1. Let $S$ be the set of all ordered pairs of real numbers. Define scalar multiplication and addition on $S$ by

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_1, 0)$$

We use the symbol $\oplus$ to denote the addition operation of this system to avoid confusion with the usual addition $x + y$ of row vectors. Show that $S$, with the operations of addition and scalar multiplication defined as above is not a vector space. Which of the eight axioms fail to hold?

2. Determine whether the set $\{(x_1, x_2)^T \mid x_1x_2 = 0\}$ is a subspace of $R^2$.

3. Determine whether the following are subspaces of $P_4$.
   
   (a) The set of polynomials in $P_4$ of even degree.
   
   (b) The set of all polynomials of degree 3.
   
   (c) The set of all polynomials $p(x) \in P_4$ such that $p(0) = 0$.
   
   (d) The set of all polynomials in $P_4$ having at least one real root.

4. Find the null space of the matrix

$$A = \begin{pmatrix} 1 & 3 & -4 \\ 2 & -1 & -1 \\ -1 & -3 & 4 \end{pmatrix}$$

5. Let $A$ be an $n \times n$ matrix. Prove that the following statements are equivalent.

   (a) $N(A) = \{0\}$
   
   (b) $A$ is nonsingular
   
   (c) For each $b \in R^n$, the system $Ax = b$ has a unique solution.

6. Let $U$ and $V$ be subspaces of a vector space $W$. Define

$$U + V = \{z \mid z = u + v \text{ where } u \in U \text{ and } v \in V\}$$

Show that $U + V$ is a subspace of $W$.

7. Determine whether the vectors $v_1 = (2, 1, -2)^T$, $v_2 = (-2, -1, 2)^T$, $v_3 = (4, 2, -4)$ are linearly independent. Describe geometrically the span of these vectors.

8. Determine whether the vectors $x^2$, $x + 2$, and $x + 1$ are linearly independent in $P_3$.

9. Prove that any finite set of vectors that containing the zero vector must be linearly dependent.