1. If $A$ is an $m \times n$ matrix of rank $r$, what are the dimensions of $N(A)$ and $N(A^T)$? Explain.

2. Let $A$ be an $m \times n$ matrix of rank $r$ and let \( \{x_1, x_2, \ldots, x_r\} \) be a basis for $R(A^T)$. Show that \( \{Ax_1, Ax_2, \ldots, Ax_r\} \) is a basis for $R(A)$.

3. Find the least squares solution to the following systems.
   
   (a) \[
   \begin{align*}
   -x - y &= 1 \\
   2x + y &= 3 \\
   x + 3y &= 0
   \end{align*}
   \]
   
   (b) \[
   \begin{align*}
   -x + 2y &= 1 \\
   x - y &= 2 \\
   3x + y &= -1
   \end{align*}
   \]

4. For each of the solutions in the previous exercise:
   (a) Determine the projection $p = A\hat{x}$.
   (b) Calculate the residual $r(\hat{x})$.
   (c) Verify that $r(\hat{x}) \in N(A^T)$.

5. (a) Find the least squares fit by a linear function to the data
   \[
   \begin{array}{c|ccccc}
   x & -2 & -1 & 0 & 1 \\
   y & 0 & 1 & 3 & 9 \\
   \end{array}
   \]
   (b) Plot your linear function from part (a) along with the data on a coordinate system.

6. Find the best least squares fit to the data in the previous exercise by a quadratic polynomial. Plot the data points and sketch the graph of the function.

7. Let $x = (1, 2, -1, 3, 1)^T$ and $y = (0, -1, 1, 1, 0)^T$. Show that $x \perp y$. Calculate $\|x\|_2$, $\|y\|_2$, and $\|x + y\|_2$ and verify that the Pythagorean Law holds.

8. Show that $\|x\|_1 = \sum_{i=1}^{n}|x_i|$ defines a norm on $\mathbb{R}^n$.

9. Show that $\|x\|_\infty = \max_{1 \leq i \leq n}|x_i|$ defines a norm on $\mathbb{R}^n$. 
10. Let $x = (3, 2, 1)^T$ and $y = (1, -1, 0)^T$. Compute $\| x - y \|_1$, $\| x - y \|_2$, and $\| x - y \|_\infty$. Under which norm are the two vectors closest together? Under which norm are they farthest apart? Give an example of a non-zero vector $v$ in $\mathbb{R}^3$ for which $\| v \|_1 = \| v \|_2 = \| v \|_\infty$. 