GALOIS REPRESENTATIONS AND TORSION COHOMOLOGY: A SERIES OF MISUNDERSTANDINGS

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1. Introduction

In November 2020, in the depths of the first year of the COVID-19 pandemic, when my life was still largely confined to a 2-mile radius around my Manhattan apartment and I was still rolling my shopping cart a mile each way to the supermarket, avoiding public transportation whenever possible, Chi-Yun Hsu wrote to invite me to give a virtual UCLA seminar talk, following up on an earlier invitation by Shekhar Khare but with a surprising request: it was to be a talk "under the style 'My Life in Mathematics.'" For example, it could be an account of

> *... a theorem* [I] *proved or a theory* [I] *developed, explained from a personal and historical perspective, like how* [I] *came up with the problem, how the ideas came into place, or what the theorem/theory meant to* [me], etc.

The other senior mathematicians who were invited to gives talks in what the organizers eventually called the CHAT series – for **Career, History, and Thoughts** – had such compelling success stories to tell that I felt free to experiment with the format by reporting on my own most compelling failure. I hinted at my intentions in the title:

Galois representations and torsion cohomology:
a series of misunderstandings

and made the stakes clear in the abstract:

> *In 2013, Peter Scholze announced his proof that Galois representations with finite coefficients could be associated to torsion classes in the cohomology of certain locally symmetric spaces. The existence of such a correspondence had been predicted by a number of mathematicians but for a long time no one had the slightest idea how to construct the Galois representations. In this talk I will review some of the history of the problem, with emphasis on the many false starts and occasional successes, and on my own intermittent involvement with this and related problems.*

2. First steps: 1988

2.0.1. *Locally symmetric spaces.* We will look at $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ or $Gal(\overline{\mathbb{Q}}/K)$, where $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field.

But I could just as well work with $Gal(\overline{\mathbb{Q}}/K)$ where K is either a *totally real number field* or a *CM field* (totally imaginary quadratic extension of a totally real field). Complete understanding of the topic of the title is only possible in this generality.

2.0.2. *Locally symmetric spaces.* We consider the locally symmetric spaces attached to $GL(n)$ over $\mathbb{Q}$ or K:

$$X_\Gamma := \Gamma\backslash GL(n, K \otimes \mathbb{R})/K_\infty \cdot Z(\mathbb{R})$$

where $\Gamma \subset GL(n, \mathcal{O}_K)$ is a (congruence) subgroup of finite index, $Z(\mathbb{R})$ is the diagonal subgroup of $GL(n, K \otimes \mathbb{R})$, and K_∞ is a maximal compact subgroup: either $SO(n)$ (over $\mathbb{Q}$) or $U(n)$ (over K).

We actually have to consider all Γ simultaneously (the adelic locally symmetric space) but I don't want to write the definition; so I just write the notation: $X_{n,\mathbb{Q}}$ or $X_{n,K}$.

2.0.3. *A theorem about cohomology.* This space has cohomology and we start with

$$H^*_!(X_{n,\mathbb{Q}}, \mathbb{Q}) := Im[H^*_c(X_{n,\mathbb{Q}}, \mathbb{Q}) \to H^*(X_{n,\mathbb{Q}}, \mathbb{Q})]$$

(likewise with $X_{n,K}$).

This space is (in some sense) finite-dimensional (depending on Γ, in some sense) and has a large commuting $\mathbb{Q}$-algebra $\mathbb{T}$ of operators. A version of *Matsushima's formula* (adapted to non-compact locally symmetric spaces) expresses $H^*_!$ in terms of automorphic forms on the group $GL(n)$.

Theorem 2.1. *Let $\alpha : \mathbb{T} \to \mathbb{Q}$ be a homomorphism. Then for any prime number ℓ, there is an n-dimensional representation*

$$\rho_{\alpha,\ell} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(n, \mathbb{Q}_\ell) \ (\rho_{\alpha,\ell} : Gal(\overline{\mathbb{Q}}/K) \to GL(n, \mathbb{Q}_\ell))$$

unramified outside a finite set of primes S, such that, for all $p \notin S$ the characteristic polynomial of Frobenius at p is determined by the values of α on the Hecke operators in $\mathbb{T}$ corresponding to p.

