Deformations of automorphic Galois representations
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The present manuscript is the fourth draft of a project begun in the summer of 1996, to apply the methods of [TW] to Galois representations of dimension > 2. We show that a $\pmod{\ell}$ representation $\overline{\rho}$ of dimension n that "comes from" automorphic forms on certain unitary groups has the property that every lifting of $\overline{\rho}$ to characteristic zero also comes from automorphic forms. In the language of [W] and [TW], we are constructing isomorphisms between ℓ-adic Hecke algebras and ℓ-adic deformation rings. The precise statements are Theorems **V**.1.6 and **V**.1.7.

Let E be a totally real field, $\mathcal{K}_0$ an imaginary quadratic field, and $\mathcal{K} = E \cdot \mathcal{K}_0$. We consider automorphic representations π of the unitary similitude group G of a division algebra D over $\mathcal{K}$ of dimension n^2 with involutions "of the second kind"; i.e., which restrict to complex conjugation on $\mathcal{K}$. To such data we can attach Shimura varieties Sh which are uniformized by the unit ball in $\mathbb{C}^{n-1}$, and which admit canonical models over $\mathcal{K}$. The automorphic representations of interest to us are those that contribute to middle-dimensional (singular or ℓ-adic) cohomology with coefficients in group-theoretic local systems over Sh; in the present article only the trivial local system is considered, but this is mainly for convenience. The Galois representation on the piece of the ℓ-adic cohomology cut out by π is rather well understood, both at (most) unramified places [K1, C1] and at places of $\mathcal{K}$ split over E and where π is supercuspidal [H1, H2]. Via base change, π can be associated to an automorphic representation Π of $GL(n, \mathcal{K})$, and, up to normalization, the L-function of the corresponding Galois representation, denoted $r_\rho(\pi)$, coincides with that of Π. In particular, $r_\rho(\pi)$ is n-dimensional and enjoys a number of arithmetically interesting properties; e.g., it is crystalline at primes dividing ℓ. Let $\mathcal{O}$ be an ℓ-adic integer ring over which $r_\rho(\pi)$ can be realized, and let k be its residue field.

We assume that the only ramification of π comes at a finite set SC of places that split in $\mathcal{K}/E$, and that π_v is full induced from supercuspidals for $v \in SC$. When D is not split at v we assume the representation of $GL(n)$ corresponding to π is supercuspidal. We also assume ℓ is prime to the order of $GL(n, k(v))$ for all v such that π_v is ramified; here $k(v)$ is the residue field. In Vignéras' terminology, ℓ is *banal* for such v. This implies that $\ell > n$. It thus follows from ℓ-adic Hodge theory [FM, F1] that the residual representation $\overline{r}_\rho(\pi)$ over k is also crystalline, i.e. it can be obtained by the construction of Fontaine-Laffaille [FL]. It can be shown that $\overline{r}_\rho(\pi)$ restricts to an absolutely irreducible representation of the decomposition group Z_v at any $v \in SC$. Using these facts, we can define a universal ring $R(\pi)$ classifying deformations of $\overline{r}_\rho(\pi)$ to crystalline representations of $Gal(\overline{\mathcal{K}}/\mathcal{K})$, via the theory developed by Mazur and Ramakrishna (cf. [DDT]). For technical reasons,

$\overline{r}_\rho(\pi)$ is replaced in the deformation theory by the representation $\overline{\rho}(\pi)$of $Gal(\overline{E}/E)$ induced from $\overline{r}_\rho(\pi)$, which we view as a homomorphism to a disconnected group with identity component $GL(n)$.

The goal is to construct a natural isomorphism between $R(\pi)$ and an appropriate Hecke algebra. The Hecke algebra acts on the space of functions on the zero-dimensional Shimura variety Sh' attached to an inner form J of G. The correspondence between functions on Sh' and cohomology of Sh, analogous to the Jacquet-Langlands correspondence for inner forms of $GL(2)$, is established in most cases in [H1], and in general in [HL]. It is relatively easy to study congruences for Hecke algebras acting on zero-dimensional Shimura varieties. Thus we can hope to apply the techniques of [TW]. The possibility of extending these techniques to groups other than $GL(2,\mathbb{Q})$ is based on the crucial fact, observed simultaneously and independently by Diamond and Fujiwara [D, Fu], that one can work directly with Hecke algebra modules, rather than with the Hecke algebras themselves.

Few general techniques are available for studying the images of Galois representations of dimension > 3. In order to carry out the sorts of calculations of Galois cohomology familiar to readers of [W], [TW], and [DDT], we need to impose several hypotheses to guarantee that the image of $\overline{\rho}$ is not too small (Hypotheses **IV**.5.2), in addition to the conditions on ℓ already mentioned. Our hypotheses on $\overline{\rho}$ are certainly not optimal, but they have the advantage of economy. We hope to relax some of them before completing the definitive version. For this reason, the present draft develops techniques in greater generality than is immediately necessary.

The article is divided into five sections. Section **I** introduces the unitary groups and the associated Shimura varieties, and summarizes results on the spaces of automorphic forms on these groups that are derived from the trace formula. Section **II** is mainly concerned with applications of Vignéras' theory of modular representations of p-adic reductive groups, especially $GL(n)$. Vignéras' results replace some of the theory of the U-operator, familiar from work on $GL(2)$. Section **III** describes the Shimura varieties attached to unitary groups as moduli spaces of abelian varieties of PEL type, and develops the properties of the Galois representations realized in the ℓ-adic cohomology of these Shimura varieties. Section **IV** is concerned with deformations of Galois representations, and extends the Galois-cohomological techniques of [W] and (especially) [TW] to dimension > 2. Finally, section **V** generalizes the arguments and the main result of [TW]. At the end of **V** we indicate how the main theorems can be applied to construct automorphic tensor products.

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I. Automorphic forms on unitary groups

I.1. Cohomological automorphic forms.

Let E be a totally real number field of degree d over $\mathbb{Q}$ and let $\mathcal{K}$ be a totally imaginary quadratic extension of E; let $c \in Gal(\mathcal{K}/E)$ denote the non-trivial automorphism. We assume $\mathcal{K}$ contains an imaginary quadratic field $\mathcal{K}_0$, so that $\mathcal{K} = \mathcal{K}_0 \cdot E$. We denote by Σ_E and $\Sigma_{\mathcal{K}}$ the sets of complex embeddings of E and $\mathcal{K}$. Let D be a central simple algebra of dimension n^2 over $\mathcal{K}$, endowed with an involution, denoted $\tilde{c}$, that induces the Galois automorphism c on $\mathcal{K}$; i.e., $\tilde{c}$ is an involution of the second kind. We will generally make the hypothesis

(D) At every finite place v of $\mathcal{K}$, D_v is either split or a division algebra.

We define algebraic groups $U(D) = U(D, \tilde{c})$ and $GU(D) = GU(D, \tilde{c})$ over $\mathbb{Q}$ such that, for any $\mathbb{Q}$-algebra R,

$$U(D)(R) = \{g \in D^{opp} \otimes_{\mathbb{Q}} R \mid g \cdot \tilde{c}(g) = 1\};$$

$$GU(D)(R) = \{g \in D^{opp} \otimes_{\mathbb{Q}} R \mid g \cdot \tilde{c}(g) = \nu(g) \text{ for some } \nu(g) \in R^{\times}\}.$$

Thus $GU(D)$ admits a homomorphism $\nu : GU(D) \to \mathbb{G}_m$ with kernel $U(D)$. There is an algebraic group $U_E(D)$ over E such that $U(D) \xrightarrow{\sim} R_{E/\mathbb{Q}} U_E(D)$, where $R_{E/\mathbb{Q}}$ denotes Weil's restriction of scalars functor. This isomorphism identifies automorphic representations of $U(D)$ and $U_E(D)$.

