Let ξ be the gauge defined by* (2.3.1). *If $t' > t$ is another number, there is $\phi' \in \mathcal{S}(F^{r-1})$ such that the gauge ξ', defined by* (2.3.1) *with t' and ϕ' instead of t and ϕ, majorizes ξ.*

This is clear.

LEMMA (2.3.4). *The sum of two gauges is majorized by a gauge.*

This follows from (2.3.3).

LEMMA (2.3.5). *If Ω is a compact subset of $G(F)$ and ξ a gauge, there is a gauge ξ' such that*

$$\xi(g\omega) \leqq \xi'(g) ,$$

for g in $G(F)$ and ω in Ω.

Proof. Write the Iwasawa decompositions of g and $g\omega$:

$$g = nak,\ g\omega = n^1a^1k^1 .$$

Then $(a^1)^{-1}(n^1)^{-1}na^1(a^1)^{-1}a \in K\Omega^{-1}K$. Since $A(F)N(F)$ is closed in G and topologically isomorphic to $A(F)\times N(F)$, it follows that $a(a^1)^{-1}$ is restricted to a compact subset of $A(F)$.

Write

$$a = \mathrm{diag}(a_1a_2 \cdots a_r,\ a_2 \cdots a_r,\ \cdots,\ a_r),\ a^1 = ab$$

and

$$b = \mathrm{diag}(b_1b_2 \cdots b_r,\ b_2 \cdots b_r,\ \cdots,\ b_r) .$$

Then

$$\xi(g\omega) = \xi(a^1) = |a_1b_1a_2b_2 \cdots a_{r-1}b_{r-1}|^{-t}\phi(a_1b_1,\ a_2b_2,\ \cdots,\ a_{r-1}b_{r-1}) .$$

Since the b_i's vary in a compact set, this is majorized by

$$|a_1a_2 \cdots a_{r-1}|^{-t}\phi^1(a_1,\ a_2,\ \cdots,\ a_{r-1})$$

with a suitable $\phi^1 \geqq 0$ in $\mathcal{S}(F^{r-1})$ and we are done.

From (2.2) and (2.3.3) we have

PROPOSITION (2.3.6). *Let π be a generic representation of $G_r(F)$ and ω its central quasi-character. Let $|\omega| = \alpha^t$. Then for any $W \in \mathcal{W}(\pi; \psi)$ there is a gauge ξ such that*

$$|W \otimes \alpha^{-t/r}| \leqq \xi .$$

(2.4) As usual more information is needed in the unramified situation. Assume ψ has exponent zero—i.e., the largest ideal on which ψ is trivial is $\mathfrak{R}$. Suppose π is a generic irreducible representation which contains the trivial representation of $K = \mathrm{GL}(r, \mathfrak{R})$ (necessarily with multiplicity one). Then there is an r-tuple of unramified quasi-characters μ_i of $F^\times$ such that π is a component of the induced representation:

$$\pi' = I(G,\ B;\ \mu_1,\ \mu_2,\ \cdots,\ \mu_r) .$$

In general if the μ_i are unramified then there is exactly one irreducible component of π' which contains the trivial representation of K. We denote this representation by $\pi(\mu_1,\ \mu_2,\ \cdots,\ \mu_r)$.

Thus the given π has the form $\pi(\mu_1,\ \mu_2,\ \cdots,\ \mu_r)$. Note that the central quasi-character ω of π is $\mu_1\mu_2 \cdots \mu_r$.

Clearly the space of K-fixed vectors in $\mathcal{W}(\pi; \psi)$ has dimension one. In [35], it is proved that if W is in that space then $W(e) \neq 0$. Let W_0 be the element of that space such that $W_0(e) = 1$. From [35], we have an explicit formula for W_0. Let $a = \mathrm{diag}(\tilde{\omega}^{m_1},\ \tilde{\omega}^{m_2},\ \cdots,\ \tilde{\omega}^{m_r})$. Then $W_0(a) = 0$ unless $m_1 \geqq m_2 \geqq \cdots \geqq m_r$. If on the contrary $m_1 \geqq m_2 \geqq \cdots \geqq m_r$ then let ρ_a be the rational representation of $\mathrm{GL}_r(\mathbf{C})$ whose highest weight is

$$\mathrm{diag}\,(a_1, a_2, \cdots, a_r) \longmapsto a_1^{m_1}(a_1a_2)^{m_2} \cdots (a_1a_2 \cdots a_r)^{m_r} .$$

Then

$$W_0(a) = \delta_B^{1/2}(a)\mathrm{Tr}\,\rho_a(C) \quad \text{where} \quad C = \mathrm{diag}\big(\mu_1(\tilde{\omega}), \mu_2(\tilde{\omega}), \cdots, \mu_r(\tilde{\omega})\big) .$$

Suppose now that $|\omega| = 1$ and that $m_r = 0$. Write $\mu_i = \alpha^{s_i}$ and suppose $-s_0 \leqq \mathrm{Re}(s_i) \leqq s_0$. Then ρ_a is a subrepresentation of $\otimes^n \rho_0$ where ρ_0 is the standard representation of $\mathrm{GL}_r(\mathbf{C})$. Here $n = m_1 + m_2 + \cdots + m_{r-1}$. It follows that every weight η of ρ_a has the form

$$\eta(t) = t_1^{n_1}t_2^{n_2} \cdots t_r^{n_r} \quad \text{with} \quad \Sigma n_i = n \quad \text{if} \quad t = \mathrm{diag}(t_1, t_2, \cdots, t_r) .$$

Thus

$$|\eta(C)| \leqq q^{s_0 n} = |\det \mathrm{a}|^{-s_0} .$$

On the other hand $\deg \rho_a \leqq r^n$. Thus if $q \geqq r$ we get $\deg \rho_a \leqq |\det a|^{-1}$ and

$$|W_0(a)| \leqq \delta_B^{1/2}(a)\,|\det a|^{-s_0-1} .$$

From this we obtain:

PROPOSITION (2.4.1). *Suppose F is such that $q \geqq r$. Suppose that the central quasi-character of π has module one and that, with the above notations, $-s_0 \leqq \mathrm{Re}(s_i) \leqq s_0$ $(1 \leqq i \leqq r)$. Let Φ be the characteristic function of $\mathfrak{R}^{r-1}$ in F^{r-1}. Then there is $t > 0$ which depends on s_0 but not on F nor π such that the gauge* (2.3.1) *majorizes W_0.*

(2.5) Finally recall the following result of [13, (5.2)]: if ϕ is any function on P^1, transforming on the left according to θ, smooth and of compact support mod N, for any non-zero P-invariant subspace $\mathcal{V}$ of $\mathcal{W}(\pi; \psi)$, there is $W \in \mathcal{V}$ such that $W|P^1 = \phi$. In fact this is true under the assumptions of (2.2.5).

3. Some auxiliary integrals

Again F is local and non-archimedean.

(3.1) If π is irreducible generic, we may regard the elements of $\mathcal{W}(\pi; \psi)$ as "generalized" matrix coefficients. Thus it is natural to introduce the integrals

$$Z(\Phi, s, W) = \int \Phi(x) W(x)\,|\det x|^s dx .$$

PROPOSITION (3.1). *Assume π is irreducible generic.*

(1) *The integrals $Z\big(\Phi, s + (r-1)/2, W\big)$ admit $L(s, \pi)$ for* "g.c.d."

(2) *They satisfy the functional equation:*

$Z(\Phi', 1-s+(r-1)/2, \tilde{W})/L(1-s, \tilde{\pi}) = \varepsilon(s, \pi, \psi)Z(\Phi, s+(r-1)/2, W)/L(s, \pi)$, *where* $\Phi'(g) = \hat{\Phi}(wg)$.

Proof. Replacing W by a gauge (cf. (2.3.6)), we see that the integrals converge in some half-space. For a given Φ there is a compact open subgroup Ω of G such that $\Phi(\omega x) = \Phi(x)$ for $\omega \in \Omega$.

With $d\omega$ normalized, the linear form

$$W \longmapsto \int_\Omega W(\omega)d\omega$$

on $\mathcal{W}(\pi; \psi)$ is smooth and thus belongs to the space of the representation contragredient to π. It follows that the function f defined by

$$f(g) = \int_\Omega W(\omega g)d\omega \qquad (3.1.3)$$

is a matrix-coefficient of π, in the usual sense. For Re(s) sufficiently large it is clear that

$$\begin{aligned} Z(\Phi, s, W) &= \int W(g)\,|\det g|^s d^\times g \int \Phi(\omega g)d\omega \\ &= \int d\omega \int \Phi(\omega g)W(g)\,|\det g|^s d^\times g \\ &= \int d\omega \int \Phi(g)W(\omega^{-1}g)\,|\det g|^s d^\times g \\ &= \int \Phi(g)\,|\det g|^s d^\times g \int W(\omega g)d\omega , \end{aligned}$$

so that

$$Z(\Phi, s, W) = Z(\Phi, s, f) . \qquad (3.1.4)$$

It follows that the integrals

$$Z(\Phi, s + (r-1)/2, W)$$

span an ideal of $\mathbf{C}[q^{-s}, q^s]$ which is contained in $\mathbf{C}[q^{-s}, q^s]L(s, \pi)$. To complete the proof of the first assertion we shall use the following lemma:

LEMMA (3.1.5). *Notations being as in* (1.2.3), *suppose* π *generic. Let* $\mathcal{W}_0$ *be the space of* W*'s in* $\mathcal{W}(\pi; \psi)$ *such that* W *vanishes outside* G^d. *Then* $\mathcal{W}_0$ *is irreducible under* G^0 *and if* $v(\det a) = 1$,

$$\mathcal{W}(\pi; \psi) = \bigoplus \pi(a^i)\mathcal{W}_0 , \quad (0 \leqq i \leqq d-1) .$$

Proof. If W is in $\mathcal{W}(\pi; \psi)$ and η is a character of $F^\times$, then $W \otimes \eta$ is in $\mathcal{W}(\pi \otimes \eta; \psi)$. If moreover η is in X_d, then $W \otimes \eta$ is in $\mathcal{W}(\pi; \psi)$. So the sum

$$\sum W \otimes \eta , \quad (\eta \in X_d)$$

is still in $\mathcal{W}(\pi; \psi)$. It vanishes outside G^d but not on G^d if $W(e) \neq 0$. Hence

$\mathcal{W}_0$ is a non-zero G^d-invariant subspace such that

$$\mathcal{W}(\pi, \psi) = \bigoplus \pi(a^i)\mathcal{W}_0 \quad (0 \leqq i \leqq d-1) .$$

By (1.2.3), $\mathcal{W}_0$ must be irreducible under G^d and G^0. Q.E.D.