The theorem has a long history. The case $n = 1$, for any number field, is just class field theory – the special case of base field $\mathbb{Q}$ is Kronecker's theorem on abelian extensions of $\mathbb{Q}$. The proof for general n evolved over about 60 years, with increasingly general hypotheses, until it reached this form just about 10 years ago.

When $n = 2$ this was proved (for $\mathbb{Q}$) by Eichler and Shimura, and this is the starting point for the proof of Fermat's Last Theorem.

When $n = 3$ this is a UCLA theorem – under the duality hypothesis of the next theorem – and is essentially due to Blasius and Rogawski. It was worked out in 1988 in Montreal and the proof is the subject of the book *The zeta functions of Picard modular surfaces.* The proof was based on

2.1. The stable trace formula.

Laurent Clozel

2.1.1. *Clozel's theorem announced in Ann Arbor, 1988.*

Theorem 2.2 (Clozel)**.** *Suppose α is contained in the self-dual part of the cohomology. (The space $H_!^*(X_{n,\mathbb{Q}}, \mathbb{Q})$ satisfies Poincaré duality and we consider a $\mathbb{T}$-eigenspace that is its own dual; or complex-conjugate to its dual for K.) Suppose S is not empty (and some additional hypotheses that were eventually relaxed). Then the $\rho_{\alpha,\ell}$ exist.*

Remark 2.2. We can take $\alpha : \mathbb{T} \to E$ for any number field E, and the theorem is still valid.

Here is a quick summary of the proof:

(A) Clozel used the *stable trace formula*, as well as to relate self-dual α to the cohomology of a *Shimura variety* $S_{n,K}$ obtained as an arithmetic quotient of the unit ball in $\mathbb{C}^{n-1}$, with a *canonical model* over the (imaginary quadratic field) K. Thus its topological cohomology can be related to its ℓ-adic cohomology, which has a Galois action that commutes with appropriate Hecke operators.

(B) The $Gal(\overline{\mathbb{Q}}/K)$-action on this cohomology had just been determined by Kottwitz (also announced in Ann Arbor).

(C) Q.E.D.

2.2.1. *The idea almost announced in Ann Arbor.* This theorem came as a great surprise to me, because at the time the case $n = 3$ was still being written up.

I spoke to Clozel after his talk and he explained that in his next talk he was going to announce his plans to solve the remaining (not self-dual) case.

The space $X_{n,K}$ is part of the (Borel-Serre) boundary of a Shimura variety $Y_{2n,K}$ related to the Lie group $U(n, n)$ (unitary group in $2n$-variables over K, with signature (n, n)).

The α-eigenspace of the cohomology of $X_{n,K}$ could then be realized as an eigenspace in the cohomology of $Y_{2n,K}$.

But this eigenspace (over $\mathbb{Q}_\ell$) also has a Galois action. What could it be if not the one predicted by the Langlands conjectures?

2.2.2. *Anti-digression: On the significance of the self-dual condition.* The distinction between the self-dual and non-self-dual parts of the cohomology, and the corresponding Galois representations, representations may seem [frivolous] in this context but it actually expresses a basic structural feature of the theory of automorphic forms. In view of the Langlands correspondence, this feature has deep implications for Galois theory. The cohomology of Shimura varieties with ℓ-adic coefficients can be related to their étale cohomology, and thus carry Galois representations. Clozel's theorem built on conjectures and results of Langlands and Kottwitz and confirmed that the Galois representations attached to the self-dual part of $H^*_!(X_{n,\mathbb{Q}}, \mathbb{Q})$ are (mostly) directly realized in the cohomology of appropriate Shimura varieties. In particular, they automatically have all the properties expected of the Galois representations on étale cohomology of algebraic varieties, notably *purity* (the generalized Ramanujan conjecture) and Fontaine's de Rham property at the restriction to a decomposition group over p.

In contrast, at the time of Clozel's Ann Arbor talks no conjectures claimed that the Galois representations attached to the non-self-dual part of the cohomology of $H^*_!(X_{n,\mathbb{Q}}, \mathbb{Q})$ could be realized in the étale cohomology of any specific algebraic variety. It was known that the non-self-dual part of the cohomology is *sparser*, in a precise sense, than the self-dual part; fewer techniques are available to construct such classes explicitly. For this reason, although Clozel's construction in the self-dual case was the culmination of more than a decade of hard work by a number of people, treating the non-self-dual case was already seen as a greater challenge.