The groups $U(D)$ (resp. $GU(D)$) are all inner forms of the same quasi-split unitary group (resp. unitary similitude group), denoted U_0 (resp. GU_0). The group U_0 is of the form $U(D_0, \tilde{\chi}(*)_0)$ where D_0 is the matrix algebra and $\tilde{\chi}(*)_0$ is an appropriate involution. Then $U_{0,\infty} \cong U(\frac{n}{2}, \frac{n}{2})^{[E:\mathbb{Q}]}$ if n is even, $U_{0,\infty} \cong U(\frac{n+1}{2}, \frac{n-1}{2})^{[E:\mathbb{Q}]}$ if n is odd.

Let G be a reductive algebraic group over the number field F. If v is a place of F we let $G_v = G(\mathbb{Q}_v)$; if v is archimedean we let $\mathfrak{g}_v = Lie(G_v)_{\mathbb{C}}$. We let G_∞ denote $\prod_{v|\infty} G_v$, the product taken over all archimedean places of F, and let $\mathfrak{g}_\infty = \prod_{v|\infty} \mathfrak{g}_v$. Let π be an irreducible automorphic representation of G; i.e., an irreducible $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbf{A}^f)$-module that embeds as a submodule of the space of automorphic forms relative to the maximal compact subgroup K_∞. We write $\pi = \pi_\infty \otimes \pi_f$ as usual, and say π is *cohomological* if π is cuspidal and if the relative Lie algebra cohomology $H^\bullet(\mathfrak{g}_\infty, K_\infty; \pi_\infty \otimes V) \neq 0$ for some finite dimensional representation V of $\mathfrak{g}_\infty$. We let $Coh(G)$ denote the set of cohomological cuspidal automorphic representations of G, $Coh(G, V) \subset Coh(G)$ the subset of π for which $H^\bullet(\mathfrak{g}_\infty, K_\infty; \pi_\infty \otimes V) \neq 0$, with V fixed. If $K \subset G(\mathbf{A}^f)$ is a compact open subgroup, σ a finite-dimensional irreducible representation of K, we let $Coh(G, K, \sigma) \subset Coh(G)$ denote the subset of π such that $Hom_K(\sigma, \pi) \neq 0$, $Coh(G, K, \sigma, V) = Coh(G, K, \sigma) \cap Coh(G, V)$. If $\sigma = 1$ we just write $Coh(G, K)$ and $Coh(G, K, V)$.

Let $\mathcal{A}_0(G)$ denote the space of cusp forms on G. Let $Rep(G)$ denote the set of equivalence classes of irreducible $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbf{A}^f)$-modules. If $\pi \in Rep(G)$, we let $m(\pi) = \dim Hom(\pi, \mathcal{A}_0(G))$, where Hom denotes the space of homomorphisms of $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbf{A}^f)$-modules. More generally, let S be a finite set of places of F, containing the archimedean places (for simplicity) and let $Rep(G)^S$ denote the set of equivalence classes of irreducible $G(\mathbf{A}^{f,S})$-modules, where $G(\mathbf{A}^{f,S}) \subset G(\mathbf{A}^f)$ is the subgroup with trivial entry at every place in S. We say $\pi^S \in Rep(G)^S$ is *automorphic* if

$$Hom_{G(\mathbf{A}^{f,S})}(\pi^S, \mathcal{A}_0(G)) \neq 0;$$

i.e., if π^S can be extended to a cuspidal automorphic representation of G. We say π^S is *cohomological* if it can be extended to a cohomological cuspidal automorphic representation of G.

We now choose a central simple algebra D, over $\mathcal{K}$ satisfying (D). Let $\tilde{c}$ be an involution of the second kind of D and define $J = GU(D, \tilde{c})$, $J' = U(D, \tilde{c})$, and view J' alternatively as a group over $\mathbb{Q}$ or E. We assume that

$$(\mathbf{I}.1.1) \qquad J'_\infty \cong U(n)^{[E:\mathbb{Q}]} \text{ (compact inner form);}$$

Moreover, if v is a finite prime, we assume that

$$(\mathbf{I}.1.3) \qquad J'_v \cong U_{0,v} \text{ if } v \text{ does not split in } \mathcal{K}/F.$$

We will be working with a variant, denoted $\tilde{G}$, of the Langlands L-group of J over E. Let

$$(\mathbf{I}.1.4) \quad \tilde{G}^0 = \{g = (g_1, g_2) \in GL(n) \times GL(n) \mid \exists a \in GL(1) \text{ such that } g_2 = a \cdot {}^t g_1^{-1}\}.$$

We let

$$(\mathbf{I}.1.5) \qquad \tilde{G} = \tilde{G}^0 \ltimes Gal(\overline{E}/E),$$

where the action of $Gal(\overline{E}/E)$ on $GL(n)$ factors through $Gal(\mathcal{K}/E)$, and the non-trivial element c acts by

$$(\mathbf{I}.1.6) \qquad c((g_1, g_2)) = (g_2, g_1).$$

We regard $\tilde{G}$ as a group scheme over $Spec(\mathbb{Z})$, in order to work with its points over finite fields. The map that to g associates the element a in the definition ($\mathbf{I}$.1.4) defines a homomorphism of group schemes $\tilde{G}^0 \to GL(1)$, and one verifies that it extends to a homomorphism $\nu : \tilde{G} \to GL(1)$ by setting $\nu(c) = (-1)^{n-1}$.

In the applications to Galois representations, we need to work with $\tilde{G}$, rather than the L-group of J' or G', because Galois representations of geometric origin are generally polarized but rarely self-dual. Moreover, $\tilde{G}$ does not incorporate the usual conjugation by a symmetric or anti-symmetric form. This simplifies our arguments. In Remark **V**.2.5, we reformulate our results in terms of the usual Langlands L-group.

I.2. Automorphic forms on a totally definite unitary group.

Let $K \subset J(\mathbf{A}^f)$ be a compact open subgroup,

$${}_K S(J) = J(\mathbb{Q}) \backslash J(\mathbf{A}) / J_\infty \times K.$$

More generally, if $U \subset J(\mathbf{A}^f)$ is any closed subgroup, we let

$$_U S(J) = \varprojlim_{K \supset U} {}_K S(J),$$

the limit over open compact subgroups K containing U. We define certain spaces of functions on $J(\mathbf{A})$. In what follows, K as above is always assumed to be of the form $\prod_v K_v$, the product taken over finite places of $\mathbb{Q}$, with K_v open compact in J_v. We fix a rational prime number ℓ and let $K^\ell = K \cap J^{\mathbf{A}^{f,\ell}}$ (finite adeles of J trivial at ℓ). All functions are taken to be continuous.

Let (ρ, W) denote a finite-dimensional algebraic representation of J, rational over the number field $L = L(\rho)$. For any place v of $\mathbb{Q}$ we use the same letter ρ to designate the representation of J_v on $W(L_v)$, where $L_v = L \otimes_{\mathbb{Q}} \mathbb{Q}_v$. For each place v we choose a finite-dimensional representation σ_v of K_v with coefficients in L, factoring through a finite quotient of K_v, so that σ_v is trivial for all v dividing ℓ and for almost all v. We let $\sigma = \otimes_v \sigma_v$, and let $W_\sigma = W \otimes \sigma$, on which $J_\infty \times K$ acts via $\rho \otimes \sigma$.

Let

(**I**.2.1)

$$\mathcal{A}_0(J, \rho, \sigma, K) = \{f \in C^\infty(J(\mathbb{Q})\backslash J(\mathbf{A}), W_\sigma(\mathbb{C})) | f(g \cdot g_\infty k) = (\rho \otimes \sigma)(g_\infty \times k)^{-1} f(g)\},$$

where $g \in J(\mathbf{A})$, $g_\infty \in J_\infty$, $k \in K$.