The lemma being proved, select $W \in \mathcal{W}_0$ with $W(e) \neq 0$. Let also Ω be a compact open subgroup which fixes W. Then the function f defined by (3.1.3) is actually in $\mathcal{A}_0$ (notations of (1.2.5)). Since $f(e) = W(e) \neq 0$ it is non-zero. By (1.2.5) there is a Φ in $\mathcal{S}(r \times r)$ so that

$$Z(\Phi, s + (r-1)/2, f) = L(s, \pi) .$$

We may assume that $\Phi(\omega g) = \Phi(g)$ for $\omega \in \Omega$. Then from (3.1.4) we get

$$Z(\Phi, s + (r-1)/2, W) = L(s, \pi) ,$$

which concludes the proof of the first assertion of (3.1).

For the second assertion we let Ω, Φ, W, and f be as before. Then

$$\Phi\hat{}(\omega^\iota g) = \hat{\Phi}(g) \quad \text{for} \quad \omega \in \Omega ,$$

and

$$f^\iota(g) = \int W^\iota(\omega^\iota g) d\omega .$$

Thus we get

$$\int \Phi\hat{}(g) W^\iota(g) |\det g|^s d^\times g = Z(\Phi\hat{}, s, f^\iota) .$$

Changing g to wg we get on the left

$$Z(\Phi', s, \tilde{W})$$

and now we apply (1.1.2). Q.E.D.

(3.2). In the unramified situation, we have the following supplementary information:

PROPOSITION (3.2). *Let ψ have exponent zero and π contain the trivial representation of K. Let W_0 be as in* (2.4.1). *Call Φ_0 the characteristic function of $M(r \times r, \mathfrak{R})$. Then:*

$$Z(\Phi_0, s + (r-1)/2, W_0) = L(s, \pi) ,$$

the Haar-measure on G being normalized by the condition that meas(K) *be one.*

Proof. If we take in (3.1.3) $W = W_0$ and $\Omega = K$, the corresponding function f is bi-invariant under K and $f(e) = 1$. Thus it is the spherical function attached to π and our assertion follows from (3.1.4) and [17, Proposition (6.1.2)]. Q.E.D.

4. A new definition of ε and L

Here again the ground field F is a p-field.

(4.1) Let π be a generic representation of G. By Section 3, we may define $L(s, \pi)$ as the g.c.d. of the rational functions

$$Z(\Phi, s+(r-1)/2, W),\ \Phi \in \mathcal{S}(r\times r, F),\ W \in \mathcal{W}(\pi; \psi)\ .$$

The factor $L(s, \pi)$ will also be the g.c.d. of other integrals which we now introduce.

For $W \in \mathcal{W}(\pi; \psi)$, $0 \leqq j \leqq r-2$, set

$$\text{(4.1.1)}\quad \Psi(s, W; j) = \iint W\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_j & 0 \\ 0 & 0 & 1_{r-j-1} \end{pmatrix}\right] |a|^{s-(r-1)/2} d^\times a\, dx \qquad (a \in F^\times,\ x \in F^j)\ .$$

For $j = 0$, this is, by an obvious convention,

$$\text{(4.1.2)}\qquad \Psi(s, W; 0) = \int W\left[\begin{pmatrix} a & 0 \\ 0 & 1_{r-1} \end{pmatrix}\right] |a|^{s-(r-1)/2} d^\times a\ .$$

Furthermore, for $0 \leqq j \leqq r-3$ and Φ in $\mathcal{S}(F)$, we define

$$\text{(4.1.3)}\quad \Psi(s, W; j, \Phi) = \iint W\left[\begin{pmatrix} a & 0 & 0 & 0 \\ x & 1_j & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-2} \end{pmatrix}\right] |a|^{s-(r-1)/2} d^\times a\, dx\, \Phi(y)\, dy \qquad (a \in F^\times,\ x \in F^j,\ y \in F)\ .$$

From (2.2), it is clear that (4.1.2) converges for $\mathrm{Re}(s)$ large and represents a rational function in q^{-s} whose denominator can be taken independent of W. Thus the fractions (4.1.2) form a fractional ideal of $\mathbf{C}[q^{-s}, q^s]$. By (25), it contains the constants and thus admits a unique generator of the form $Q^{-1}(q^{-s})$, $Q \in \mathbf{C}[X]$, $Q(0) = 1$.

PROPOSITION (4.1.4). *For a given j, the integrals* (4.1.1) *and* (4.1.3) *also admit $Q^{-1}(q^{-s})$ for a "g.c.d."*

Proof. We shall use the following lemma repeatedly:

LEMMA (4.1.5). *Let H be a function on G, smooth on the right, and satisfying*

$$H(ng) = \theta(n)H(g)\ ,\quad (n \in N,\ g \in G)\ .$$

Then the support of the function

$$x \longmapsto H\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_{r-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] \quad (x \in F^{r-2})$$

is contained in a compact set independent of $a \in F^\times$. *For* $1 \leqq j \leqq r-2$ *and* Φ *in* $\mathcal{S}(F)$, *set*

$$H^1(g) = \int H\left[g\begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-2} \end{pmatrix}\right]\Phi\hat{\ }(-z)dz\ .$$

Then

$$\int H^1\left[\begin{pmatrix} a & 0 & 0 & 0 \\ b & 1_{j-1} & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-1} \end{pmatrix}\right]dx = \int H\left[\begin{pmatrix} a & 0 & 0 & 0 \\ b & 1_{j-1} & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-1} \end{pmatrix}\right]\Phi(x)dx\ .$$

Proof. Let $t' = \operatorname{diag}(a, 1, \cdots, 1)$ and $\bar{n} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1_{r-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Write $\bar{n} = ntk$ according to the Iwasawa decomposition. From the assumptions on H, if $H(t'\bar{n}) = \theta(t'n(t')^{-1})H(t'tk) \neq 0$, $t = \operatorname{diag}(t_1, t_2, \cdots, t_{r-1}, 1)$, there is a constant c so that $|t_2t_3^{-1}| \leqq c, \cdots, |t_{r-1}| \leqq c$. Since $t^{-1}n^{-1}\bar{n} \in K$, we find $t_2^{-1}, \cdots, t_{r-1}^{-1} \in \mathfrak{R}$. Hence $t_2, \cdots, t_{r-1}$ vary in a compact set in $F^\times$. Since $|\det t| = 1$, t_1 also does. Hence t varies in a compact set independent of a. Thus $n^{-1}\bar{n}$ has bounded entries. It follows that n, and thus $\bar{n}$, varies in a compact set independent of $a \in F^\times$.

Thus in the second assertion, the left hand side is convergent and equal to

$$\int dx \int dz \hat{\Phi}(-z)H[huh^{-1}h]\ ,$$

where

$$h = \begin{pmatrix} a & 0 & 0 & 0 \\ b & 1_{j-1} & 0 & 0 \\ x & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-1} \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 & z & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-2} \end{pmatrix}.$$

But then huh^{-1} is in N and $\theta(huh^{-1}) = \psi(zx)$. Thus our assertion follows from the Fourier inversion formula.

This being so, the first part of the lemma shows that (4.1.1) and (4.1.3) are really finite sums of integrals (4.1.2) where W is replaced by suitable translates. It easily follows that (4.1.1) and (4.1.3) admit a g.c.d.

The proposition will be proved as soon as we have the following inclusions of ideals of $\mathbf{C}[q^{-s}, q^{s}]$, for $1 \leqq j \leqq r-2$:

$$\begin{aligned}[\Psi(s, W; j)] &\subseteq [\Psi(s, W; j-1, \Phi)] \\ &\subseteq [\Psi(s, W; j-1)] \subseteq [\Psi(s, W; j-1, \Phi)] \\ &\subseteq [\Psi(s, W; j)] .\end{aligned}$$

By the first part of (4.1.5), given W, we may choose for Φ the characteristic function Φ_m of some fractional ideal $\mathfrak{P}^m$ so that

$$\Psi(s, W; j) = \Psi(s, W; j-1, \Phi_m) .$$

Hence the first inclusion. Since W is smooth, we have the second. For the third, we note again since W is smooth, that

$$\Psi(s, W; j-1) = \Psi(s, W; j-1, \Phi_m)[\mathrm{vol}\, \mathfrak{P}^m]^{-1}$$

for m sufficiently large. Finally, we apply the lemma to $H = W$. Then $H^1 = W^1$ is in $\mathcal{W}(\pi; \psi)$ and $\Psi(s, W; j-1, \Phi) = \Psi(s, W^1; j)$. The last inclusion and Proposition (4.1.4) follow.

(4.2) We remark the following. Suppose $\mathcal{W}$ is a space of functions satisfying the conditions of Remark (2.2.5). Then with $\mathcal{W}$ replacing $\mathcal{W}(\pi; \psi)$ the conclusions of (4.1.4) hold (cf. (7.2)). Similarly the following theorem is also true, the factor $L(s, \pi)$ being replaced by the "g.c.d." of the integrals $Z(\Phi, s + (r-1)/2, W)$.

THEOREM (4.3). *The integrals* (4.1.1) *and* (4.1.3) *admit* $L(s, \pi)$ *for their* g.c.d.

We introduce first certain complex measures which play a role in what follows.

For $\Phi \in \mathcal{S}(n \times n, F)$, $u \in F$, $u_2, v_2, \cdots, u_r, v_r \in F$, $k \in K$, set

$$\theta_\Phi(u, u_2, v_2, \cdots, u_r, v_r; k) = \int \Phi\left[\begin{pmatrix} u & x_{12} & \cdots & x_{1r} \\ & u_2 & & \vdots \\ & & \ddots & x_{r-1,r} \\ & & & u_r \end{pmatrix} k\right] \psi(v_2 x_{12} + \cdots + v_r x_{r-1,r}) \otimes dx_{jk} .$$

θ_Φ depends only on the class of k modulo some open compact subgroup and is, for each k, a Schwartz-Bruhat function of $(u, u_2, v_2, \cdots, u_r, v_r)$. We denote by K_Φ the partial co-Fourier-transform:

$$\theta_\Phi(u, u_2, v_2, \cdots, u_r, v_r; k) = \int K_\Phi(x, u_2, v_2, \cdots, u_r, v_r; k)\psi(xu)dx .$$

Then K_Φ is a function of the same type. We then define a complex measure ρ_Φ on $\mathrm{SL}(r, F)$ by

$$\int F(h)d\rho_\Phi(h)$$

$$= \int F\left[\operatorname{diag}(a_2 \cdots a_r, 1, \cdots, 1)^{-1}\begin{pmatrix} 1 & v & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1\end{pmatrix}\operatorname{diag}(1, a_2, \cdots, a_r)k\right]$$

$$K_\Phi(v, a_2, a_2^{-1}, \cdots, a_r, a_r^{-1}; k) \bigotimes_{i=2}^{r} |a_i|^{i-1}d^\times a_i dv dk .$$

Here k ranges over $K^1 = \mathrm{SL}(r, \mathfrak{R})$.

