Midterm Review
Linear Algebra

At least one of the following theorems will appear on the midterm

**Theorem 1** A system \( Ax = b \) of \( n \) linear equations in \( n \) unknowns has a unique solution if and only if the matrix \( A \) is nonsingular.

**Theorem 2** If \( A \) is a \( k \times k \) triangular matrix, then the determinant of \( A \) is the product of the diagonal entries.

**Theorem 3** The nullspace \( N(A) \) of a matrix \( A \) is a subspace.

**Theorem 4** Let \( V \) be a vector space and \( v_1, v_2, \ldots, v_n \in V \). Then a vector \( v \in \text{Span}\{v_1, v_2, \ldots, v_n\} \) can be written uniquely as a linear combination of the vectors \( v_1, v_2, \ldots, v_n \) if and only if \( v_1, v_2, \ldots, v_n \) are linearly independent.

Also, there will be a section of True/False questions. The best preparation for these is to work through the T/F chapter tests in the Leon book. Because not everyone has this book I am handing out copies of these questions in class. Let me know if you did not get one (or borrow the book from the math library).

Here’s a brief outline of the techniques and concepts we have covered so far

**Matrix algebra**
1. Denoting entries of sums and products of matrices. Also, solving matrix equations and proving properties of matrices by equating entries.
2. Different types of matrices: transpose of a matrix, symmetric matrix, upper and lower triangular matrices, diagonal matrices
3. Block form of a matrix

**Principle of mathematical induction, proofs by induction**

**Systems of equations**
1. Solutions without linear algebra (by manipulating the equation itself).
2. Using an augmented matrix and elementary row operations to solve a system of equations \( Ax = b \).
3. Using the inverse of a matrix to solve a system of equations
4. When is a system of equations consistent (for a system \( Ax = b \) a solution exists in \( b \) is a linear combination of the column vectors of \( A \)), when does it have a unique solutions, when are there infinitely many solutions.
5. Theorem: For an \( n \times n \) matrix \( A \) TFAE: a) \( A \) is nonsingular, b) \( Ax = 0 \) has a unique solution, c) \( A \) is row equivalent to \( I \).
6. Cramer’s rule

**Elementary matrices**
Inverse of a matrix
1. Definition of the inverse and how to show that a particular matrix is the inverse, when does an inverse exist.
2. How to find the inverse using row operations.
3. Using the adjoint of a matrix to calculate the inverse

Determinants
1. The definition of the determinant as the cofactor expansion along any row or column.
2. Determinant of $A^{-1}$, $A^T$, and the determinant of triangular matrices
3. How row operations effect the determinant
4. Calculating the determinant using row operations
5. An $n \times n$ matrix is singular if and only if its determinant is 0.
6. $\det(AB) = \det(A) \det(B)$

Vector spaces, subspaces
1. Definition of a vector space and closure properties (although you do not have to remember the eight axioms for the exam)
2. Definition of a subspace (closure properties)
3. The nullspace of a matrix

Span, linear independence
1. Definition of span and linear independence
2. Using the determinant to show that $n$ vectors in $\mathbb{R}^m$ are linearly independent or dependent.
3. An $n \times n$ matrix is nonsingular if and only if its column vectors are linearly independent.
4. Applications to differential equations, the Wronskian

Basis and dimension
1. Definition of a basis, how can you show that a set of vectors is a basis
2. Theorems regarding dimension (for example: if $\{v_1, ..., v_n\}$ span $V$ then in $m > n$ and $\{u_1, ..., u_m\}$ is any other set of vectors, then they are linearly dependent)

Finding the transition matrix between two sets of basis vectors