For any L-algebra L', we let

(**I**.2.2)

$$\mathcal{A}_f(J, \rho, \sigma, K)(L') = \{f : J(\mathbf{A}^f) \to W(L') | f(\gamma \cdot gk) = (\rho \otimes \sigma)(\gamma \times k^{-1}) f(g)\},$$

where now $g \in J(\mathbf{A})$, $\gamma \in J(\mathbb{Q})$, $k \in K$. If L' is an L_ℓ algebra, we define

(**I**.2.3)

$$\mathcal{A}_\ell(J, \rho, \sigma, K)(L') = \{f : {}_{K^\ell} S(J) \to W(L'_\ell) | f(gk) = (\rho \otimes \sigma)(k_\ell \times k)^{-1} f(g)\},$$

for $g \in J(\mathbf{A})$, $k \in K$, k_ℓ its J_ℓ-component. Here the standing continuity hypothesis needs to be specified: we assume $f \in \mathcal{A}_\ell(J, \rho, K)(L')$ to be locally constant on $J(\mathbf{A}^{f,\ell})$. Continuity of f in G_ℓ in the ℓ-adic topology is guaranteed by the functional equation.

For any L-algebra L', there is a canonical isomorphism

$$\mathcal{A}_f(J, \rho, K)(L') = \mathcal{A}_f(J, \rho, K)(L) \otimes_L L'. \tag{I.2.4}$$

Restriction to $J(\mathbf{A}^f)$ defines a natural isomorphism

$$res : \mathcal{A}_0(J, \rho, \sigma, K) \xrightarrow{\sim} \mathcal{A}_f(J, \rho, \sigma, K)(\mathbb{C})$$

and hence, via (**I**.2.4), a canonical L-structure on $\mathcal{A}_0(J, \rho, \sigma, K)$. Similarly, for any L_ℓ-algebra L' we have an isomorphism

$$res_\ell : \mathcal{A}_\ell(J, \rho, \sigma, K)(L') \xrightarrow{\sim} \mathcal{A}_f(J, \rho, \sigma, K)(L'), \tag{I.2.5}$$

given by

$$f \mapsto \{g \mapsto \rho(g_\ell)^{-1} \cdot f(g), g \in J(\mathbf{A}^f)\}.$$

Letting K vary, it is clear that these isomorphisms commute with the action of the prime-to-ℓ finite adeles of J.

When ρ or σ is the trivial one-dimensional representation, we drop it from the notation.

In what follows, we assume K^ℓ is sufficiently small so that

$$\text{(I.2.6)} \qquad K_\ell \text{ acts without fixed points on } J(\mathbb{Q})\backslash J(\mathbf{A})/K_\infty \cdot K^\ell.$$

Let $\mathcal{O}$ denote the ring of integers of L_ℓ, and let Λ_{W_σ} be a K_ℓ-stable $\mathcal{O}$-lattice in $W_\sigma(L)$. We can define

$$\text{(I.2.7)} \quad \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K)$$
$$= \{f : {}_{K^\ell}S(J) \to \Lambda_{W_\sigma} | f(gk) = (\rho \otimes \sigma)(k_\ell \times k)^{-1} f(g), g \in J(\mathbf{A}), k \in K\},$$

with the continuity hypothesis as above. Then there is a natural isomorphism

$$\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K) \otimes_{\mathcal{O}} L_\ell \cong \mathcal{A}_\ell(J, \rho, \sigma, K)(L_\ell).$$

When ρ and σ are trivial, we take the lattice $\mathcal{O} \subset L_\ell$ and write $\mathcal{A}_\ell(J, K)(\mathcal{O})$.

I.2.8. Now we will make specific choices of (K_v, σ_v). We choose disjoint finite sets SC, Q, and $\{\mathfrak{r}\}$ of finite places of E dividing disjoint sets of rational primes that split in $\mathcal{K}_0$; $\{\mathfrak{r}\}$ is a single prime. The set SC is assumed to include all primes divisible by primes of $\mathcal{K}$ at which D is not split. For any rational prime q that splits in $\mathcal{K}_0$, we choose a maximal compact subgroup of J_q in the form

$$\text{(I.2.8.1)} \qquad \mathbb{Z}_q^\times \times \prod_{v \mid q} J_v,$$

the product being taken over divisors v of q in E. Here if w is a divisor of q in $\mathcal{K}_0$ lying above v and if D_w is isomorphic to $GL(a, B_w)$ for some factorization $n = ab$ and some division algebra B_w of degree b^2 over $\mathcal{K}_w$, then J_v can be taken in the form $GL(a, O_{B_w})$, where O_{B_w} is the maximal order of B_w.

To define the (K_v, σ_v) at ramified places, we assume D satisfies hypothesis (D) of §**I**.1. For $w \in SC$, we fix an irreducible admissible representation π_w of $GL(n, E_w)$. We assume there is a (not necessarily proper) parabolic subgroup P of G, with Levi quotient L_P, such that π_w can be realized as the full induced representation from a supercuspidal representation of L_P, inflated to P. Equivalently, π_w is associated to a fully decomposable representation of the Weil-Deligne group of E_w – i.e., a representation of the Weil group W_{E_w} of E_w – under the local Langlands correspondence $\pi \mapsto \mathcal{L}(\pi)$ [HT1, He]. Such representations will be called *non-monodromic.* If the algebra D is split at w we let (K_w, σ_w) be a semisimple type of π_w in the sense of Bushnell and Kutzko [BK]. Thus any irreducible admissible representation π' of $GL(n, E_w)$ whose restriction to K_w contains σ_w is inertially equivalent to π_w [BK, Theorem 8.2]. In other words, if $\mathcal{L}(\pi_w)$ is the representation $\mathfrak{r}_w = \oplus_{i=1}^r \mathfrak{r}_{i,w}$ of W_{E_w}, with each $\mathfrak{r}_{i,w}$ irreducible, and if π' contains the type (K_w, σ_w), then $\mathcal{L}(\pi')$ is of the form $\oplus_{i=1}^r \mathfrak{r}_{i,w} \otimes \chi_i$, where each χ_i is an unramified character of W_{E_w}.

If D_w is a division algebra we assume π_w is supercuspidal, and let $JL(\pi_w)$ be the irreducible representation of $D_w^\times$ associated to π_w by the Jacquet-Langlands correspondence [DKV,R]. We let K_w be the maximal compact subgroup of $D_w^\times$ and let σ_w be any irreducible component of the restriction of $JL(\pi_w)$ to K_w (since $JL(\pi_w)$ is irreducible, all choices of σ_w are conjugate under $D_w^\times$. If q is a rational prime divisible by some $w \in SC$, let

$$K_q = \mathbb{Z}_q^\times \times \prod_{w \notin SC} GL(n, \mathcal{O}_w) \times \prod_{w \in SC} K_w,$$

the products running only over divisors of q. We also take

$$\sigma_q = \xi_q \otimes \bigotimes_{w \in SC} \sigma_w,$$

where ξ_q is a character of $\mathbb{Z}_q^\times$ and $\prod_{w \notin SC} GL(n, \mathcal{O}_w)$ acts trivially.

Primes $\mathfrak{q} \in Q$ are assumed to have the property that, if q is the rational prime divisible by $\mathfrak{q}$, then q splits completely in E. We let $Q(\mathbb{Q})$ denote the set of rational primes divisible by primes in Q, and assume each $q \in Q(\mathbb{Q})$ is divisible by a unique $\mathfrak{q} \in Q$. For $\mathfrak{q} \in Q$ we define

$$\Gamma_{0,\mathfrak{q}} = \{k \in GL(n, \mathbb{Z}_q) \mid k \equiv \begin{pmatrix} *_1 & * \\ 0 & *_{n-1} \end{pmatrix} \pmod{q}\}$$

$$\Gamma_{1,\mathfrak{q}} = \{k \in GL(n, \mathbb{Z}_q) \mid k \equiv \begin{pmatrix} 1 & * \\ 0 & *_{n-1} \end{pmatrix} \pmod{q}\}$$

as subgroups of the $\mathfrak{q}$-factor of **I**.2.8.1. Then we let

$$K_{0,q} = \mathbb{Z}_q^\times \times \prod_{v \neq \mathfrak{q}, v \mid q} GL(n, \mathcal{O}_v) \times \Gamma_{0,\mathfrak{q}};$$

$$K_{1,q} = \mathbb{Z}_q^\times \times \prod_{v \neq \mathfrak{q}, v \mid q} GL(n, \mathcal{O}_v) \times \Gamma_{1,\mathfrak{q}},$$

viewed as subgroups of J_q.