Clearly ρ_Φ has compact support. We will use the following lemma:

LEMMA (4.3.1). *Let $\mathcal{V}$ be the space of all functions H on G, smooth on the right, satisfying*

$$H(ng) = \theta(n)H(g) \quad (n \in N,\ g \in G)$$

and such that for all Φ in $\mathcal{S}(n \times n, F)$ the integral

$$\int H(g)\Phi(g)d^\times g$$

is convergent. Then given Φ there is a compactly supported measure μ_Φ on $\mathrm{SL}(r, F)$ such that

$$\iint H[\operatorname{diag}(a, 1, \cdots, 1)h]\,|a|^{-(r-1)}d^\times a\, d\mu_\Phi(h)$$

is convergent and equal to the above integral, for all $H \in \mathcal{V}$.

Proof. Note that $\mathcal{V}$ is stable by right translations. By Iwasawa, the first integral is

$$\int (H\cdot\Phi)\left[\begin{pmatrix} a_1 & x_{12} & \cdots & x_{1,r} \\ & a_2 & & \\ & & \ddots & \\ & & & a_r\end{pmatrix} k\right] \bigotimes_{i=1}^{r} |a_i|^{-(r-i)}d^\times a_i \otimes dx_{jk}dk . \tag{4.3.2}$$

Here k varies in K^1. Since H and Φ are smooth on the right, we find, by Fubini, that the analogous integral over B is also convergent. Taking a suitable Φ, we find that the integral

$$\int H[\operatorname{diag}(a, 1, \cdots, 1)h]\,|a|^{-(r-1)}d^\times a \tag{4.3.3}$$

is convergent for all $h \in G$.

By the hypotheses on H, we may write (4.3.2) as the absolutely convergent integral

$$\int H[\operatorname{diag}(a_1, a_2, \cdots, a_r)k]\theta_\Phi(a_1, a_2, a_2^{-1}, \cdots, a_r, a_r^{-1}; k) \bigotimes_{i=1}^{r} |a_i|^{-(r-i)} d^\times a_i dk ,$$

which we may accordingly rewrite as the iterated integral

$$\int_{(F^\times)^r \times K^1} \int_F H\left[\operatorname{diag}(a_1, 1, \cdots, 1) \begin{bmatrix} 1 & v & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \operatorname{diag}(1, a_2, \cdots, a_r)k\right]$$

$$K_\Phi(v, a_2, a_2^{-1}, \cdots, a_r, a_r^{-1}; k) dv \bigotimes_{i=1}^{r} |a_i|^{-(r-i)} d^\times a_i dk .$$

Since K_Φ is a Schwartz-Bruhat function and H is smooth on the right, this integral is dominated by a finite sum of integrals of the form (4.3.3). Hence we may change a_1 to $a_1(a_2 \cdots a_r)^{-1}$ to find

$$\int H[\operatorname{diag}(a, 1, \cdots, 1)h] |a|^{-(r-1)} d^\times a d\rho_\Phi(h) .$$

Thus in fact we may take $\mu_\Phi = \rho_\Phi$.

For $m \geqq 1$, let K_m be the subgroup of $k \in K^1$, congruent to 1 mod $\mathfrak{P}^m$.

LEMMA (4.3.4). *Suppose the exponent to ψ is zero. Then we can choose $\Phi \in \mathcal{S}(r\times r, F)$ so that* (4.3.1) *is true with μ_Φ equal to the normalized Haar-measure of K_m.*

Proof. Let S be the set of $r \times r$ matrices (x_{jk}) such that $x_{11} \in \mathfrak{P}^{-m}$, $x_{jj} \in 1 + \mathfrak{P}^m$ for $j \geqq 2$, $x_{ij} \in \mathfrak{R}$ for $i < j$, $x_{ij} \in \mathfrak{P}^m$ for $i > j$. Of course S is right invariant under K_m. We take for Φ the characteristic function of S. Clearly $\Phi \in \mathcal{S}(r\times r, F)$.

We also make a simple remark. Let A be any commutative ring. Let $e_1, e_2, \cdots, e_r$ be the canonical basis of $C = A^r$ (column vectors). In each module $\Lambda^j C$, $1 \leqq j \leqq r-1$, we have the natural basis $e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_j}$ $(k_1 < k_2 < \cdots < k_j)$ ordered lexicographically. Then the upper triangular matrices t in $\mathrm{GL}(r, A)$ are characterized by the fact that each matrix $\Lambda^j t$ has a last row of the form $(0, 0, \cdots, a)$ with $a \in A^\times$.

Now we claim that $b \in B$, $k \in K$, $bk = s \in S$ imply that $k \in (B \cap K)K_m$. Let $\mathcal{R}$ be the last row of K. Then $b_{rr}\, \mathcal{R}$ has integral entries. The elements of $\mathcal{R}$ being relatively prime, we see that b_{rr} is integral. Since s_{rr} is a unit, in fact so is b_{rr}. It then follows that k_{rl} is in $\mathfrak{P}^m$ for $1 \leqq l \leqq r-1$. We contend that for $j \geqq 2$, the last row of $\Lambda^{r-j+1}(s)$ has the form $(0, 0, \cdots, a) \bmod \mathfrak{P}^m$ with a a unit. Let c be the coefficient of $e_j \wedge e_{j+1} \wedge \cdots \wedge e_r$ in $se_{k_1} \wedge se_{k_2} \wedge \cdots \wedge se_{k_l}$ where $k_1 < k_2 < \cdots < k_l$ and $l = r - j + 1$. Then c has the form

$$c = \sum_\pi \pm s_{\pi(j)k_1} \cdot s_{\pi(j+1)k_2} \cdots s_{\pi(r)k_l} \,,$$

where π runs over the permutations of $\{j, j+1, \cdots, r\}$. Since $\pi(n) \neq 1$, the elements $s_{\pi(j)k_1}, s_{\pi(j+1)k_2}, \cdots, s_{\pi(r)k_l}$ are integral. Moreover the corresponding contribution to c is in $\mathfrak{P}^m$, unless $\pi(j) \leqq k_1, \pi(j+1) \leqq k_2, \cdots, \pi(r) \leqq k_l$. But then we must have $j = k_1, j+1 = k_2, \cdots, r = k_l$ and π is trivial. Our contention follows.

As above, it follows that the last row of $\Lambda^{r-j+1}(k)(j \geqq 2)$ has the same form. By the remark, K is upper triangular mod $\mathfrak{P}^m$ or equivalently is in $(B \cap K)K_m$.

Now let $H \in \mathfrak{V}$ and Φ be the characteristic function of S. Then

$$\begin{aligned}\int H(g)\Phi(g)d^\times g &= \iint_{B\times K} H(bk)\Phi(bk)d_l b dk \\ &= \int_{B\times B\cap K\times K_m} H(bb_1k)\Phi(bb_1k)d_l b db_1 dk = \int_{B\times K_m} H(bk)\Phi(bk)d_l b dk \,,\end{aligned}$$

since in fact the left Haar-measure of B is $B \cap K$-invariant on the right. This is also

$$\int_B \Phi(b)d_l b \int_{K_m} H(bk)dk \,.$$

Let $f(b) = \int_{K_m} H(bk)dk$. Then as before,

$$\begin{aligned}&\int f(b)\Phi(b)d_l b \\ &\quad = \int f[\operatorname{diag}(a_1, a_2, \cdots, a_r)]\theta_\Phi(a_1, a_2, a_2^{-1}, \cdots, a_r, a_r^{-1}; e) \otimes |a_i|^{-(r-i)} d^\times a_i \,.\end{aligned}$$

But here we find that, for an appropriate normalization of measures,

$$\theta_\Phi(u, u_2, v_2, \cdots, u_r, v_r; e) = 1$$

if $u \in \mathfrak{P}^{-m}$, $u_i \in 1 + \mathfrak{P}^m$, $v_i \in \mathfrak{R}$, and is zero otherwise. Thus our integral is, up to a non-zero constant,

$$\int_{\mathfrak{P}^{-m}} f[\operatorname{diag}(a, 1, \cdots, 1)]\,|a|^{-(r-1)} d^\times a \,.$$

Since the support of $f[\operatorname{diag}(a, 1, \cdots, 1)]$ is actually contained in $\mathfrak{P}^{-m}$, we can write this as

$$\int_{F^\times \times K_m} H[\operatorname{diag}(a, 1, \cdots, 1)k]\,|a|^{-(r-1)} d^\times a dk \,,$$

and we are done.

We pass to the proof of (4.3). We may assume the exponent of ψ is zero. $L(s, \pi)$ is the g.c.d. of the integrals

$$\int W(g)\Phi(g)|\det g|^{s+(r-1)/2}d^{\times}g = Z(\Phi,\, s+(r-1)/2,\, W)$$
$$\big(W\in \mathcal{W}(\pi;\psi),\ \Phi\in \mathcal{S}(r\times r,\, F)\big)\ .$$

If we apply Lemma (4.3.1) to

$$H(g) = W(g)|\det g|^{s+(r-1)/2}$$

(for Re(s) large), we get that

$$Z\big(\Phi,\, s+(r-1)/2,\, s,\, W\big) = \Psi(s,\, W';\, 0)$$

where $W'(g) = \int W(gh)d\mu_{\Phi}(h)$. Thus $L(s,\pi)$ belongs to $[Q^{-1}(q^{-s})]$. By (4.3.4), we can choose Φ so that μ_{Φ} is the Haar measure of K_m. Then for appropriate m, $W' = W$. Thus $Q^{-1}(q^{-s})$ belongs to $[L(s,\pi)]$ and we are done.

(4.4). We will now prove an analogous result for the ε-factor. By (3.1), we have

(4.4.1) $\quad Z\big(\Phi',\, 1-s+(r-1)/2,\, \widetilde{W}\big) = \varepsilon'(s,\pi,\psi)Z\big(\Phi,\, s+(r-1)/2,\, W\big)\ .$

We may accordingly take this as a definition of $\varepsilon'(s,\pi,\psi)$. Note that if we *assume* (4.4.1) for the space $\mathcal{W}$ of Remark (4.2), the factor ε' being replaced by the appropriate factor, then the following theorem is true in this greater generality (cf. (7.3)).

Note that in Theorem (4.5) $\pi(w')W$ stands for the right translate of W: $\pi(w')W(h) = W(hw')$. It will be interpreted in a similar way in the situation of Remark (4.2).

THEOREM (4.5). *Suppose $j+k = r-2$, $j\geqq 0$, $k\geqq 0$. Then*

$$\Psi\big(1-s,\, (\pi(w')W)^{\sim};\, k\big) = \varepsilon'(s,\pi,\psi)\Psi(s,\, W;\, j)\ .$$

Suppose $j < r-2$ and Φ is in $\mathcal{S}(F)$; then

$$\Psi\big(1-s,\, (\pi(w')W)^{\sim};\, k-1,\, \Phi'\big) = \varepsilon'(s,\pi,\psi)\psi(s,\, W;\, j,\, \Phi)\ ,$$

where $\Phi'(u) = \hat{\Phi}[(-1)^{j+r-1}u]$.