I call this section an *anti-digression* because it introduces the principal subject of this talk. The account of the self-dual case, though it will continue through §3.1, is in fact a digression from the talk's main theme; the discussion of the non-self-dual case here serves as a transition to the case of torsion coefficients.

2.2.3. *Right cohomology, wrong Galois action.* Here is where my role in the story begins, primarily as an engaged spectator. I had been thinking about the boundary cohomology of Shimura varieties in terms of the *toroidal compactification*. The boundary cohomology attached to $X_{n,K}$ all comes from *rational varieties*. I explained to Clozel that Galois action looked too simple to be the one predicted by Langlands.

I worked this all out in detail over the next few years with Zucker – starting in Ann Arbor, in fact. In particular, we showed that boundary cohomology had a *weight filtration* corresponding to mixed Hodge theory.

Meanwhile, back in Ann Arbor, Clozel checked with Kottwitz, who agreed with my diagnosis. So Clozel took the theorem for non self-dual representations off the agenda for the time being.

Steve Zucker

2.3. **Digression: why was I thinking about boundary cohomology?** Beilinson had formulated his conjectures on K-theoretic regulators and special (non-critical) values of L-functions in a paper published in 1980. He had illustrated his conjectures by constructing explicit classes in the motivic cohomology of some low-dimensional Shimura varieties and related the classes to integral expressions for L-functions. Over the next few years a good deal of energy was devoted to these conjectures; the results were summarized in Ramakrishnan's talk in Ann Arbor. All of these results made use at some point of the motivic classes that had appeared in Beilinson's first papers, whose automorphic counterparts were special values of Eisenstein series on modular curves.

At the same time, Harder was developing the foundations of Eisenstein cohomology, and he hoped to use his theory to construct new examples of extensions of motives – or more precisely, of their cohomological realizations – in the cohomology of Shimura varieties. I saw Harder's program as an opportunity to apply my recent work on coherent cohomology to construct motivic extension classes that went beyond Beilinson's constructions on modular curves. My first paper with Zucker aimed to construct Eisenstein classes in coherent cohomology, and this required an extended analysis of the classes with non-trivial restriction to the toroidal boundary.

One of the main results of my first paper with Zucker was the construction of non-trivial rational classes in higher coherent cohomology attached to Eisenstein series in some generality. However, as far as I know, it is still the case that every special instance of Beilinson's conjectures for Shimura varieties still depends on Beilinson's original constructions on modular curves. A few years after the Ann Arbor conference I spent two weeks with Ramakrishnan at Caltech, attempting unsuccessfully to work out an example of Beilinson's conjectures for Siegel modular threefolds. Years later Francesco Lemma revived and extended our project: with his collaborators he has constructed motivic cohomology classes for higher-dimensional Siegel modular varieties – always starting with Beilinson's Eisenstein classes; in some low-dimensional cases these have been related to Euler systems and special values of L-functions.

Only two of my papers – one with Scholl and one with Esnault, both quite short – have anything to do with Beilinson's conjectures. They make no reference to the classes I constructed with Zucker, which have never been applied to any question in number theory. I would be delighted to learn that they are good for something.

Meanwhile, Beilinson's conjectures as well as my earlier work on Shimura varieties had sensitized me to the interplay between the rational structure on de Rham cohomology of varieties over number fields and the Hodge structure of their cohomology. A few simple calculations of boundary cohomology led me to suspect that the Hodge structures expected to be related to Eisenstein series decomposed as tensor products of a purely topological part, whose Hodge structures were all of Tate type, and an "arithmetic" part, corresponding to the Hodge structure of the Shimura variety that appeared in the minimal (Baily-Borel-Satake) compactification.[1] One day Zucker and I took refuge in a bar from a freak Michigan heat wave and began to work on our second paper,[2] which eventually proved a precise version of this suspicion. I had no way to compute the Galois representation on the boundary cohomology but I could determine their Hodge structures in low dimension. The motivic perspective indicated that the corresponding Galois representations should also be of Tate type – this was completely settled some years later by Sophie Morel – and this is what I told Clozel.

2.4. **The 1990s.**

2.4.1. *Collaboration with Taylor, $n = 2$.* I met Richard Taylor for the first time in Ann Arbor. A few years later, Soudry and I provided the missing piece in his project to prove the theorem for non-self-dual representations of $GL(2)$ over K, using the theta correspondence; this was my first collaboration with Taylor.