Let $q(\mathfrak{r})$ denote the residue characteristic of $\mathfrak{r}$. Let N_0 denote the upper triangular unipotent subgroup of $GL(n)$ and let

$$I_1(\mathfrak{r}) = \{k \cdot n \in GL(n, \mathcal{O}_\mathfrak{r}) \mid k \equiv 1 \pmod{\mathfrak{r}}, n \in N_0(\mathcal{O}_\mathfrak{r})\};$$

$$I(\mathfrak{r}) = \mathbb{Z}_{q(\mathfrak{r})}^\times \times \prod_{v \neq \mathfrak{r}, v \mid q(\mathfrak{r})} GL(n, \mathcal{O}_w) \times I_1(\mathfrak{r}).$$

We let

$$K_{q(\mathfrak{r})} = I_\mathfrak{r}. \tag{I.2.8.2}$$

Finally, for the remaining primes q, we take maximal compact subgroups K_q that are *very special* in the sense of [L]. In particular, K_q is hyperspecial whenever J_q contains a hyperspecial maximal compact subgroup; i.e., if q is unramified in $\mathcal{K}_0$. If q is ramified in $\mathcal{K}_0$ and n is odd, there are two conjugacy classes of special maximal compact subgroups, only one of which is very special. Here we are using the hypothesis (**I**.1.3). We let

$$K_{0,Q} = \prod_{q \notin Q(\mathbb{Q})} K_q \times \prod_{q \in Q(\mathbb{Q})} K_{0,q};$$

$$K_{1,Q} = \prod_{q \notin Q(\mathbb{Q})} K_q \times \prod_{q \in Q(\mathbb{Q})} K_{1,q}.$$

Lemma I.2.9. *For $q(\mathfrak{r})$ sufficiently large, the subgroups $K_{0,Q}$ and $K_{1,Q}$ satisfy (**I**.2.6). Moreover, for any $s \in J(\mathbf{A}^f)$ the groups $s^{-1}J(\mathbb{Q})s \cap K_{0,Q}$ and $s^{-1}J(\mathbb{Q})s \cap K_{1,Q}$ are trivial.*

Proof. The first assertion follows from the second. Let $K = K_{0,Q}$ or $K_{1,Q}$. Let $x \in s^{-1}J(\mathbb{Q})s \cap K$ for some $s \in J(\mathbf{A}^f)$. The subgroup of $J(\mathbf{A})$ generated by x is both discrete and compact, hence finite. The group $I_{\mathfrak{r}}$ is pro-$q(\mathfrak{r})$ and it follows that x is a root of unity of order a power of $q(\mathfrak{r})$, lying in some extension field $\mathcal{K}'$ of $\mathcal{K}$ that admits an embedding in D. The degree of $\mathcal{K}'$ over $\mathbb{Q}$ is bounded by $n[\mathcal{K} : \mathbb{Q}]$, hence for $q(\mathfrak{r})$ sufficiently large we must have $x = 1$. Condition (**I**.2.6) is now immediate.

Let π be an automorphic representation of J. The restriction of π to the unitary group J', which can be viewed as an algebraic group over E, decomposes as a direct sum of irreducible automorphic representations. Any two summands have the same local components at any finite place w dividing a rational prime q that splits in $\mathcal{K}_0$, since at such places the similitude map splits as a product $J(\mathbb{Q}_q) \simeq J'(\mathbb{Q}_q) \times \mathbb{Q}_q^\times$. For such a place w, we say π is **non-monodromic at w** if one component (hence every component) of the restriction of π to the unitary group J' is non-monodromic at w (if J_w is split) or corresponds to a supercuspidal representation of $GL(n, E_w)$ by the Jacquet-Langlands correspondence (if J_w is the multiplicative group of a division algebra).

Proposition I.2.10. *Let π be an automorphic representation such that $Hom_{K_{1,Q}}(\sigma, \pi) \neq 0$. Then π_w is non-monodromic for all $w \in SC$ and, for every $w \in Q$, π_w is either (a) unramified; (b) principal series attached to an n-tuple $(\alpha, \beta_1, \ldots, \beta_{n-1})$ of characters of $E_w^\times$, with α tamely ramified and each β_i unramified; or (c) the Langlands sum of a special representation of $GL(2)$ and an unramified representation of $GL(n-2)$. In cases (a) and (c), but not in case (b), π_w has a $\Gamma_{0,w}$-fixed vector.*

Moreover, if SC is non-empty, then π_q is generic for every q that splits in $\mathcal{K}_0$.

Proof. The assertion regarding $w \in SC$ is a consequence of the properties of Bushnell-Kutzko types. The first assertion regarding $w \in Q$ follows easily from the Bernstein-Zelevinsky classification of admissible irreducible representations of $GL(n)$ and from the theory of the conductor [JPSS]. The second assertion follows from the existence of base change of π to $J_{\mathcal{K}_0} \simeq GL(n)_{\mathcal{K}} \times GL(1)_{\mathcal{K}_0}$ [CL], cf. [HT1,VI.2]. Let Π denote the base change. Then Π is cuspidal, hence globally generic by Shalika's theorem. The second assertion of (**I**.2.10) then follows from the properties of base change, especially [L, Proposition IV.4.1].

Let $\mathbb{K} = \prod_v \mathbb{K}_v$ be a subgroup of finite index in $K_{1,Q}$, contained in the kernel of σ and normal in $K_{0,Q}$. Let $\Lambda_W = \Lambda_{W_\sigma}$ with σ replaced by the trivial representation. We can write

$$J(\mathbf{A}^f) = \coprod_{i=1}^{r} J(\mathbb{Q}) g_i \mathbb{K}, \qquad (\mathbf{I}.2.11)$$

for some finite set $\Xi = \{g_1, \ldots, g_r\}$ of elements of $J(\mathbf{A}^f)$. Then the map $f \mapsto (f(g_i))$ defines an injection

$$\mathcal{A}_\ell(J, \Lambda_W, \mathbb{K}) \hookrightarrow \oplus_{i=1}^r \Lambda_W. \qquad (\mathbf{I}.2.12)$$

As in [DT,p. 442], this is an isomorphism provided $g_i^{-1}J(\mathbb{Q})g_i \cap \mathbb{K}$ is trivial for all i. In particular, this is true when $q(\mathfrak{r})$ is sufficiently large, by Lemma **I**.2.9, even when SC is empty. We assume $q(\mathfrak{r})$ to be sufficiently large from now on.

Now $K_{1,Q}$ satisfies the same properties as $\mathbb{K}$ above, hence $\mathcal{A}_\ell(J, \Lambda_W, K_{1,Q})$ has the decomposition (**I**.2.12). If $\Xi' = \{g_1, \ldots, g_{r'}\}$ is a set of double coset representatives for $J(\mathbb{Q})\backslash J(\mathbf{A}^f)/K_{1,Q}$ then we can take $\Xi = \Xi' \cdot [K_{1,Q}/\mathbb{K}]$ in (**I**.2.12). It follows that

(**I**.2.13) $\qquad$ $\mathcal{A}_\ell(J, \Lambda_W, \mathbb{K})$ is a free $\mathcal{O}[K_{1,Q}/\mathbb{K}]$ – module.

We can apply the same argument to $K_{0,Q}$ instead of $K_{1,Q}$. Since

$$K_{0,Q}/\mathbb{K} \cong K_{1,Q}/\mathbb{K} \times \prod_{q \in Q} (\mathbb{Z}/q\mathbb{Z})^\times,$$

we obtain:

Lemma I.2.14. *Assume $q(\mathfrak{r})$ to be sufficiently large. Then $\mathcal{A}_\ell(J, \Lambda_W, \mathbb{K})$ is a free $\mathcal{O}[K_{1,Q}/\mathbb{K} \times \prod_{q\in Q}(\mathbb{Z}/q\mathbb{Z})^\times]$-module.*

In what follows, we assume Λ_{W_σ} to be of the form $\Lambda_W \otimes \Lambda_\sigma$, where Λ_σ is a $K_{1,Q}/\mathbb{K}$-invariant lattice in σ.