We shall use the following lemmas.

LEMMA (4.5.1). *Let Φ belong to $\mathcal{S}(r\times r;\, F)$. Then, for all $h\in \mathrm{SL}(r,F)$,*

$$\int\hat{\Phi}\left[\, n^{\iota}w^{-1}\begin{pmatrix} a & 0\\ 0 & 1_{r-1}\end{pmatrix}h\right]\bar{\theta}(n)dnda = \int\Phi\left[n\begin{pmatrix} a & 0 & 0\\ x & 1_{r-2} & 0\\ 0 & 0 & 1\end{pmatrix}w'h^{\iota}\right]\bar{\theta}(n)dndadx\ ,$$

both integrals absolutely convergent.

Proof. Clearly the integrals are convergent. Since the Fourier transform interchanges h and h^{ι}, we may assume that $h = (w')^{-1}$. In that case $w'h^{\iota} = e$. Recall that w and w' have been defined in (0.4.2). Thus:

$$w^{-1}\begin{pmatrix} a & 0 \\ 0 & 1_{r-1}\end{pmatrix}(w')^{-1} = \begin{pmatrix} 0 & -1_{r-1} \\ (-1)^r a & 0\end{pmatrix}.$$

After a simple change of variables, the formula reads

$$\int\hat{\Phi}\left[\begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & x_{12} & -1 & \cdots & 0 \\ 0 & x_{13} & x_{23} & -1\cdots & 0 \\ \vdots & \vdots & \vdots & & -1 \\ a & x_{1r} & x_{2r} & \cdots & x_{r-1r}\end{pmatrix}\right]\psi[-x_{12} - x_{23} - \cdots - x_{r-1r}]\otimes dx_{ij}\otimes da$$

$$= \int\Phi\left[\begin{pmatrix} a & x_{12} & x_{13} & \cdots & x_{1r} \\ \xi_1 & 1 & x_{23} & \cdots & x_{2r} \\ \xi_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_{r-1} & 0 & 0 & \cdots 1 & x_{r-1r} \\ 0 & 0 & 0 & \cdots & 1\end{pmatrix}\right]\psi[-x_{12} - x_{23} - \cdots - x_{r-1r}]da\otimes d\xi_i\otimes dx_{ij}.$$

This in fact follows directly from the Fourier inversion formula.

LEMMA (4.5.2). *With the notations of* (4.3.1), *for each* Φ *in* $\mathcal{S}(r\times r, F)$, *there is a compactly supported (complex) measure* ν_Φ *on* SL(r, F) *such that, for any* H *in* $\mathcal{V}$,

$$\int H(g)\Phi^{\wedge}(wg)d^\times g = \iint H[\mathrm{diag}\,(a, 1, \cdots 1)h]\,|a|^{-(r-1)}d^\times a d\nu_\Phi(h).$$

In addition, for any H *in* $\mathcal{V}$,

$$\iiint H\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_{r-2} & 0 \\ 0 & 0 & 1\end{pmatrix}w'h^l\right]|a|^{-(r-1)}d^\times a d\mu_\Phi(h)dx \tag{4.5.3}$$
$$= \iint H\left[\begin{pmatrix} a & 0 \\ 0 & 1_{r-1}\end{pmatrix}h\right]|a|^{-(r-1)}d^\times a d\nu_\Phi(h),$$

all integrals being absolutely convergent. Here the measure μ_Φ *is any measure satisfying the condition of* (4.3.1).

Proof. Indeed if $\Phi''(g) = \hat{\Phi}(wg)$, we may take $\nu_\Phi = \mu_{\Phi''}$. Then convergence of the left side of (4.5.3) follows from (4.1.5) and (4.3.1); similarly for the right side. Hence by applying the dominated convergence theorem, we may assume in proving (4.5.3) that H has compact support mod N. Then H can be represented by an integral

(4.5.4) $$H(g) = |\det g|^r \int \Phi_1(ng)\bar{\theta}(n)dn$$

with Φ_1 in $C_c^\infty(G)$.

We note that for Φ and Φ_1 in $\mathcal{S}(r\times r, F)$ the double integral

$$\iint \Phi_1(ng)\Phi(g)|\det g|^r d^\times g dn$$

is convergent. In fact, taking $\Phi_1 \geqq 0$, $\Phi \geqq 0$ and using the Iwasawa decomposition, this follows easily. Thus we might as well take Φ_1 in $\mathcal{S}(r\times r, F)$, define an element of $\mathcal{V}$ by (4.5.4) and establish (4.5.3) for that function.

On the right side of (4.5.3) we have

$$\int H(g)\hat{\Phi}(wg)d^\times g = \int dg\hat{\Phi}(wg)\int \Phi_1(ng)\bar{\theta}(n)dn = \int \bar{\theta}(n)dn\int \hat{\Phi}(wg)\Phi_1(ng)dg$$

or, by Fourier inversion,

$$= \int \bar{\theta}(n)dn \int \Phi(g)\hat{\Phi}_1(n^t w^{-1}g)dg = \int \Phi(g)K(g)d^\times g\ ,$$

where

(4.5.5) $$K(g) = |\det g|^r \int \hat{\Phi}_1(n^t w^{-1}g)\bar{\theta}(n)dn\ .$$

K is again in $\mathcal{V}$. Thus choosing μ_Φ as in (4.3.1), the previous integral is

$$\int K[\mathrm{diag}(a, 1, \cdots, 1)h]\,|a|^{-(r-1)}d^\times a d\mu_\Phi(h)\ ,$$

or, by Lemma 4.5.1,

$$\int H\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_{r-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} w' h^t\right] |a|^{-(r-1)}d^\times a dx d\mu_\Phi(h)\ .$$

The identity (4.5.3) follows.

We pass to the proof of (4.5). We remark that $(\pi(w')W)^\sim(g) = \widetilde{W}(gw')$. Set again $\Phi''(g) = \hat{\Phi}(wg)$. Choose Re$(s)$ so small that the integral

$$Z(\Phi'', 1-s+(r-1)/2, \widetilde{W}) = \int \widetilde{W}(g)\hat{\Phi}(wg)|\det g|^{1-s+(r-1)/2}d^\times g$$

is convergent. By Lemma (4.5.2) applied to $H(g) = \widetilde{W}(g)|\det g|^{1-s+(r-1)/2}$, this is

$$\iint \widetilde{W}\left[\begin{pmatrix} a & 0 \\ 0 & 1_{r-1}\end{pmatrix} h\right] |a|^{1-s-(r-1)/2}d^\times a d\nu_\Phi(h)$$

$$= \iiint \widetilde{W}\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_{r-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} w' h^t\right] |a|^{1-s-(r-1)/2}d^\times a d\mu_\Phi(h)dx\ .$$

On the other hand if we choose Re(s) so large that the integral

$$Z\left(\Phi, s + (r-1)/2, W\right) = \int W(g)\Phi(g)\,|\det g|^{s+(r-1)/2} d^\times g$$

is convergent, we obtain by (4.3.1),

$$Z\left(\Phi, s + (r-1)/2, W\right) = \iint W\left[\begin{pmatrix} a & 0 \\ 0 & 1_{r-1} \end{pmatrix} h\right] |a|^{s-(r-1)/2} d^\times a\, d\mu_\Phi(h) .$$

Finally suppose W (and hence $\tilde{W}$) is K_m-invariant ($m \geqq 1$). Choosing Φ and μ_Φ as in (4.3.4), we arrive at

$$Z\left(\Phi, s + (r-1)/2, W\right) = \Psi(s, W; 0)$$

and similarly

$$Z\left(\Phi'', 1 - s + (r-1)/2, \tilde{W}\right) = \Psi\left(1 - s, \left(\pi(w')W\right)^\sim; r - 2\right) .$$

Thus for $j = 0$ the first assertion of (4.5) follows from (3.1).

Assume it is true for j, where $0 \leqq j < r - 2$. Then, if Φ is in $\mathcal{S}(F)$, we can write

$$\Psi(s, W; j, \Phi) = \Psi(s, W'; j) ,$$

where W' is the element of $\mathcal{W}(\pi; \psi)$ defined by

$$W'(g) = \int_F W\left[g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1_j & 0 & 0 \\ y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j-2} \end{pmatrix}\right] \Phi(y) dy .$$

By hypothesis, we have then

$$\Psi(s, W; j, \Phi)\varepsilon'(s, \pi, \psi) = \Psi\left(1 - s, \left(\pi(w')W'\right)^\sim; k\right) .$$

The last integral reads

$$\int |a|^{1-s-(r-1)/2} d^\times a \int (W')^\sim \left[\begin{pmatrix} a & 0 & 0 \\ x & 1_k & 0 \\ 0 & 0 & 1_{r-k-1} \end{pmatrix} w'\right] dx .$$

The inner integral however is

$$\iint \tilde{W}\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_k & 0 \\ 0 & 0 & 1_{r-k-1} \end{pmatrix} w' \begin{pmatrix} 1 & 0 & -y & 0 \\ 0 & 1_j & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-j+2} \end{pmatrix}\right] \Phi(y) dy dx$$

$$= \iint \tilde{W}\left[\begin{pmatrix} a & 0 & 0 \\ x & 1_k & 0 \\ 0 & 0 & 1_{r-k-1} \end{pmatrix}\begin{pmatrix} 1 & 0 & (-1)^{j+r-1}y & 0 \\ 0 & 1_k & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-k-2} \end{pmatrix} w'\right]\Phi(y)dydx\ .$$

Since $\tilde{W}(ng) = \theta(n)\tilde{W}(g)$, $n \in N$, $g \in G_r$, this is (see (4.1.5)),

$$\int \tilde{W}\left[\begin{pmatrix} a & 0 & 0 & 0 \\ x & 1_{k-1} & 0 & 0 \\ u & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{r-k-1} \end{pmatrix} w'\right]\hat{\Phi}((-1)^{j+r-1}u)dudx\ .$$

Thus if $\Phi'(u) = \hat{\Phi}((-1)^{j+r-1}u)$, we obtain

$$\Psi(1 - s, (\pi(w')W')^{\sim}; k) = \Psi(1 - s, (\pi(w')W)^{\sim}; k - 1, \Phi')\ ,$$

and

$$\Psi(s, W; j, \Phi)\varepsilon'(s, \pi, \psi) = \Psi(1 - s, (\pi(w')W')^{\sim}; k - 1, \Phi')\ .$$

Now let Φ be the characteristic function of $\mathfrak{P}^{-m}$ with m so large that

$$\Psi(s, W; j, \Phi) = \Psi(s, W; j + 1)\ .$$

Φ' is the characteristic function of $\mathfrak{P}^m$ times (vol $\mathfrak{P}^{-m}$). If m is large, $(\pi(w')W)^{\sim}$ is invariant by K_m. Hence, for such m, we find

$$\Psi(1 - s, (\pi(w')W)^{\sim}; k - 1, \Phi') = \Psi(1 - s, (\pi(w')W)^{\sim}; k - 1)\ .$$

Combining, we obtain

$$\Psi(s, W; j + 1)\varepsilon'(s, \pi, \psi) = \Psi(1 - s, (\pi(w')W)^{\sim}; k - 1)\ .$$

This is the first assertion of (4.5) for $j + 1$, $0 \leqq j < r - 2$. Hence by induction that assertion is true for all j, $0 \leqq j \leqq r - 1$. The second assertion follows from the proof.