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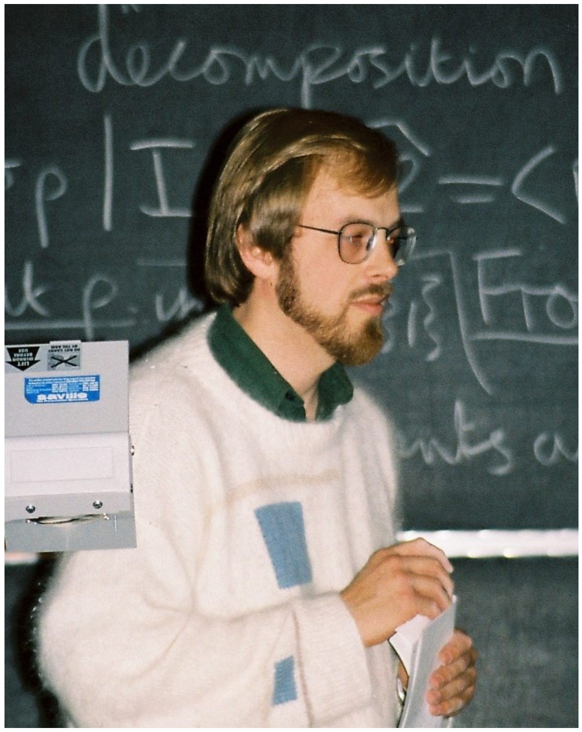
[1]Although I didn't know it in Ann Arbor, Harder's student Richard Pink, working with ℓ-adic rather than algebraic de Rham cohomology, was coming to a similar conclusion at the very same time.

[2]The second paper was completed after the first paper but we began working on them in the opposite order.



Richard Taylor 8 years after Ann Arbor

(Our work was generalized by Chung-Pang Mok 20 years later to $GL(2)$ over any CM field.)

2.4.2. *Collaboration with Taylor, $n > 2$.* A few years after our paper with Soudry, Taylor and I refined Clozel's theorem using p-adic uniformization of Shimura varieties (at the primes in S), obtaining the local Langlands correspondence as a corollary.

The result was slowly refined over the next ten years, with important contributions by Taylor-Yoshida, Labesse, Clozel, Shin, Chenevier, and Caraiani, leading to the removal of successive ramification conditions.

(Removal of the final ramification conditions will be mentioned a few pages from now.)

2.5. **Skinner suggests using congruences.** I met Chris Skinner in 2000. At some point after we met, probably in 2002, he returned to Clozel's idea. He had written a paper with Eric Urban on eigenvarieties, and he suggested that the boundary cohomology of $Y_{2n,K}$, coming from $X_{n,K}$, could be deformed p-adically to classes in the interior of $Y_{2n,K}$.

Chris Skinner, undated photo

These classes had the right kind of $2n$-*dimensional* Galois representation to be split up into two n-dimensional representations, one of which was the one predicted by the Langlands correspondence (he had checked).

I filed this away in my mind: Chris Skinner, possibly in collaboration with Urban, was going to use eigenvarieties to prove the non self-dual case of the theorem I stated.

I was happy for them although I knew it would take a long time, because they were busy writing up their proof of the main conjecture for the p-adic L-functions of elliptic curves, using (of course) eigenvarieties.

3. Torsion classes

3.1. **The Montreal conference, September 2005.** In the fall of 2005 I flew to Montreal for a conference, organized by Darmon and Iovita, billed as a Workshop on p-adic representations. This was the conference that introduced Colmez's work on the p-adic representation theory of the group $GL(2,\mathbb{Q}_p)$ (the so-called *Montreal functor*). Nearly all the talks were about p-adic modular forms, attempts to construct a p-adic Langlands correspondence, or both.

My talk was practically the sole exception. It was my first trip to Montreal since the 1988 program that included the Blasius-Rogawski work on

The Oratoire Saint-Joseph, in Montréal

3-dimensional Galois representations. I arrived at the conference in Montreal with the aim of building interest in the **Paris Book Project** (after climbing the Oratoire stairs on my knees).

Laumon and Ngô had proved the Fundamental Lemma for unitary groups; soon Ngô would prove it in general. The Book Project would work to remove the ramification conditions that persisted in my book with Taylor, using the full stable trace formula, thus obtaining the generalization of the Blasius-Rogawski theorem for all n. Here is the beginning of my notes for this talk.