Corollary I.2.15. *Assume $q(\mathfrak{r})$ to be sufficiently large. Suppose Λ_σ can be realized as a direct summand in $\mathcal{O}[K_{1,Q}/\mathbb{K}]$ as $\mathcal{O}[K_{1,Q}/\mathbb{K}]$-module. (For example, suppose ℓ does not divide the order of $\prod_{w\in SC} K_w/\mathbb{K}_w$.) Then $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q})$ is a free $\mathcal{O}[\prod_{q\in Q}(\mathbb{Z}/q\mathbb{Z})^\times]$-module.*

Let $D_Q = \prod_{q\in Q}(\mathbb{Z}/q\mathbb{Z})^\times$. For any $\mathbb{Z}_\ell[D_Q]$-module N let N_{D_Q} denote the module of coinvariants.

Corollary I.2.16. *Let ℓ^N be the exact power of ℓ dividing the order of D_Q. Under the hypotheses of Corollary* **I**.*2.15, the natural inclusion*

$$\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{0,Q}) \to \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}),$$

followed by the canonical map

$$\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}) \to \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q})_{D_Q},$$

induces an isomorphism between $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{0,Q})$ and $\ell^N \cdot \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q})_{D_Q}$.

Proof. Indeed, Corollary **I**.2.15 reduces the assertion of **I**.2.16 to the corresponding statement for the group ring $\mathcal{O}[D_Q]$ itself; but this is clear.

Let $K_{1,Q}^{[\ell]} \subset K_{0,Q}$ be the largest subgroup containing $K_{1,Q}$ such that the quotient $\Delta_Q = K_{0,Q}/K_{1,Q}^{[\ell]}$ is an ℓ-group, necessarily abelian. (In other words, $K_{1,Q}^{[\ell]}$ is generated by $K_{1,Q}$ and by the diamond operators of order prime to ℓ; cf. **III**.1.3.8, below). We will be using Corollaries **I**.2.15 and **I**.2.16 with $K_{1,Q}$ replaced by $K_{1,Q}^{[\ell]}$:

Corollary I.2.17. *Assume the hypotheses of Corollary* **I**.*2.15. Then*

(i) $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]})$ *is a free* $\mathcal{O}[\Delta_Q]$*-module.*

(ii) The natural inclusion

$$\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{0,Q}) \rightarrow \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]}),$$

followed by the canonical map

$$\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]}) \rightarrow \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]})_{\Delta_Q},$$

induces an isomorphism between $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{0,Q})$ *and* $\ell^N \cdot \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]})_{\Delta_Q}$.

Proof. Since the kernel of $D_Q \rightarrow \Delta_Q$ is of order prime to ℓ, this follows immediately from the previous corollaries.

I.3. Functoriality and multiplicities

We collect here some applications of the stable trace formula, established in [C1,CL,L,HL]. We let $\dot{J}'$ denote the group $R_{\mathcal{K}/\mathbb{Q}}GL(n)_{\mathcal{K}}$, $\dot{J} = \dot{J}' \times R_{\mathcal{K}_0/\mathbb{Q}}\mathbb{G}_{m,\mathcal{K}_0}$, both viewed as algebraic groups over $\mathbb{Q}$; $\dot{J}'$ will also be viewed as an algebraic group over $\mathcal{K}$ or over E. Let J and J' be the groups defined at the end of **I**.1. Then we have

(**I**.3.1) $\dot{J}$ is an inner form of $R_{\mathcal{K}_0/\mathbb{Q}}G_{\mathcal{K}_0}$ and $R_{\mathcal{K}_0/\mathbb{Q}}J_{\mathcal{K}_0}$;

(**I**.3.2) $\dot{J}'$ is an inner form of $R_{\mathcal{K}/E}G'_{\mathcal{K}}$ and $R_{\mathcal{K}/E}J_{\mathcal{K}}$.

Let $\pi \in Coh(J)$, resp. $\pi \in Coh(J')$, and let S be the set of places of $\mathbb{Q}$, resp. of E, containing all archimedean places and all finite places where either π or the group J is ramified. Then we can formally define $BC^S(\pi) \in Rep(\dot{J})^S$ (resp. $Rep(\dot{J}')^S$) to be the representation such that $BC^S(\pi)_w = BC(\pi_w)$ for all $w \notin S$, with $BC(\pi_w)$ the local unramified base change map with respect to the isomorphism **I**.3.1.

Theorem I.3.3 [C1,L,HL]. *Let* $\pi \in Coh(J)$ *(resp.* $Coh(J')$*). Suppose either*

(i) J *is the unitary similitude group of a division algebra; or*

(ii) There is a rational prime p *split in* $\mathcal{K}_0$ *and a prime* v_0 *of* E *dividing* p *such that* π *is supercuspidal at* v_0*, in the sense of* **I**.*2.10.*

Then $BC^S(\pi)$ *is the prime-to-S part of a cohomological automorphic representation of* $\dot{J}$ *(resp.* $\dot{J}'$*).*

Proof. The proof in case (i) is basically contained in [C1,C2], and is completed in [CL] using the results of [L]. Case (ii) is worked out in [HL].

Under the hypotheses of Theorem **I**.3.3, strong multiplicity one implies that $BC^S(\pi)$ extends to a unique cuspidal automorphic representation of $\dot{J}$ (resp. $\dot{J}'$), which we denote $BC(\pi)$; it is necessarily cohomological. In the case of $\dot{J}$, we write $BC(\pi) = BC(\pi)_1 \times BC(\pi)_2$, where $BC(\pi)_1$ is an automorphic representation of $\dot{J}'$ and $BC(\pi)_2$ is a Hecke character of $\mathcal{K}_0$.

Proposition I.3.4 [L]. *We retain the hypotheses of Theorem* **I***.3.3. Let q be a rational prime that ramifies in $\mathcal{K}_0$, and suppose π_q is spherical with respect to a very special maximal compact subgroup of $\mathbf{G}_q$ (resp. $\mathbf{G}'_q$). Then $BC(\pi)$ is unramified at all primes dividing q.*

Moreover, in the case of $\mathbf{G}'$, let q be a rational prime that splits in $\mathcal{K}_0$, $q = v_1 \cdot v_2$. Then we have

$$BC(\pi)_q = BC(\pi)_{v_1} \otimes BC(\pi)_{v_2} \simeq \pi_q \otimes \check{\pi}_q$$

via the isomorphism $\mathbf{G}'(\mathcal{K}_{0,q}) \xrightarrow{\sim} \mathbf{G}'(\mathbb{Q}_q) \times \mathbf{G}'(\mathbb{Q}_q)$.

Proof. The first assertion follows from [L, Proposition IV.6.4], the second from [L,Lemma IV.4.1]. The application of these results to unitary groups is carried out in detail in [HL].

Let E'/E be a totally real cyclic extension, $\mathcal{K}' = \mathcal{K}_0 \cdot E'$, and let

$$BC_{\mathcal{K}'}(\pi) = BC(\pi)_{1,\mathcal{K}'} \times BC(\pi_2),$$

where $BC(\pi)_{1,\mathcal{K}'}$ is the base change, as in [AC], of $BC(\pi)_1$, for the cyclic extension $\mathcal{K}'/\mathcal{K}$. We let $J^{E'}$, (resp. $J^{',E'}$) be unitary similitude groups (resp. unitary groups) defined as in §**I**.1 relative to $\mathcal{K}'/E'$ and satisfying the analogues of (**I**.1.1)-(**I**.1.2).

Theorem I.3.5 [C1]. *Suppose $BC(\pi)_{1,\mathcal{K}'}$ is supercuspidal at a non-empty set of places, including every finite place w at which $J^{',E'}$ is anisotropic. Then there is an automorphic representation $BC_{E'}(\pi) \in Coh(J^{E'})$ whose base change to $\mathcal{K}'$, in the sense of Theorem* **I***.3.3, is isomorphic to $BC_{\mathcal{K}'}(\pi)$.*

Moreover, $BC_{E'}(\pi)_\infty$ belongs to the discrete series.