(4.6). In the unramified situation, we have the following supplementary information:

PROPOSITION (4.6). *Let the notations be as in* (3.2) *and let* Φ_1 *be the characteristic function of* $\mathfrak{R}$ *in* F. *Then*:

$$\Psi(s, W_0; j) = \Psi(s, W_0; j, \Phi_1) = L(s, \pi)\ .$$

Proof. If $\Phi = \Phi_0$ (notations of (3.2)), then ρ_Φ is the normalized Haar-measure on the group SL(3, $\mathfrak{R}$). Then:

$$Z(\Phi_0, s, W_0) = \Psi(s, W_0; 0)$$

and the assertion for $\Psi(s, W_0; 0)$ follows directly from (3.2). The functional equation (4.5) implies then the assertion for $\Psi(s, W_0; r - 2)$. We leave the remaining assertions to the reader.

5. Complements on "Local zeta-functions": the ε-factor

(5.1) In this section the ground field F is a p-field. We let H be a simple algebra of dimension r^2 over F. We denote by ν the reduced norm of H. If π is an irreducible admissible representation of $H^\times$, the multiplicative group of H, the factors $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ have been defined in [17], just as in the case $H = M(r \times r, F)$ recalled in Section 1. If χ is a quasi-character of $F^\times$, we denote, just as before, by $\pi \otimes \chi$ the representation $g \mapsto \pi(g)\chi(\nu(g))$. For the applications we have in mind (cf. § 14) the following result is essential:

PROPOSITION (5.1). *Suppose π is an irreducible admissible representation of $H^\times$ whose central quasi-character is ω. Let χ_i, $1 \leqq i \leqq r$, be r quasi-characters of $F^\times$ whose product is ω. If k is a sufficiently large integer and χ a quasi-character of $F^\times$ of conductor $\mathfrak{P}_F^k$ then*

(5.1.1) $$L(s, \pi \otimes \chi) = 1 ,$$

(5.1.2) $$\varepsilon(s, \pi \otimes \chi, \psi) = \prod_{1 \leqq i \leqq r} \varepsilon(s, \chi\chi_i, \psi) .$$

A proof is given in [25]. We simply observe here that in view of (3.5) in [17], it suffices to prove the proposition when π is supercuspidal. Then $L(s, \pi) = 1$ unless H is a division algebra and π has the form

(5.1.3) $$\pi(g) = |\nu(g)|^t .$$

Thus the assertion about the L-factor is clear. The assertion about the ε-factor is similar to some well known assertions on Gauss-sums; the proof is also similar.

6. Problems of classification

In this section, the ground field F is again a p-field. We shall be primarily interested in classifying the irreducible admissible representations of GL(3, F). We first state without proof some general results having a bearing on classification; they are or will soon be in the literature.

(6.1) In (6.1) and (6.2) the integer r is arbitrary. We need some results on induced representations.

PROPOSITION (6.1.1). *Let R be a standard parabolic subgroup of $G = \mathrm{GL}(r, F)$. Identify R/U_R with the group $M = \prod_{1 \leqq i \leqq k} M_i$ where $M_i = \mathrm{GL}(r_i)$. Let σ_i be an irreducible admissible representation of M_i, $(1 \leqq i \leqq k)$ and*

$$\xi = I(G, R; \sigma_1, \cdots, \sigma_k) .$$

Then if ξ admits a generic irreducible component, each σ_i is generic. Conversely, if each σ_i is generic, then in any composition series for ξ there is exactly one irreducible quotient which is generic.

There is an analogous result in the context of unitary representations. Indeed by Mackey's theory the representation of P^1,

$$\tau = I(P^1, N; \theta)\ ,$$

is irreducible unitary. We shall say that an irreducible unitary representation π of $\mathrm{GL}(r, F)$ is *strongly generic* if $\pi|P^1$ is equivalent to τ.

PROPOSITION (6.1.2). *With the notation of* (6.1.1), *let the σ_i be irreducible and strongly generic representations. Let ξ be the unitarily induced representation. Then ξ is irreducible and strongly generic.*

For proofs and detailed discussions we refer, for instance, to [18]. For example, if π is irreducible unitary and square-integrable mod Z, then π is strongly generic (loc. cit.). Note that this applies trivially to $r = 1$, that is to all characters of $F^\times$. We may then apply (6.1.2) when the σ_i are unitary and square-integrable mod Z_{r_i}. The irreducible induced representations we obtain are by definition the *tempered* representations. They are strongly generic. *The essentially tempered* representations are those of the form $\pi \otimes \mu$ where π is tempered and μ a quasi-character.

The relation between the notions of generic and strongly generic representations is as follows. Recall first that if π is unitary and (topologically) irreducible then π determines an irreducible admissible representation π_0: π_0 is the restriction of π to the space of smooth vectors. Of course π_0 is pre-unitary. Conversely every admissible irreducible pre-unitary representation is of the form π_0 for a unique irreducible representation π.

Now suppose π is strongly generic. Every G-smooth vector is also P^1-smooth and thus defines a smooth function on P^1. Evaluation at e is then a non-zero element of $(\pi_0)^*_\theta$. Thus π_0 is generic. We conjecture that the converse is also true. It is so, in any case, for $r = 2$ or 3.

From now on, for notational purposes, we disregard the distinction between π and π_0. For instance we will say that π_0 rather than π is unitary (or strongly generic).

(6.2) We recall the "Langlands' classification" of the representations of $\mathrm{GL}(r, F)$. This has been obtained by Silberger ([36]) and Wallach for arbitrary reductive groups.

Let R be a standard upper parabolic subgroup of G_r (including G_r itself) and let M and the M_i be as in (6.1.1). Let $D(R)$ be the set of positive valued quasi-characters of R/U_R. They have the form

(6.2.1) $$\chi(m) = \prod_{1\leq i\leq k} |\det m_i|_F^{s_i},\ m \in M,\ s_i \in \mathbf{R} .$$

Let $D^+(R)$ be the subset of these for which

(6.2.2) $$s_1 > s_2 > \cdots > s_k .$$

Let σ_0 be a tempered (unitary) representation of M and $\chi \in D^+(R)$. Then if $\sigma = \sigma_0 \otimes \chi$, the induced representation

(6.2.3) $$\xi = I(G, R; \sigma)$$

admits a unique irreducible quotient that we shall denote by $\pi_R(\sigma)$ or, if

(6.2.4) $$\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_k ,$$

by

(6.2.5) $$\pi_R(\sigma_1, \sigma_2, \cdots, \sigma_k) .$$

Remark (6.2.6). The outer automorphism $g \mapsto wg^tw^{-1}$ takes R to another upper standard parabolic subgroup $R^\vee$, $\chi \in D^+(R)$ to $\chi^\vee \in D^+(R^\vee)$, $\sigma = \sigma_0 \otimes \chi$ to $\sigma^\vee = \sigma_0^\vee \otimes \chi^\vee$, the representation $\xi = I(G, R; \sigma)$ to $\xi^\vee = I(G, R^\vee, \sigma^\vee)$ and hence $\pi_R(\sigma)$ to $\pi_{R^\vee}(\sigma^\vee)$. In particular

$$\pi_R(\sigma)^\sim \cong \pi_{R^\vee}(\sigma^\vee) .$$

However in general we can only say that $\xi^\sim$ and $\xi^\vee$ have the same semi-simplification.

PROPOSITION (6.2.7). *To every irreducible admissible representation π of $G_r(F)$ there corresponds exactly one upper parabolic subgroup R and one representation of the form $\sigma = \sigma_0 \otimes \chi$ with σ_0 tempered and $\chi \in D^+(R)$ such that*

$$\pi \cong \pi_R(\sigma) .$$

(6.3) If $\pi = \pi_R(\sigma)$ and $\xi = I(G, P; \sigma)$ with $\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_k$, then we recall ([22]) that $L(s, \xi)$, $L(s, \tilde{\xi})$ and $\varepsilon(s, \xi, \psi)$ are defined even when ξ is reducible and that

$$L(s, \pi) = L(s, \xi) = \prod_i L(s, \sigma_i) ,$$
$$L(s, \tilde{\pi}) = \prod_i L(s, \tilde{\sigma}_i) ,$$
$$\varepsilon(s, \pi, \psi) = \varepsilon(s, \xi, \psi) = \prod_i \varepsilon(s, \sigma_i, \psi) .$$

(6.4) We now apply the above results to $r = 2$ and $r = 3$. Suppose first $r = 2$. Let σ_2 be the Steinberg representation of G_2, that is, the square-integrable component of

$$I(G_2, B_2; \alpha^{1/2}, \alpha^{-1/2}) .$$

Recall that $\tilde{\sigma}_2 \cong \sigma_2$. The essentially tempered representations are the supercuspidal ones, those of the form $\sigma_2 \otimes \mu$ with μ a quasi-character, and those of the form

$$I(G_2, B_2; \mu_1, \mu_2)$$

where $\mu_1\mu_2^{-1}$ is a character (the principal series). The latter are denoted also by $\pi(\mu_1, \mu_2)$. The other representations are of the form

$$\pi_B(\mu_1, \mu_2)$$

where $\mu_1 = \chi_1\alpha^{s_1}$, $\mu_2 = \chi_2\alpha^{s_2}$, $s_1 > s_2$ and χ_i is a character. These are also denoted by $\pi(\mu_1, \mu_2)$. Actually unless $\mu_1\mu_2^{-1} = \alpha$ the representation

$$\xi = I(G_2, B_2; \mu_1, \mu_2)$$

is irreducible and therefore equivalent to $\pi_B(\mu_1, \mu_2)$. If on the other hand $\mu_1\mu_2^{-1} = \alpha$ then we have an exact sequence

$$0 \longrightarrow \sigma_2 \otimes \mu_2\alpha^{1/2} \longrightarrow \xi \longrightarrow \pi_B(\mu_1, \mu_2) \longrightarrow 0$$

and $\pi_B(\mu_1, \mu_2)$ is just $\mu_2\alpha^{1/2} \circ \det$.

By (6.1.1) we see that the infinite dimensional representations are generic. The unitary representations are the tempered representations, those of the form $\chi \circ \det$ with χ a character, and those of the form $\pi_B(\chi\alpha^s, \chi\alpha^{-s})$ with χ a character and $0 < s < 1/2$ (the complementary series). Since the representations of the complementary series are known to be strongly generic ([16], Chap. II, Appendix 2), we see that any unitary irreducible generic representation is also strongly generic.