On Wikipedia I found the following explanation of the term "infomercial:"

Definition 3.2 (Informercials)**.** *Infomercials are television commercials that run as long as a typical television program (roughly thirty minutes or an hour), ... often made to closely resemble actual television programming.*

An infomercial is designed to solicit a direct response which is specific and quantifiable. The delivery of the response is direct between the viewer and the advertiser.

My presentation, which will run as long as a typical lecture, is meant to resemble closely an actual lecture, but is in fact a *commercial* for the Book Project of the automorphic forms group in Paris, and is designed to solicit a direct, specific response from the viewer. The Book Project, which we are hoping to publish with International Press, is itself a response to a specific situation. I last came to Montreal in 1988, when Langlands organized a special program at the CRM on Picard Modular Surfaces, a family of two-dimensional Shimura varieties whose cohomology was used by Blasius and Rogawski to construct a large class of three-dimensional compatible systems of ℓ-adic Galois representations, as well as certain two-dimensional representations that apparently could not be constructed otherwise. The occasion for this conference was Rogawski's work on the stable trace formula for the group $U(3)$, which explains the presence of three-dimensional representations. At the time it was generally believed that the results could be extended without much difficulty to n-dimensional representations, for any n, if one assumed a certain case of the so-called Fundamental Lemma. That lemma has now been proved by Laumon and Ngô. The first purpose of the Book Project is to carry out the program of the 1988 Montreal conference for construction of the largest possible class of n-dimensional Galois representations, for all n, that can be obtained by means of Shimura varieties. The second purpose – and here is where I, as the advertiser, am hoping for a direct response from the viewer – is to make this material as accessible as possible to number theorists who are familiar with the arithmetic of modular curves but uncomfortable with Shimura varieties and automorphic forms in higher dimensions. In the setting of the present conference, I would hope that this Book Project will contribute to the development of p-adic representation theory for groups other than $GL(2)$, as well as to the theory of p-adic families of automorphic forms of higher dimension, which is curiously more advanced than the local theory.

The week's exposure to p-adic methods must nevertheless have burned itself into my unconscious thinking. In the middle of my talk I surprised myself by blurting out that Chenevier's work with Bellaïche on eigenvarieties would provide the final step, although I only realized several weeks later that this is what I had done. After Clozel, Labesse, Ngô, and I completed the first volume of the Book Project, Chenevier and I wrote a paper applying eigenvarieties to the construction of the missing even-dimensional p-adic Galois representations. This was in turn based on the paper Chenevier published a few years later in the second volume of the Book Project. Fintzen and Shin have more recently found a way to complete the construction, based on congruences of types, rather than on eigenvarieties.

3.2.1. *Breakfast in Montreal, 2005.* However, I was still on Paris time, and I came down very early for breakfast every day. Barry Mazur was staying at the same hotel and he is an early riser. So over breakfast, he explained to me something that was on his mind. He had long been convinced that most of the Galois representations attached to the cohomology of $GL(n, \mathbb{Q})$ for $n > 2$, or $GL(n, K)$ for $n \geq 2$, came from *torsion* classes. His recent paper with Calegari had confirmed this in the first non-trivial case.

The trace formula methods of the Paris book project knew nothing about torsion cohomology. But Mazur was convinced that there had to be Galois representations attached to torsion classes. I later learned that Serre and Taylor, and Ash and Stevens, among others, had also come to this conclusion. I filed this away in my mind as a mystery.

I assumed Skinner and Urban would eventually work out the non self-dual case. But the idea that Galois representations could be attached to torsion classes seemed to me so far-fetched that I could not get it out of my mind.

4. First misunderstandings

4.1. **A uniquely satisfying idea.** At some point in 2006, I realized that my work with Zucker showed that torsion classes in $H^*(X_{n,K})$ could also be realized in $H^*(Y_{2n,K})$.

This was the first and (to all intents and purposes) the last idea I had in connection with this question. But it was uniquely satisfying for three reasons.

Barry Mazur and friend

- It suggested an unexpected application of my work with Zucker.
- My work on the Sato-Tate conjecture (with Clozel, Shepherd-Barron, and Taylor) had just come to a successful conclusion, and I was looking for a new problem.
- It reminded me of my pleasant breakfast conversations with Barry Mazur in Montreal. My main motivation in giving the present talk is to stress how important that experience was for everything that followed. Such considerations are unfortunately rarely preserved in the published record.

