1. Find the eigenvalues and the corresponding eigenspaces for the following matrices

(a) \[
\begin{pmatrix}
3 & 4 \\
0 & -1
\end{pmatrix}
\]

(b) \[
\begin{pmatrix}
1 & 2 & 1 \\
0 & -1 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

2. Let $A$ be a nonsingular matrix and let $\lambda$ be an eigenvalue of $A$. Show that $1/\lambda$ is an eigenvalue of $A^{-1}$.

3. Let $\lambda$ be an eigenvalue of $A$ and let $x$ be an eigenvector belonging to $\lambda$. Use mathematical induction to show that $\lambda^m$ is an eigenvalue of $A^m$ and $x$ is an eigenvector of $A^m$ belonging to $\lambda^m$ for $m = 1, 2, \ldots$.

4. An $n \times n$ matrix $A$ is called nilpotent if $A^k = O$ for some positive integer $k$. Show that all eigenvalues of a nilpotent matrix are 0.

5. Let $A$ be an $n \times n$ matrix and let $\lambda$ be an eigenvalue of $A$. If $A - \lambda I$ has rank $k$, what is the dimension of the eigenspace corresponding to $\lambda$?

6. Let $A$ be a matrix whose columns all add up to a fixed constant $\delta$. Show that $\delta$ is an eigenvalue of $A$.

7. Verify which of the following sets form an orthonormal basis for $\mathbb{R}^3$.

   $A = \{(0, 1, 0)^T, (1, 0, 0)^T, (0, 0, 1)^T\}$, $B = \{(\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2})^T, (-1, 0, 0)^T\}$

   $C = \{(1, 1, 1)^T, (1, -1, 1)^T, (0, 0, 1)^T\}$

8. Show that the functions $\{\sin(mx), m = 1, 2, \ldots, -1, -2, \ldots\}$ form an orthonormal set of vectors in $C[-\pi, \pi]$ with respect to the inner product

   $< f, g > = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) \, dx$

9. Use the result of the previous exercise to calculate the integral

   $\int_{-\pi}^{\pi} \sin^6 x \, dx$

   (Hint: use Parseval’s formula)
10. Find an orthonormal basis for $P^3$, the space of polynomial of degree less than three, and find the best least squares approximation of the function $e^x$ on the interval $[0, 1]$ by elements of $P^3$ with respect to the inner product

$$< f, g > = \int_0^1 f(x) g(x) \, dx$$

Plot the function $e^x$ and the approximating polynomial.