(6.5) We now apply the results of (6.1) to (6.3) to the case $r = 3$. Let P and Q be the upper parabolic subgroups of type (2, 1) and (1, 2) respectively. Let σ_3 be the Steinberg representation of $G_3(F)$, that is, the square-integrable component of

$$I(G_3, B_3; \alpha, 1, \alpha^{-1}) .$$

Then the essentially tempered representations are the supercuspidals, those of the form $\sigma_3 \otimes \mu$ with μ a quasi-character, and those of the form

$$I(G, P; \tau_1, \mu_3) = I(G, Q; \mu_3, \tau_1) ,$$

where $\tau_1 = \tau \otimes \rho$, $\mu_3 = \chi\rho$, τ is a tempered unitary representaion of $G_2(F)$, χ a character, and ρ a quasi-character of $F^\times$. The latter are denoted by $\pi(\tau_1, \mu)$ or $\pi(\mu, \tau_1)$. If $\tau_1 = \pi(\mu_1, \mu_2)$ is in the principal series then $\pi(\tau_1, \mu)$ is equivalent to

$$I(G, B; \mu_1, \mu_2, \mu_3)$$

(for any order of the triple (μ_1, μ_2, μ_3)). We now describe the representations of the form

$$\pi = \pi_R(\sigma)$$

(R proper). As before we set

$$\xi = I(G, R; \sigma) .$$

The case when ξ is irreducible, that is, $\xi = \pi_R(\sigma)$, will be especially important.

(6.5.1) *The case* $R = Q$. Then $\xi = I(G, Q; \chi\alpha^{s_1}, \tau \otimes \alpha^{s_2})$ with χ a character of $F^\times$, τ a tempered representation of $G_2(F)$ and $s_1 > s_2$.

If τ is supercuspidal ξ is irreducible.

If $\tau = \sigma_2 \otimes \mu$ with μ a character then ξ is irreducible unless $s_1 - s_2 = 3/2$ and $\chi = \mu$. In that case we have an exact sequence

$$0 \longrightarrow \sigma_3 \otimes \mu\alpha^{s_2+1/2} \longrightarrow \xi \longrightarrow \pi_Q(\chi\alpha^{s_1}, \tau \otimes \alpha^{s_2}) \longrightarrow 0$$

and $\sigma_3 \otimes \mu\alpha^{s_2+1/2}$ is generic (as an essentially tempered representation).

If $\tau = \pi(\mu_1, \mu_2)$ where μ_1 and μ_2 are characters, then ξ is irreducible unless $s_1 - s_2 = 1$ and $\chi = \mu_1$ or μ_2. Assume then $s_1 = s_2 + 1$ and $\chi = \mu_1$ ($\mu_2\mu_1^{-1}$ trivial or not). We then have an exact sequence

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \pi \longrightarrow 0 , \qquad \xi' = I(G, P; \sigma_2 \otimes \mu_1\alpha^{s_2+1/2}, \mu_2\alpha^{s_2}) .$$

Applying the automorphism $g \mapsto wg^t w^{-1}$ we see from the preceding case that ξ' is irreducible. It is generic by (6.1.1). The case $\chi = \mu_2$ is similar.

(6.5.2) *The case* $R = P$. Since $g \mapsto wg^t w^{-1}$ carries P to Q we are immediately reduced to the previous case by Remark (6.2.6).

(6.5.3) *The case* $R = B$. Then

$$\xi = I(G, B; \chi_1, \chi_2, \chi_3)$$

with $\chi_i = \mu_i\alpha^{s_i}$, μ_i a character and $s_1 > s_2 > s_3$. ξ is irreducible unless one of the following holds:

$$\chi_1\chi_2^{-1} = \alpha , \quad \chi_2\chi_3^{-1} = \alpha , \quad \chi_1\chi_3^{-1} = \alpha .$$

If $\chi_1\chi_2^{-1} = \alpha$ but $\chi_2\chi_3^{-1} \neq \alpha$, we have again an exact sequence

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \pi \longrightarrow 0 , \qquad \xi' = I(G, Q; \mu_3\alpha^{s_3}, \sigma_2 \otimes \mu_2\alpha^{s_2+1/2}) .$$

By (6.5.1), $\xi'^\sim$ is irreducible and is a π_Q. Thus by (6.2.6), ξ' is a π_P, again generic by (6.1.1). The irreducible representation π is equivalent to

$$I(G, Q; \mu_3\alpha^{s_3}, \mu_2\alpha^{s_2+1/2} \circ \det) .$$

If $\chi_2\chi_3^{-1} = \alpha$, $\chi_1\chi_2^{-1} \neq \alpha$, then in a similar fashion $\xi' = I(G, Q; \mu_1\alpha^{s_1}, \sigma_2 \otimes \mu_3\alpha^{s_3+1/2})$ is a π_Q and π is equivalent to $I(G, Q; \mu_1\alpha^{s_1}, \mu_3\alpha^{s_3+1/2} \circ \det)$.

Suppose $\chi_1\chi_3^{-1} = \alpha$. Again we have a composition series

$$0 \longrightarrow \xi' \longrightarrow \xi \longrightarrow \pi \longrightarrow 0 , \qquad \xi' = I(G, Q; \mu_2\alpha^{s_2}, \sigma_2 \otimes \mu_3\alpha^{s_3+1/2}) .$$

The irreducible representation ξ' has been described earlier as either a π_G, π_P or π_Q. As for π, it is equivalent to

$$I(G, Q; \mu_2\alpha^{s_2}, \mu_3\alpha^{s_3+1/2}\circ\det) .$$

Note for later purposes, that if $s_2 = 0$, $s_3 = -1/2$, then this representation is irreducible and unitary.

Finally suppose $\chi_1\chi_2^{-1} = \chi_2\chi_3^{-1} = \alpha$. After replacing ξ by $\xi \otimes \mu$ for a suitable μ we may assume $\chi_1 = \alpha$, $\chi_2 = 1$, $\chi_3 = \alpha^{-1}$. For any parabolic subgroup R of $G_3(F)$, let V_R be the space of smooth functions on $G_3(F)$ invariant on the left under R. Then V_B is the space of $\tilde{\xi}$ and contains V_P and V_Q. The space V of ξ contains the respective annihilators $V_P^\perp$ and $V_Q^\perp$ and admits a composition series

$$\{0\} \subset V_P^\perp \cap V_Q^\perp \subset V_Q^\perp \subset V_P^\perp + V_Q^\perp \subset V .$$

The representation on $V_P^\perp \cap V_Q^\perp$ is σ_3, that on $V_Q^\perp / V_P^\perp \cap V_Q^\perp$ is $\pi_Q(\alpha, \sigma_2 \otimes \alpha^{-1/2})$, that on $V_P^\perp / V_P^\perp \cap V_Q^\perp$ is $\pi_P(\sigma_2 \otimes \alpha^{1/2}, \alpha^{-1})$ and $V / V_P^\perp + V_Q^\perp$ affords the trivial representation. Note that $V_Q^\perp$ is equivalent to $\mathrm{Ind}(G, Q; \alpha, \sigma_2 \otimes \alpha^{-1/2})$ and $V_P^\perp$ to $\mathrm{Ind}(G, P; \sigma_2 \otimes \alpha^{1/2}, \alpha^{-1})$.

To a large extent the proofs of these facts can be derived from [2], [4], [44], [45].

(6.6) According to (6.2.7), any irreducible admissible representation π of $G_3(F)$ is the unique irreducible quotient of an induced representation

$$\xi = I(G, R; \sigma) ,$$

where R is an upper standard parabolic subgroup and σ an essentially tempered representation. The pair (R, σ) is uniquely determined. We call ξ *the induced representation attached to* π. The automorphism $g \mapsto wg^t w^{-1}$ takes π to $\tilde{\pi}$ and the pair (R, σ) to the pair $(R^\vee, \sigma^\vee)$. Thus the induced representation attached to $\tilde{\pi}$ is

$$\xi^\vee = I(R, R^\vee; \sigma^\vee) .$$

In any case ξ admits a unique generic component by (6.1.1). From the explicit composition series for ξ given in (6.5), it follows that if $\xi \neq \pi$, then π is not generic. Otherwise said, $\xi = \pi$ if and only if π is generic.

(6.7) The irreducible unitary representations of $G_3(F)$ are described in the following:

PROPOSITION (6.7). *Let π be any irreducible unitary representation of $G_3(F)$. Then either π is square-integrable, or one-dimensional, or of the form*

$$\eta = I(G, P; \sigma, \nu) ,$$

where σ is an irreducible unitary representation of $G_2(F)$ and ν a character of $F^\times$.

Proof. First note that $g \mapsto wg^t w^{-1}$ carries η to $\eta^\vee = I(G, Q; \nu^{-1}, \sigma^\vee)$. If σ is one-dimensional we have noted in (6.5.3) that $\eta^\vee$, and hence η, is irreducible. Otherwise σ is strongly generic. Thus, in this case, by (6.1), η is also irreducible. Conversely let π be an irreducible unitary representation of $G_3(F)$. Suppose π is not tempered. Then $\pi = \pi_R(\sigma)$ and $\tilde{\pi} = \pi_{R^\vee}(\sigma^\vee)$ (see 6.2.6) where R is proper. On the other hand, $\bar{\pi} = \pi_R(\bar{\sigma})$ as is easy to see. Thus $R = R^\vee$ by (6.2.7). Thus $R = B$ and $\bar{\sigma} = \sigma^\vee$. Hence π is the irreducible quotient of

$$\xi = I(G, B; \chi_1, \chi_2, \chi_3)$$

with $\chi_1 = \mu\alpha^s$, $\chi_3 = \mu\alpha^{-s}$, $s > 0$, χ_2, μ characters. Then either $\chi_1\chi_3^{-1} = \alpha$ or $\chi_1\chi_2^{-1} = \chi_2\chi_3^{-1} = \alpha$ or ξ is irreducible. In the first case $\pi = I(G, Q; \chi_2, \mu\circ\det)$ and in the second $\pi = \mu \circ \det$.

Assuming then that ξ is irreducible we have $\pi = \xi = I(G, P; \sigma, \mu_2)$, $\sigma = \pi_{B_2}(\mu\alpha^s, \mu\alpha^{-s})$. Clearly $\tilde{\sigma} = \bar{\sigma}$ so there is an essentially unique invariant Hermitian form $\beta \neq 0$ on the space of σ. Then the integral

$$B(f_1, f_2) = \int_{P\backslash G} \beta\big(f_1(g), f_2(g)\big) dg$$

defines a non-zero invariant Hermitian form on the space of π. Since π is unitary, B is proportional to a positive-definite form. We may then assume β so chosen that $B(f, f) \geqq 0$ for f in the space of π. Since $G = PK$, this implies $\beta(v, v) \geqq 0$ for any v in the space of σ. Hence σ is unitary. Q.E.D.