4.2. **Weights.** The idea, then, was to apply Skinner's suggestion to deform *torsion* cohomology classes of $X_{n,K}$ p-adically to interior classes of $Y_{2n,K}$. For large p we could even hope to use the mixed Hodge weights to lift torsion boundary classes to the cohomology of the Shimura variety (as in my work with Zucker).

I talked about it mainly with Skinner, but also with Urban, Calegari, and Emerton. I mentioned it to Taylor in passing. I hoped that, even at the cost of imposing highly restrictive hypotheses, we could construct *at least one* Galois representation attached to a torsion class.

4.3. **My idea will not work.** In the spring of 2007 I flew from Paris to work with Skinner in Princeton. On the second day Urban came down from New York to join us, and to explain why the idea would not work. In fact, Skinner's original idea, to deform characteristic zero cohomology classes, could not work with the known constructions of eigenvarieties (Euler characteristics are constant).

Eric Urban, undated photo

Frank Calegari and Matt Emerton, 2016, probably in Chicago

Urban proposed a more complicated construction, based on his ongoing work with Skinner. I returned to Paris fully discouraged and ready to forget about the project.

4.4. **Completed cohomology.** In 2009 I visited Calegari and Emerton in Chicago. They wanted to talk about a completely different approach, for $GL(4)$, using their conjectures on completed cohomology. Assuming their conjectures and also an extension of the Arthur conjectures to torsion classes, Galois representations could be attached to some torsion classes for $GL(4)$.

The Institute Woods, photo by Mark Goresky

This involved a specific classification problem that was not at all obvious but that was at least concrete.

It looked more hopeful at this point than the idea with Skinner and Urban. But both ideas were extremely technical and could only be applied in low dimension.

5. Finding the right framework

5.1. **Constructions in characteristic zero.**

5.1.1. *A walk in the woods.* By the time I visited the IAS in the winter of 2011 I was giving little thought to the torsion question, and none at all to the non self-dual representations in characteristic zero.

But a few days after I arrived, Richard Taylor invited me on a walk through the frozen Institute Woods.

5.1.2. *p-adic modular forms.* Few methods were available at the time for deforming topological cohomology. But the theory of overconvergent p-adic modular forms was well understood.

Taylor's idea was to use the control provided by the overconvergent theory to compute the global p-adic de Rham cohomology $H^*_{dR}(Y_{2n,K})$ on the ordinary locus. The latter is (close to) affine, so its cohomology can be computed by global sections of the de Rham complex. This gives a complex of p-adic Banach spaces but the overconvergent theory provides enough control.

The Hecke eigenvectors of $H^*_{dR}(Y_{2n,K})$ could then be approximated p-adically by eigenspaces on holomorphic cusp forms of various weights. Galois representations are attached to the latter, and because they are cuspidal these have the right properties.

5.1.3. *The collaboration with Lan, Taylor, and Thorne.* Lan and Thorne were also at the IAS, and Taylor proposed that we work this out together. This involved solving the following problems (among others):

- Replacing the ordinary locus in the toroidal compactification (not affine) by one in the minimal compactification (affine). (This argument was discovered independently by Andreatta, Iovita, and Pilloni.)
- Finding a cohomology theory that related p-adic de Rham cohomology to p-adic modular forms, and with a weight formalism.
- Showing that the boundary classes contributed non-trivially to the cohomology of the ordinary locus.
- Doing all of this for Kuga-Sato varieties, not least because the p-adic approximation involved cohomology with arbitrarily twisted coefficients.
- Relating the coherent cohomology of Kuga-Sato varieties to that of the base Shimura variety (where the Galois representations were defined).

5.1.4. *Dagger spaces.* Taylor decided that we would use *rigid cohomology* of *dagger spaces*. This had a weight formalism (Chiarellotto) and a relation to coherent cohomology of rigid analytic or dagger spaces (Le Stum, Grosse-Klönne).

Crucially, we had to look at de Rham (coherent) cohomology of the ordinary locus that was compactly supported near the boundary but with no support condition away from the ordinary locus. I don't know whether this can now be done with less esoteric theories of p-adic analytic spaces.