Proof. The first assertion for the unitary group $J^{',E'}$ is a special case of [C1, Proposition 4.11]. Indeed, it suffices to show that $BC_{\mathcal{K}'}(\pi)$ is dual to its $Gal(\mathcal{K}'/E')$-conjugate, and this is immediate from the corresponding fact for $BC(\pi)$. The assertion for the similitude groups follows easily from this case.

The final assertion is [C1, Corollaire 5.6].

(**I**.3.6) Clozel's theorem actually asserts that $BC(\pi)_{1,\mathcal{K}'}$ descends to an L-packet of cohomological automorphic representations of $J^{E'}$, which is stable at archimedean places in the sense that the one can switch freely within the archimedean discrete series L-packet. Under additional hypotheses we can say more. Let $\pi \in Coh(J)$. We now suppose

(**I**.3.6.1) The extension $\mathcal{K}/E$ is unramified at all finite primes.

(**I**.3.6.2) π_q is spherical for every rational prime q that does not split in $\mathcal{K}_0$.

The proof of the following theorem will appear in [HL]:

Theorem I.3.7 [HL]. *Under hypotheses (***I***.3.6.1) and (***I***.3.6.2), the multiplicity $m(\pi) = 1$.*

**[THIS IS NO LONGER REALLY NECESSARY]

II. Hecke operators and modular representation theory

II.*1. Calculations with Jacquet modules.*

In the present section we fix a rational prime q and a q-adic field F, with maximal order $\mathcal{O}_F$, maximal ideal m_F, and residue field $k(F)$, and uniformizing parameter ϖ. Let $H = GL(n,F)$ and let B be its standard upper triangular Borel subgroup. Let $P \subset GL(n)$ be the standard parabolic subgroup with Levi decomposition $P = LN$, $L = GL(1) \times GL(n-1)$ with its standard embedding in H. Let $\Gamma = GL(n, \mathcal{O}_F)$ and let $r : \Gamma \to GL(n, k(F))$ denote the reduction map. We define $\Gamma_1 \subset \Gamma_0 \subset \Gamma$ as in §**I**.2: the parahoric subgroup Γ_0 is the inverse image under r of $P(k(F))$, and

$$\Gamma_1 = \{g \in \Gamma_0 : r(g) = \begin{pmatrix} 1 & * \\ 0 & *_{n-1} \end{pmatrix}\}.$$

We fix a commutative ring R in which q is invertible and let $\mathcal{H}$, $\mathcal{H}_0$, and $\mathcal{H}_1$ denote the Hecke algebras of H with respect to Γ, Γ_0, and Γ_1, respectively, in each case with R-coefficients. We also let $\mathcal{H}_L$ denote the Hecke algebra of $L(F)$ with respect to its maximal compact subgroup $\Gamma_L = \Gamma \cap L(F)$.

Let $z : F^\times \to H$ denote the embedding of $F^\times$ as diagonal matrices. The map

$$a \to < a >= z(a)K,$$

where $K = \Gamma$, Γ_0, or Γ_1, maps $F^\times$ to a subgroup of the multiplicative group of the corresponding Hecke algebra. The map $<>$ identifies its image with $F^\times/\mathcal{O}_F^\times$ in the case of $\mathcal{H}$ and $\mathcal{H}_0$ and with $F^\times/(1+\mathcal{O}_F)$ in the case of $\mathcal{H}_1$.

Let $\chi = (\chi_1, \chi_2, \ldots, \chi_n)$ be an ordered n-tuple of R-valued characters of $F^\times$. We may regard χ as a character of $T = GL(1,F)^n$, viewed as the Levi factor of B. Then, by composition with the natural projection, χ defines a character of B. We let $i_{H,T}\chi$ denote the non-normalized induced representation

$$i_{H,T}\chi = \{f : H \to R \mid f(bh) = \chi(b)f(h), \forall b \in B, h \in H\}.$$

Let $|\ |_F : F^\times \to q^{\mathbb{Z}}$ be the absolute value character:

$$|x|_F = |k(F)|^{-v_F(x)},$$

where v_F is the valuation on F. Composing with the natural map $q^{\mathbb{Z}} \to R^\times$ we obtain a character $\nu : F^\times \to R^\times$. Define the modulus character $\delta : B \to R^\times$ as the composition $\nu \circ det_{ad}$ where $det_{ad} : B \to F^\times$ is the determinant of the adjoint representation of B on its Lie algebra. We assume q to be a square in $R^\times$ and choose a square root $\sqrt{q}$. Then we can define $\delta^{\frac{1}{2}} : B \to R^\times$, which we write as an n-tuple of characters of $F^\times$:

$$\delta^{\frac{1}{2}} = (\delta_1, \delta_2, \ldots, \delta_n).$$

The Weyl group W of $GL(n)$, which we identify with the symmetric group $\mathfrak{S}_n$, acts on the set $X_R(T)$ of $R^\times$-valued characters of T by permutation:

$$w(\chi) = (\chi_w(1), \chi_w(2), \ldots, \chi_w(n)).$$

We define the twisted action

$$w * \chi = w(\chi) \cdot \delta^{\frac{1}{2}} w(\delta^{-\frac{1}{2}}).$$

The product $\delta^{\frac{1}{2}} w(\delta^{-\frac{1}{2}})$ does not depend on the choice of $\sqrt{q}$.

We will need the following theorem, due to Bernstein and Zelevinski when R is a field of characteristic zero and to Vignéras when R is a field of positive characteristic $\ell \neq q$ [BZ;V1, III. 1.15]:

Theorem II.1.1. *Suppose $R = \overline{\mathbb{Q}}_\ell$ or $R = \overline{\mathbb{F}}_\ell$. Suppose $\chi \in X_R(T)$ has the property that, for all $1 \leq i, j \leq n$, we have*

$$\chi_i \delta_i^{-1} \neq \chi_j \delta_j^{-1} \nu. \qquad (\mathbf{II}.1.1.2)$$

Then $i_{H,T}\chi$ is irreducible.

We note that the irreducibility criterion of Theorem **II**.1.1 does not depend on the choice of $\sqrt{q}$.

Let Q be a standard parabolic subgroup, with Levi decomposition $Q = L_Q N_Q$. For any smooth $R[G]$-module π we define the (non-normalized) Jacquet module $r_{H,L_Q}\pi$ to be π_{N_Q}. Then $r_{H,L_Q}\pi$ is a smooth L_Q module and is admissible if π is.

The parahoric subgroup Γ_0 admits an Iwahori decomposition

$$\Gamma_0 = (\Gamma_0 \cap \overline{N})(\Gamma_L)(\Gamma_0 \cap N). \qquad (\mathbf{II}.1.2)$$

Note that $\Gamma_L \simeq GL(1, O_F) \times GL(n-1, O_F)$. Let α_0 be an $R^\times$-valued character of $GL(1, O_F)/(1 + m_F)$. Then α_0 can be extended to an $R^\times$-valued character σ of Γ_0 that is trivial on $(\Gamma_0 \cap \overline{N}) \times GL(n-1, O_F) \times (\Gamma_0 \cap N)$. Let $\chi \in X_R(T)$ have the property that χ_i is unramified for $i > 1$, and such that χ_1 is trivial on $1 + m_F$. Write $I(\chi) = i_{H,T}(\chi)$. We are interested in $Hom_{\Gamma_0}(\sigma, I(\chi))$.