(6.8) Let us denote by $\pi(\chi_1, \chi_2, \chi_3)$ the irreducible representations of $G_3(F)$ of the following form:

$$\begin{aligned}
&\pi_B(\mu_1\alpha^{s_1}, \mu_2\alpha^{s_2}, \mu_3\alpha^{s_3}) , && s_1 > s_2 > s_3 , \\
&\pi_Q(\mu\alpha^{s_1}, \tau \otimes \alpha^{s_2}) , && \tau = \pi(\mu_2, \mu_3) ,\ s_1 > s_2 , \\
&\pi_P(\tau \otimes \alpha^{s_1}, \mu_3\alpha^{s_2}), && \tau = \pi(\mu_1, \mu_2) ,\ s_1 > s_2 , \\
&I(G, B; \mu_1\alpha^{s_1}, \mu_2\alpha^{s_2}, \mu_3\alpha^{s_3}) , && s_1 = s_2 = s_3
\end{aligned}$$

with μ_i unitary and $\chi_i = \mu_i\alpha^{s_i}$.

The irreducible admissible representations of $G_3(F)$ having a K-fixed vector are then the representations $\pi(\chi_1, \chi_2, \chi_3)$ where μ_i is unramified. Of course our notation is consistent with that of (2.4).

7. Complements on GL(3)

In this section the ground field F is a p-field and $r = 3$.

(7.1) Our main task will be the extension of the results of Section 4 to all irreducible representations of $G = G_3$, generic or not. Accordingly let π be any irreducible, admissible representation of G_3.

Write π in the form $\pi_R(\sigma)$. Then it is the unique irreducible quotient of the attached induced representation $\xi = I(G, R; \sigma)$ (6.7). By (6.1), $\dim \xi_\theta^* = 1$. Choose $\lambda \neq 0$ in ξ_θ^* and for each f in the space of ξ set $W_f(g) = \lambda(\pi(g)f)$. If π is generic then $\pi = \xi$ (6.6) and $f \mapsto W_f$ is a bijection of the space of ξ onto $\mathcal{W}(\pi; \psi)$. Accordingly we may define $\mathcal{W}(\pi; \psi)$ in all cases to be the space of all functions W_f. Then in all cases we have the following result:

LEMMA (7.1.1). *The map $f \mapsto W_f$ is injective.*

Proof. We may assume $R \neq G$ and π is not generic. Then ξ has length 2 or 4 (cf. (6.5)). If it has length 2, then we have an exact sequence

$$0 \longrightarrow \tau \longrightarrow \xi \longrightarrow \pi \longrightarrow 0 ,$$

where τ is the space of the unique generic component of ξ. Since $\dim \tau_\theta^* = 1$, the exactness of the functor shows that the restriction of λ to τ is $\neq 0$. Thus if K is the kernel of the map (7.1.1) we find $K \cap \tau = \{0\}$. Since π is the unique irreducible quotient of ξ we find $K \subset \tau$ and so $K = \{0\}$.

Assume ξ has length 4. Replacing π by a suitable $\pi \otimes \mu$, we may assume $\pi = \pi_B(\alpha, 1, \alpha^{-1})$ (trivial representation) and $\xi = I(G, B; \alpha, 1, \alpha^{-1})$. Then we have the composition series (notations of (6.5)):

$$\{0\} \subset V_P^\perp \cap V_Q^\perp \subset V_Q^\perp \subset V_P^\perp + V_Q^\perp \subset \xi .$$

Recall that in fact $V_Q^\perp$ is the space of the induced representation attached to the irreducible representation of G on $V_Q^\perp / V_P^\perp \cap V_Q^\perp$ while $V_P^\perp \cap V_Q^\perp$ is the space of special representation. Again the restriction of λ to $V_Q^\perp$ is non-zero. So if K is the kernel of the map (7.1.1) we find $K \cap V_Q^\perp = \{0\}$ by the previous case. On the other hand we have $K \subset V_P^\perp + V_Q^\perp$, again because π is the unique irreducible quotient of ξ. Suppose $K \neq \{0\}$. Then the representation of G on K is equivalent to a subrepresentation of the representation of G on $V_P^\perp + V_Q^\perp / V_Q^\perp$ or $V_Q^\perp / V_P^\perp \cap V_Q^\perp$. Similarly it is equivalent to a subrepresentation of the representation of G on $V_P^\perp / V_P^\perp \cap V_Q^\perp$. Thus we find that the irreducible representations of G on $V_P^\perp / V_P^\perp \cap V_Q^\perp$ and $V_Q^\perp / V_P^\perp \cap V_Q^\perp$ are equivalent. This is not so however and so $K = \{0\}$. Q.E.D.

(7.2) Returning to the general case, we note that ξ is finitely generated. Since by (7.1.1) we may regard ξ as the representation of G on $\mathcal{W}(\pi; \psi)$ by right translations, we see we are in the conditions of applications of Remark (4.2). In particular the fractional ideals $[Z(\Phi, s+1, W)]$ and $[Z(\Phi, s+1, f)]$, where $W \in \mathcal{W}(\pi; \psi)$, f is a matrix coefficient of ξ, and $\Phi \in \mathcal{S}(r \times r, F)$, coincide. Indeed, as in (3.1) we have the inclusion $[Z(\Phi, s+1, W)] \subseteqq [Z(\Phi, s+1, f)]$. To obtain the reverse inclusion we first prove the following lemma:

LEMMA (7.2.3). *Let ζ be a cubic, unramified, non-trivial character of $F^\times$. Suppose π and $\pi \otimes \zeta$ are inequivalent. Then the spaces $\mathcal{W}(\pi; \psi)$, $\mathcal{W}(\pi \otimes \zeta; \psi)$, and $\mathcal{W}(\pi \otimes \zeta^2; \psi)$ are linearly independent.*

Proof. Let τ be the unique generic component of ξ. We may regard ξ as acting by right translations on $\mathcal{W}(\pi; \psi)$. Hence every invariant irreducible subspace of ξ is generic. Thus τ is the unique minimal component of ξ and $\tau \otimes \zeta^i$ the unique minimal component of $\xi \otimes \zeta^i$. Assume the three spaces are not linearly independent. Then two of them must share an irreducible minimal component and we find that $\tau \cong \tau \otimes \zeta$. Write τ in the form $\tau = \pi_R(\sigma)$ (Proposition 6.2.7). Then $\tau \otimes \zeta = \pi_R(\sigma \otimes \zeta)$. If τ is not tempered, that is $R \neq G$, then $\sigma = \sigma \otimes \zeta$. This gives $\zeta = 1$, a contradiction. If τ is tempered and equal to ξ then in fact $\tau = \pi$ and again we get a contradiction. So we may assume $\tau \neq \xi$. Then from 6.5, τ is either of the form

$$\tau = \sigma_3 \otimes \mu,\ \tau = I(G, P, \sigma_2 \otimes \mu_1\alpha^{s_1}, \mu_2\alpha^{s_2}) \quad \text{or} \quad \tau = I(G, Q; \mu_1\alpha^{s_1}, \sigma_2 \otimes \mu_2\alpha^{s_2}) ,$$

where μ, μ_1, and μ_2 are characters and s_1 and s_2 real. In any case we have $L(s, \tau \otimes \rho) = L(s, \tau \otimes \rho\zeta)$ for all quasi-characters ρ. Using Theorems 3.4 and 7.11 of [17] to compute both sides in the three cases, we arrive again at a contradiction. For example, if $\tau = \sigma_3 \otimes \mu$, taking $\rho = \mu^{-1}$, we obtain $L(s, \sigma_3) = L(s, \sigma_3 \otimes \zeta)$ from which we get immediately $\zeta = 1$.

The lemma being proved, let us assume that $\pi \cong \pi \otimes \zeta$. Then if we write $\pi = \pi_R(\sigma)$ we shall find by the same argument that $R = G$. Then $\pi = \xi$ and the equality of the ideals has been proved in (3.1).

So we may assume that π and $\pi \otimes \zeta$ are inequivalent. Let G^0 be as before the set of g with $|\det g| = 1$ and K_m the set of $k \in K$, $k \equiv 1 \bmod \mathfrak{P}^m$. We contend then that the linear forms

$$W \longmapsto \int_{K_m} W(\omega h) d\omega\ , \quad (m \geqq 0,\ h \in G^0)$$

on $\xi = \mathcal{W}(\pi, \psi)$ span $\tilde{\xi}$. For if W is annihilated by these forms it vanishes on $G^0 Z$. But then $W(g)(1 + \zeta(g) + \zeta^2(g)) = 0$ for all $g \in G$. Thus by the preceding, $W = 0$. Hence our claim. Hence if

$$f_{m,h}(g) = \int_{K_m} W(\omega h g) d\omega\ ,$$

the functions $f_{m,h}(m \geqq 0,\ h \in G^0)$ span the space of coefficients of ξ. The equality of ideals follows immediately.

We note that by (6.3), $L(s, \pi) = L(s, \xi)$ and hence that (see 4.2), for $r = 3$, $L(s, \pi)$ is also the g.c.d. of the integrals (4.1.1) and (4.1.3), for $W \in \mathcal{W}(\pi; \psi)$.

(7.3) We turn to the functional equation. We want to prove (4.4.1) for the representation ξ on $\mathcal{W}(\pi; \psi)$. In fact this is very simple. As in Section 3, given $W \in \mathcal{W}(\pi; \psi)$ and Φ, choose m large and let, for $d\omega$ normalized,

$$f(g) = \int_{K_m} W(\omega g) d\omega .$$

Then f is a coefficient of ξ. For m large, we have

$$Z(\Phi, s, W) = Z(\Phi, s, f)$$

and

$$Z(\Phi', s, \tilde{W}) = Z(\hat{\Phi}, s, f^{\iota}) .$$

By [17], with

$$\varepsilon'(s, \xi, \psi) = \varepsilon'(s, \pi, \psi) ,$$

we have

$$Z\big(\Phi^{\wedge}, 1 - s + (r - 1)/2, f^{\iota}\big) = \varepsilon'(s, \xi, \psi) Z\big(\Phi, s + (r - 1)/2, f\big) .$$

Referring to (4.4), we thus have (4.5) for any irreducible admissible representation π of GL(3, F). Before stating this formally, we abbreviate our earlier notations.