The contribution of the boundary classes consisted in the weight zero subspace of cohomology with compact support. This all came from the rational varieties (as I had explained to Clozel more than 20 years earlier) but was by far the simplest part of my computations with Zucker.

5.1.5. *Division of labor.* My contribution was essentially nil, beyond my (much) earlier work with Zucker.

Concretely, I was sometimes asked to explain the relevant portions of my papers with Zucker (which even Zucker and I found hard to read). I also carried out some calculations in that framework that turned out to be unnecessary for the final results and did not appear in the paper.

Most of the technical parts of the paper were written by Lan and Taylor working closely together. Lan also had to write a second 500+ page book to justify the claims about the compactification of the ordinary loci of Kuga-Sato varieties.

While my three collaborators were busy writing the paper, I was consulting all the experts I knew in p-adic Hodge theory in the hope of finding an integral structure that could replace rigid cohomology in our construction.

5.1.6. *Making the results public.* The coefficients were all in characteristic zero – no integral theory, and therefore no torsion classes, could be treated by this theory.

Another potential advantage of an integral structure was that the finite coefficients became trivial over finite covers of $Y_{2n,K}$, eliminating the need for the mass of notation needed to work with Kuga-Sato varieties.

Taylor announced the results in the spring of 2012, before the writing was complete. I decided to talk about the project that summer at Oberwolfach, where I knew I would be surrounded by specialists in p-adic cohomology.

5.2. **Scholze's breakthrough.**

5.2.1. *Beer.* My own talk described the method of [HLTT]. From the Oberwolfach report:

> An important observation is that the relevant Eisenstein cohomology classes can be realized geometrically in the weight 0 subspace of *rigid cohomology* of the ordinary locus of the special fiber of $\mathfrak{X}_U$, with compact supports in the direction of the toroidal boundary. This can in turn be calculated by a spectral sequence whose $E_1^{r,s}$ terms are given by coherent cohomology of automorphic vector bundles $\mathfrak{X}_U$, extended in a certain way to $\mathfrak{X}_U^{tor}$, and then to $\mathfrak{X}_U^*$. Using the fact that the ordinary locus in the special fiber of $\mathfrak{X}_U^*$ is affine, the higher coherent cohomology all vanishes, which implies that the Eisenstein classes can be approximated modulo arbitrarily high powers of p by (holomorphic) cusp forms. Standard techniques due to Taylor and others then show that the systems of Hecke eigenvalues on the Eisenstein classes are approximated in a similar way by cuspidal Hecke eigenvalues. This gives a first construction of the pseudorepresentations predicted by Skinner, and by refining this construction one obtains the desired n-dimensional p-adic Galois representation.

My most vivid memory of that summer's Oberwolfach meeting, however, was of the largest collection of bottles of beer I had ever seen on a single table, when I went to bed at around 2 AM. Somehow they had all disappeared in time for breakfast the following morning.

5.2.2. *More beer.* My Oberwolfach report ended optimistically:

> Many questions remain open; the most intriguing is whether this technique can be extended to attach Galois representations to *torsion* cohomology of the locally symmetric spaces attached to $GL(n)$.

The talk itself ended (or began?) with a prophesy: that in 5 (or 3?) years the construction would be carried out in an integral Hodge theory, with no need for the Kuga-Sato varieties. At that point I couldn't help looking at Peter Scholze, who was sitting (as usual) at the back of the room.

I next saw Scholze at the Fields Institute in Toronto that fall. Together with Matt Emerton and a few others, we went to a pub, and I asked him about his current projects. He mentioned that he knew how to construct perfectoid Shimura varieties, and described the Hodge-Tate period morphism. He also explained that he could compute completed cohomology as coherent cohomology of these perfectoid spaces.

I reminded him of my Oberwolfach prophesy and asked whether this might be the missing integral theory.

5.2.3. *Scholze "made some progress".* Lan and I were both invited to Bonn the following spring to talk about [HLTT], which was not yet available as a preprint.

I have to confess that the prospect of spending time in Bonn has never appealed to me. However, I hoped to chat about the integral theory with Scholze, who had just defended his thesis – after having been named the youngest professor in German history. We did have a conversation in his office at the university.

And of course Harder, who had initiated the study of boundary cohomology and spent decades developing the theory, was in Bonn; so it was natural to explain a novel application of his theory in his presence.