Lemma II.1.3. *Suppose $R = \overline{\mathbb{Q}}_\ell$ or $R = \overline{\mathbb{F}}_\ell$. Let $\chi \in X_R(T)$ be as above, and suppose $\chi_1|_{GL(1,O_F)} = \alpha_0$. If χ_1 is unramified then $\dim_R Hom_{\Gamma_0}(\sigma, I(\chi)) = n$; if χ_1 is ramified then $\dim_R Hom_{\Gamma_0}(\sigma, I(\chi)) = 1$.*

Proof. The calculation is standard but we include it for completeness. By restriction to Γ we obtain an isomorphism

$$I(\chi) \xrightarrow{\sim} \{f : \Gamma \to R \mid f(bk) = \chi(b) f(k), \forall b \in B \cap \Gamma, k \in \Gamma\}.$$

Let

$$M(\sigma) = \{f : \Gamma \to R \mid f(bk\beta) = \chi(b)\sigma(\beta) f(k), \forall b \in B \cap \Gamma, k \in \Gamma, \beta \in \Gamma_0\}.$$

One checks easily that $\dim_R Hom_{\Gamma_0}(\sigma, I(\chi)) = \dim_R M(\sigma)$. In the unramified case σ is trivial and we find that $\dim_R M(\sigma) = \#(B \cap \Gamma \backslash \Gamma / \Gamma_0) = n$ by the Bruhat decomposition for $GL(n, k(F))$. Indeed, let W_{n-1} denote the Weyl group of $GL(n-1)$, viewed as a subgroup of W. Let $B(m_F) = r(B \cap \Gamma) \subset GL(n, k(F))$, $P(m_F) = r(\Gamma_0) \subset GL(n, k(F))$. Then the Bruhat decomposition yields

$$B \cap \Gamma \backslash \Gamma / \Gamma_0 \simeq B(m_F) \backslash GL(n, k(F)) / P(m_F) \simeq W / W_{n-1}.$$

Let $W^P \subset W$ denote a set of coset representatives of W/W_{n-1}, so that

$$\Gamma = \cup_{w \in W^P} C(w); \quad C(w) = (B \cap \Gamma) \cdot w \cdot \Gamma_0.$$

Thus $f \in M(\sigma)$ is determined by its restrictions f_w to each $C(w)$. Let a be the diagonal matrix $diag(a_1, \ldots, a_n) \in B \cap \Gamma$, with $a_i \in O_F^\times$. In the ramified case we have

$$\chi_1(a_1) f(w) = f(aw) = f(wa^w) = \sigma(a^w) f(w) = f(w)$$

unless $w \in W_{n-1}$. Thus $f_w \equiv 0$ except on the identity coset and $\dim M(\sigma) = 1$.

Let $N_0 \subset Lie(N(F))$ be a lattice in the abelian Lie algebra of $N(F)$. Let $\mathcal{H}_L^- \subset \mathcal{H}_L$ denote the subspace of Hecke operators supported on the union of cosets $\Gamma_L T^- \Gamma_L$ where

$$T^- = \{t \in T \mid ad(t^{-1})(N_0) \subset N_0\}.$$

The elements of T^- are said to be **expanding**.

More generally, we let

$$\mathcal{H}_L(\sigma) = End_L ind_{L,\Gamma_L} \sigma;$$

here *ind* denotes compact induction. Here and in what follows, all endomorphisms are assumed to commute with R. We can identify $\mathcal{H}_L(\sigma)$ with the R-module of compactly-supported functions $b : L(F) {\rightarrow} R$ such that $b(khk') = \sigma(k) b(h) \sigma(k')$ for $h \in L(F)$, $k, k' \in \Gamma_L$. Then the natural algebra structure on $\mathcal{H}_L(\sigma)$ inherited from its definition as endomorphism ring induces the convolution algebra structure on the space of functions. Similarly, we let

$$\mathcal{H}_0(\sigma) = End_H ind_{H,\Gamma_0} \sigma;$$

this is isomorphic to the R-module of compactly-supported functions

$$b : H(F) {\rightarrow} R; b(khk') = \sigma(k) b(h) \sigma(k'), h \in H(F), k, k' \in \Gamma_0.$$

Then we let $\mathcal{H}_L(\sigma)^- \subset \mathcal{H}_L(\sigma)$ be the R-submodule of functions supported on $\Gamma_L T^- \Gamma_L$; this algebra is stable under product. (For all this, cf. [V1,I.8.6]).

For any smooth $R[H]$ module π, the space $Hom_H(ind_{H,\Gamma_0} \sigma, \pi)$ is naturally a right module over $\mathcal{H}_0(\sigma)$. By the universal property of compact induction,

$$Hom_H(ind_{H,\Gamma_0} \sigma, \pi) = Hom_{\Gamma_0}(\sigma, \pi),$$

and therefore the right-hand side is also a right $\mathcal{H}_0(\sigma)$-module. In the same way, $Hom_{\Gamma_L}(\sigma, \pi)$ is a right $\mathcal{H}_L(\sigma)$-module when π is a smooth $R[L]$-module.

The proof of the first part of the following proposition was provided by M.-F. Vignéras, as were the references for the second part.

Proposition II.1.4. *The notation is as in Lemma* **II***.1.3.*

(i) The natural projection $I(\chi) \to r_{H,L}I(\chi)$ *induces an isomorphism of* R*-modules*

$$p : Hom_{\Gamma_0}(\sigma, I(\chi)) \to Hom_{\Gamma_L}(\alpha_0 \otimes 1, r_{H,L}I(\chi)). \tag{II.1.4.1}$$

(ii) The subset $\mathcal{H}_L^- \subset \mathcal{H}_L$ *is a subalgebra and there exists a homomorphism* $T_{H,L}^- : \mathcal{H}_L(\sigma)^- \to \mathcal{H}_0(\sigma)$ *such that*

$$p(fT_{H,L}^-(b)) = p(f)b.$$

Here the right actions of the Hecke algebras are defined as above. The homomorphism is given on functions $b \in \mathcal{H}_L^-$ *by*

$$T_{H,L}^-(b)(khk') = \begin{cases} 0 \text{ if } khk' \notin \Gamma_0 T^- \Gamma_0 \\ \sigma(k)b(h)\sigma(k') \text{ if } h \in T^-, k,\ k' \in \Gamma_0. \end{cases}$$

Proof. (i) We denote by p the natural map (**II**.1.4.1). Maps like p are known to be surjective in great generality, and in particular in the present case [V1,II.3.3]. Thus we have to prove injectivity. We first prove injectivity over $\overline{\mathbb{Q}}_\ell$. Note that the dimension of the left-hand side is n for χ_1 unramified, 1 otherwise. On the other hand, write

$$L(F) = GL(1, F) \times H_{n-1}, \quad H_{n-1} = GL(n-1, F), \tag{II.1.4.2}$$

and let $T_{n-1} = T \cap H_{n-1}$, the intersection taking place in $L(F)$. With this notation, the semisimplification of $r_{H,L}I(\chi)$ as admissible H-module is the direct sum

$$r_{H,L}I(\chi)_{ss} \xrightarrow{\sim} \oplus_{i=1}^n \chi_i \otimes i_{H_{n-1},T_{n-1}}(\chi^i)_{ss}, \tag{II.1.4.3}$$

where $\chi^i = (\chi_1, \ldots, \hat{\chi_i}, \ldots, \chi_n)$, the tensor product corresponding to the decomposition (**II**.1.4.2). In the unramified case each summand has a one-dimensional Γ_L-fixed subspace. Over $\overline{\mathbb{Q}}_\ell$ the action of Γ_L is semisimple, hence the right hand side of (**II**.1.4.1) has dimension n, and p is an isomorphism for reasons of dimension. In the ramified case we have

$$\dim Hom_{\Gamma_L}(\alpha_0 \otimes 1, \chi_1 \otimes i_{H_{n-1},T_{n-1}}(\chi^1)_{ss}) = 1;$$
$$\dim Hom_{\Gamma_L}(\alpha_0 \otimes 1, \chi_i \otimes i_{H_{n-1},T_{n-1}}(\chi^i)_{ss}) = 0,\ i \neq 1.$$

This is because χ^i is ramified for $i \neq 1$. Thus again we have equality of dimensions in (**II**.1.4.1). This completes the proof of (i) over $\overline{\mathbb{Q}}_\ell$.