Given π an irreducible admissible representation of GL(3, F), set for $W \in \mathcal{W}(\pi, \psi)$ and Re(s) large,

(7.3.1) $$\Psi(s, W) = \int W\left[\begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right] |a|^{s-1} d^{\times}a ,$$

(7.3.2) $$\tilde{\Psi}(s, W) = \int \tilde{W}\left[\begin{pmatrix} a & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w'\right] |a|^{s-1} d^{\times}a dx .$$

As before $\tilde{W}$ is defined by (2.1.3). It follows from (6.2.6) and (6.6) that $W \mapsto \tilde{W}$ is a bijection of $\mathcal{W}(\pi; \psi)$ onto $\mathcal{W}(\tilde{\pi}; \psi)$, the latter space affording the representation ξ^{ι}, equivalent to the induced representation $\xi^{\vee}$ attached to $\tilde{\pi}$. Also set, as before, for $\Phi \in \mathcal{S}(F)$,

(7.3.3) $$\Psi(s, W; \Phi) = \int W\left[\begin{pmatrix} a & 0 & 0 \\ x & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\right] \Phi(x) |a|^{s-1} d^{\times}a dx ,$$

and finally

$$\tag{7.3.4} \widetilde{\Psi}(s, W; \Phi) = \int \widetilde{W}\left[\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w'\right] \Phi(x) |a|^{s-1} d^\times a dx .$$

(7.4) With the above definitions, we have:

THEOREM (7.4). *Let π be an irreducible admissible representation of G. Then:*

(7.4.1) *$L(s, \pi)$ is the* g.c.d. *of the integrals $\Psi(s, W)$ and $\Psi(s, W, \Phi)$. $L(s, \tilde{\pi})$ is the* g.c.d. *of the integrals $\widetilde{\Psi}(s, W)$ and $\widetilde{\Psi}(s, W, \Phi)$.*

$$\tag{7.4.2} \widetilde{\Psi}(1 - s, W) = \varepsilon'(s, \pi, \psi)\Psi(s, W) ,$$

$$\tag{7.4.3} \widetilde{\Psi}(1 - s, W; \hat{\Phi}) = \varepsilon'(s, \pi, \psi)\Psi(s, W; \Phi) ,$$

for all W in $\mathcal{W}(\pi; \psi)$ and $\Phi \in \mathcal{S}(F)$.

We also observe that results similar to the ones in (4.6) are true in this extended result.

(7.5) For $r = 3$, we can draw the following consequences.

PROPOSITION (7.5.1). *The map $W \mapsto W|P^1$ is injective.*

Proof. Let $\mathcal{V}$ (resp. $\widetilde{\mathcal{V}}$) be the space of W in $\mathcal{W}(\pi; \psi)$ (resp. $\mathcal{W}(\tilde{\pi}; \psi)$) whose restriction to P^1 vanishes. Clearly $\mathcal{V}$ is P-invariant. On the other hand, if we apply the functional equation (7.4.3) (to $\pi \otimes \chi$ for all χ), we see that W vanishes on all matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ x & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

if and only if $(\xi(w')W)^{\sim}$ does. Now the images of these matrices form a dense subset of $N\backslash P$. Hence

$$W \in \mathcal{V} \Longleftrightarrow (\xi(w')W)^{\sim} \in \widetilde{\mathcal{V}} .$$

Suppose W is in $\mathcal{V}$ and p' is in the group

$$P' = w'P^1(w')^{-1} .$$

Then, setting $W_1 = \xi(p')W$, we see clearly that $(\xi(w')W_1)^{\sim}$ is in $\widetilde{\mathcal{V}}$. Thus $W_1 \in \mathcal{V}$. In other words $\mathcal{V}$ is stable by both P and P'. Since they generate G, the space $\mathcal{V}$ is stable under G and therefore trivial.

Suppose now that π is generic. If $\pi = \pi_R(\sigma)$, then as we have seen (6.6), $\pi = I(G, R; \sigma) = \xi$. Thus π is realized as right translations acting on $\mathcal{W}(\pi; \psi)$. By (7.5.1) we may think of π as acting on a space $\mathcal{V}$ of functions on P^1 transforming under N like θ on the left. Of course P^1 operates by right translations. Also $\mathcal{V}$ contains the space $\mathcal{V}_0$ of all smooth functions on P^1 which transform under N like θ on the left and are of compact support modulo N (2.4).

PROPOSITION (7.5.2). *Let π_1 and π_2 be generic representations of G_3 with the same central quasi-character ω. If for all χ, all s, and some ψ,*

$$\varepsilon'(s, \pi_1 \otimes \chi, \psi) = \varepsilon'(s, \pi_2 \otimes \chi, \psi) ,$$

then π_1 and π_2 are equivalent.

Proof. Let T be the direct sum

$$\mathcal{W}(\pi_1; \psi) \oplus \mathcal{W}(\pi_2; \psi) .$$

Let S be the subspace of those pairs (W_1, W_2) such that $W_1|P^1 = W_2|P^1$ or equivalently $W_1|P = W_2|P$. By the preceding remarks, S contains at least the pairs (W_1, W_2) for which $W_1|P^1$ and $W_2|P^1$ are in $\mathcal{V}_0$ and agree. Thus $S \neq \{0\}$. Denote by $\tilde{S}$ and $\tilde{T}$ the analogous spaces for the representations $\tilde{\pi}_1$ and $\tilde{\pi}_2$. G operates on T (and $\tilde{T}$) by sending (W_1, W_2) to $(\pi_1(g)W_1, \pi_2(g)W_2)$. Again by (7.1.4) applied to $\pi \otimes \chi$, χ arbitrary, we find that

$$(W_1, W_2) \in S \Longleftrightarrow \left(\tilde{\pi}(w')\tilde{W}_1 ,\quad \tilde{\pi}(w')\tilde{W}_2\right) \in \tilde{S} .$$

As before P' and hence G leave S stable. Hence for (W_1, W_2) in S, $W_1 = W_2$. Since G acts irreducibly on $\mathcal{W}(\pi_1; \psi)$ and $\mathcal{W}(\pi_2; \psi)$, these two spaces, having an element in common, must coincide.

Using this we have

LEMMA (7.5.3). *Let π_1 and π_2 be irreducible admissible representations of G_3 with the same central quasi-character. If for all χ, all s, and some ψ, $\varepsilon(s, \pi_1 \otimes \chi, \psi) = \varepsilon(s, \pi_2 \otimes \chi, \psi)$ and $L(s, \pi_1 \otimes \chi) = L(s, \pi_2 \otimes \chi)$, then π_1 and π_2 are equivalent.*

Proof. Write $\pi_i = \pi_{R_i}(\sigma_i)$ according to Proposition (6.2.7). Let $\xi_i = I(G, R_i; \sigma_i)$ and let τ_i be the unique minimal component of ξ_i. Then

$$\varepsilon'(s, \tau_1 \otimes \chi, \psi) = \varepsilon'(s, \tau_2 \otimes \chi; \psi) .$$

Since τ_1 and τ_2 are generic, by (7.5.2), τ_1 and τ_2 are equivalent. One argues then as in (6.5).

Remark (7.5.4). Proposition (7.5.1) is valid for all $r > 2$ (Bernstein [2]). One can give a proof similar to the one given here which involves generic representations of $\mathrm{GL}_{r-2}(F)$ [29].

On the contrary (7.5.2) is true for $r = 2$, 3 but false for $r > 4$. Let us give a counterexample for $r = 4$. Let σ be a unitary supercuspidal representation of G_2. Set $\sigma_t = \sigma \otimes \alpha^t$ where t is purely imaginary. Set

$$\pi_t = I(G_4, Q; \sigma_t, \sigma_{-t}) ,$$

where Q is the standard parabolic subgroup of G_4 of type (2, 2). Each π_t is irreducible generic. We have

$$L(s, \pi_t \otimes \chi) = L(s, \alpha_t \otimes \chi)\, L(s, \alpha_{-t} \otimes \chi) = 1 ,$$
$$\varepsilon(s, \pi_t \otimes \chi, \psi) = \varepsilon(s, \sigma_t \otimes \chi, \psi)\, \varepsilon(s, \sigma_{-t} \otimes \chi, \psi) .$$

But there is an n (dependent on χ) so that $\varepsilon(s, \sigma_t \otimes \chi, \psi) = q^{-nt}\varepsilon(s, \sigma \otimes \chi, \psi)$. Thus $\varepsilon(s, \pi_t \otimes \chi, \psi) = \varepsilon(s, \sigma \otimes \chi, \psi)^2$. Hence all the π_t have the same central quasi-character and the same ε'-factor. However $\pi_t = \pi'_t$ implies [2] that σ_t is equivalent to either σ'_t or σ'_{-t}. This in turn implies $\alpha^{2(t-t')}$ or $\alpha^{2(t+t')} = 1$. Hence a non-countable infinity of the π_t are inequivalent and we have a counterexample to (7.5.2).

Another property special to $r = 2$ or 3 is the following:

PROPOSITION (7.5.5). *Suppose $r = 2$ or 3 and π is an irreducible, admissible representation of $G_r(F)$ such that $\varepsilon'(s, \pi \otimes \chi, \psi)$ is a monomial for all χ. Then π is supercuspidal.*

For if π is not, it is a component of an induced representation. The ε'-factor can thus be computed and is found not to be a monomial for all χ.

(7.6) Finally we shall also require, for the global theory, the following technical lemma:

LEMMA (7.6). *Let π be an irreducible, admissible representation of G_3, of central quasi-character ω.*

Suppose ψ has exponent zero. For $a \geqq 1$, let K^a be the subgroup of h in GL(3, $\mathfrak{R}$) *such that:*

$$h_{12} \in \mathfrak{P},\ h_{31} \in \mathfrak{P}^a,\ h_{32} \in \mathfrak{P}^{a+1} .$$

Then there is an $a \geqq 1$ and a W in $\mathcal{W}(\pi; \psi)$ such that

(1) $W(gh) = \omega(h_{33})W(g)$ *for* $g \in G_3$, $h \in K^a$,

(2) $W(e) = 1$,

(3) $\int_{\mathfrak{P}^{-1}} W\left[g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right] dx = 0$ *for all* $g \in G_3$.

Proof. By (2.4), there is a W in $\mathcal{W}(\pi; \psi)$ such that

$$W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] = \psi(x) , \quad \text{if} \quad g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} ,$$

with

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathrm{GL}(2, \mathfrak{R}),\ q \in \mathfrak{P}\ ;$$

$$W\left[\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\right] = 0 \qquad \text{otherwise}\ .$$

Then W satisfies (1) if a is so large that W is invariant under K_{a+1}. In fact, by (7.5.1), it is enough to check (1) for g of the form $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ with $x \in G_2$. Then since W is K_{a+1}-invariant we may assume $h \in K^a \cap P^1$. In that case since the exponent of ψ is zero our assertion is clear. To prove (3), one may assume again by (7.5.1) that

$$g = \begin{pmatrix} p & q & 0 \\ r & s & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the integral is just

$$\int_{\mathfrak{P}^{-1}} \psi(sx)dx \cdot W(g)\ .$$

Since $W(g) = 0$ unless $s \in \mathfrak{R}^\times$, our assertion is clear.

(To be continued in the next issue)