I gave two talks and returned to Paris; Lan also gave a few talks. About a month later, when I saw Scholze in Paris, he told me, "I've made some progress."

5.2.4. *A few words about Scholze's proof.* The only overlap of Scholze's *Annals* paper with the strategy of [HLTT] – which was only posted on our websites a few months later, and on arXiv a year after that! – was in the recovery of the n-dimensional Galois representations attached to torsion classes from the $2n$-dimensional representations. As far as I can tell, the most difficult material in Scholze's paper had to do with the proof of the results he had mentioned in Toronto but had not yet written up. The rest of the paper was the sort of display of abundant originality that most mathematicians don't manage in an entire lifetime, and that number theorists are still trying to digest, although Scholze himself has progressed through at least three landmark contributions in the meantime.

To mention just one example, the Hodge-Tate morphism, which gave Scholze the benefits of an affine covering while sidestepping all the constructions of compactifications and related notation – and, as I anticipated, the need to introduce Kuga-Sato varieties – that took up at least 1/3 of [HLTT], has since become a fundamental object of study in its own right.

5.2.5. *Refinements.* Scholze's method, as anticipated, is based on realizing torsion classes in $H^*(X_{n,K})$, viewed as classes in the cohomology of the Borel-Serre boundary, as classes in $H^*(Y_{2n,K})$. But the cohomology of the boundary is the cone on the map from cohomology with compact support of $Y_{2n,K}$ to the cohomology of $Y_{2n,K}$. Since these two are dual to each other, the contribution of the torsion classes of one or the other is nontrivial. But Scholze's method, as written, doesn't show that all the Galois representations attached to torsion classes are on torsion $\mathbb{Z}_\ell$-modules of maximal length. A refinement of Scholze's result was obtained by Newton and Thorne, shortly after Scholze's breakthrough, under a regularity hypothesis that allows them to apply a vanishing theorem of Lan and Suh.

6. Aftermath

6.1. **Applications.**

6.1.1. *A few applications.*

- Pilloni and Stroh very quickly proved that Scholze's construction also provided a general construction of Galois representations attached to coherent cohomology of Shimura varieties (of Hodge type, for the specialists). This extends the construction that began with Clozel to automorphic forms whose infinity type belongs to the (non-degenerate) limit of discrete series.
- Caraiani and Scholze initiated a study applying p-adic Hodge theory, in Scholze's version, to prove vanishing of torsion cohomology under rather general conditions.
- This program continues, but it had an immediate application in the landmark *ten author paper* that proves potential automorphy of elliptic curves over CM fields.
- Most recently, Lue Pan combined the Hodge-Tate morphism with considerations inspired by the p-adic Simpson correspondence, and representation theory of enveloping algebras, to give a new proof of the Artin conjecture for odd 2-dimensional representations of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. These ideas have been developed further by Pilloni, Rodriguez Camargo, and Pan himself.

6.2. **Alternatives.**

6.2.1. *Boxer, Goldring, Koskivirta.* A year after Scholze's paper was released – and a year before the 300+ pages of [HLTT] were finally published – George Boxer explained an alternative construction of the Galois representations attached to torsion classes. Boxer defined higher Hasse invariants in the setting of EGA-style algebraic geometry – he even cites EGA in his (unpublished) thesis – and works with the integral structure provided by the coherent cohomology of schemes. The construction of Galois representations has not yet appeared, as far as I know.

Boxer's ideas overlap with constructions discovered independently by Goldring and Koskivirta, who have published a complete construction of Galois representations for *coherent cohomology* along these lines, including for torsion classes. These ideas have apparently influenced the development of *higher Hida theory* by Pilloni, and *higher Coleman theory* by Boxer and Pilloni. This is still very much in flux.

6.3. **It is always too early to draw any lessons.**

6.3.1. *What remains of [HLTT].* As I expected, the painstaking constructions of [HLTT] have largely proved unnecessarily complicated for the purpose, though this happened rather more quickly than I predicted.

On the other hand, Lan's 500+ page book has been indispensable for applications of the Hida theory of holomorphic modular forms in higher dimensions. My paper on p-adic L-functions with Eischen, Li, and Skinner makes extensive use of Lan's book, and it would probably have been impossible to complete otherwise.

The most interesting applications of [HLTT], as of any long and difficult paper, are still the ones that no one has yet anticipated.

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