We complete the proof for $\overline{\mathbb{F}}_\ell$ in the unramified case, the ramified case being similar. Note first that the result for $\overline{\mathbb{Q}}_\ell$ implies the corresponding result over $R = \overline{\mathbb{Z}}_\ell$, the integral closure of $\mathbb{Z}_\ell$ in $\overline{\mathbb{Q}}_\ell$. Indeed, let χ take values in $\overline{\mathbb{Z}}_\ell^\times$ and write $I(\chi)_{\overline{\mathbb{Z}}_\ell}$ for the corresponding induced representation, viewed as a free $\overline{\mathbb{Z}}_\ell$-module. Let $I(\chi)_{\overline{\mathbb{Q}}_\ell}$ denote the induced representation with $\overline{\mathbb{Q}}_\ell$-coefficients. It is known that $I(\chi)_{\overline{\mathbb{Z}}_\ell}^{\Gamma_0}$ is a lattice in $I(\chi)_{\overline{\mathbb{Q}}_\ell}^{\Gamma_0}$, hence is free of rank n. It is also known that $r_{H,L}I(\chi)_{\overline{\mathbb{Z}}_\ell}^{\Gamma_0}$

is a lattice in $r_{H,L}I(\chi)^{\Gamma_0}_{\overline{\mathbb{Q}}_\ell}$ [V1, II.4.14(d)], hence is also free of rank n. Moreover, the map p with coefficients in $\overline{\mathbb{Z}}_\ell$ is still surjective (cf. [V1, II 3.3]), hence is an isomorphism by considerations of rank.

Finally, we suppose $R = \overline{\mathbb{F}}_\ell$. Reduction modulo ℓ defines an injection

$$(\mathbf{II}.1.4.4) \qquad r_{H,L}I(\chi)^{\Gamma_0}_{\overline{\mathbb{Z}}_\ell} \otimes \overline{\mathbb{F}}_\ell \quad \to \quad r_{H,L}I(\chi)^{\Gamma_0}_{\overline{\mathbb{F}}_\ell}.$$

But we have seen that $r_{H,L}I(\chi)^{\Gamma_0}_{\overline{\mathbb{Z}}_\ell}$ is free of rank n. On the other hand, the right-hand side of (**II**.1.4.4) is of dimension $\leq n$. Indeed, the analogue of (**II**.1.4.3) remains true over $\overline{\mathbb{F}}_\ell$, and the Γ_L-fixed subspace of the right-hand side of (**II**.1.4.3) is always of dimension n. Thus $\dim r_{H,L}I(\chi)^{\Gamma_0}_{\overline{\mathbb{F}}_\ell} = n$. But the dimension over $\overline{\mathbb{F}}_\ell$ of the left-hand side of (**II**.1.4.1) still equals n, and we have already seen that p is surjective. This completes the proof of (i).

Assertion (ii) is a special case of [V2,Theorem II.4] and [V2,Lemma II.10.1].

For $i = 1, \ldots, n$, we write $t(\varpi)_i$ for the diagonal matrix with entry ϖ^{-1} in the ith place and 1 elsewhere, and let $U_L \in \mathcal{H}_L$ denote the function equal to 1 on $\Gamma_L t(\varpi)_1 \Gamma_L$ and 0 elsewhere. Then $t(\varpi)_1$ belong to T^-, hence $U_L \in \mathcal{H}_L^-$. We let $U = T^-_{H,L}(U_L) \in \mathcal{H}_?(\sigma)$, for $? = 0$, 1.

For the remainder of this section we assume $q \equiv 1 \pmod{\ell}$. Then the characters ν and δ_i, $i = 1, \ldots, n$ are trivial mod ℓ. We restrict attention to the case of unramified χ for the time being, so σ is the trivial representation. We also work over $R = \overline{\mathbb{F}}_\ell$. Let χ and $I(\chi)$ be as in the statement of Proposition **II**.1.4. The operator U_L has support in the center of $L(F)$, and acts on each constituent of the right-hand side of **II**.1.4.3 as the scalar $\chi_i(\varpi)^{-1}$. It follows from (ii) of Proposition **II**.1.4 that U, viewed as a linear operator on the n-dimensional space $Hom_{\Gamma_0}(\sigma, I(\chi))$ has the n generalized eigenvalues $\chi_i(\varpi)^{-1}$ (counted with multiplicity). Let $\beta_1, \ldots, \beta_r$ denote the r distinct χ_i, m_r the multiplicity of β_r in χ, and assume the χ_i are ordered so that

$$(\mathbf{II}.1.5) \qquad \chi_1 = \cdots = \chi_{m_1} = \beta_1; \ldots, \chi_{m_1+\cdots+m_{j-1}} = \cdots = \chi_{m_1+\cdots+m_j} = \beta_j; \ldots$$

For any R-valued character β of $F^\times$, and any positive integer m, let $\beta[m]$ be the 1-dimensional representation $\beta \circ det$ of $GL(m, F)$. Following the notation of [V2], we write

$$\chi_1 \times \chi_2 \times \cdots \times \chi_n = i_{H,T}\chi.$$

More generally, we let $n = m_1 + \cdots + m_r$ be a partition, $\mathcal{P} = P(m_1, \ldots, m_r)$ the corresponding standard parabolic subgroup of H, and let

$$\beta_1[m_1] \times \ldots \times \beta_r[m_r]$$

denote the full induced representation (when $q \equiv 1 \pmod{\ell}$ normalized and non-normalized induction coincide) from the representation $\beta_1[m_1] \otimes \cdots \otimes \beta_r[m_r]$ of the standard Levi factor $GL(m_1, F) \times \ldots GL(m_r, F)$ of $\mathcal{P}$.

A representation of H is *spherical* if it contains a non-zero fixed vector under Γ. Obviously $\beta_1[m_1] \times \ldots \times \beta_r[m_r]$ is a spherical representation. Conversely,

Proposition II.1.6 [V2]. *Assume $R = \overline{\mathbb{F}}_\ell$ and $q \equiv 1 \pmod{\ell}$. Assume $\beta_i \neq \beta_j$ for $i \neq j$. Then*

(i) $\beta_1[m_1]\times\ldots\times\beta_r[m_r]$ is irreducible;

(ii) Every irreducible spherical R-representation of H is obtained in this way.

(iii) $\dim_R Hom_{\Gamma_L}(1, r_{H,L}\beta_1[m_1]\times\ldots\times\beta_r[m_r]) = r$, and the operator U_L has the r distinct eigenvalues $\beta_i(\varpi)^{-1}$, $i = 1,\ldots,r$.

(iv) The natural projection $\beta_1[m_1]\times\ldots\times\beta_r[m_r] \to r_{H,L}\ \beta_1[m_1]\times\ldots\times\beta_r[m_r]$ induces an isomorphism of R-modules

$$Hom_{\Gamma_0}(1, \beta_1[m_1]\times\ldots\times\beta_r[m_r]) \xrightarrow{\sim} Hom_{\Gamma_L}(1, r_{H,L}\beta_1[m_1]\times\ldots\times\beta_r[m_r])$$

and the operator U on the left-hand side has the r-distinct eigenvalues $\beta_i(\varpi)^{-1}$, $i = 1,\ldots,r$.

Proof. Assertions (i) and (ii) are proved in VI.3 of [V2]. We prove (iii) and (iv) together. First, we can lift β_i to a character $\tilde{\beta}_i$ with values in $\overline{\mathbb{Z}}_\ell^\times$ and define the representations $\tilde{\beta}_i[m_i]$ and $\tilde{\beta}_1[m_1]\times\ldots\times\tilde{\beta}_r[m_r]$ over $\overline{\mathbb{Q}}_\ell$, by the same procedure as above. It is well known that, for appropriate characteristic zero lifts $\tilde{\chi}_j$ of the characters χ_j of (**II**.1.5), there is an embedding

$$f : \tilde{\beta}_1[m_1]\times\ldots\times\tilde{\beta}_r[m_r] \hookrightarrow \tilde{\chi}_1\times\ldots\times\tilde{\chi}_n.$$

Write $\tilde{\pi} = \tilde{\beta}_1[m_1]\times\ldots\times\tilde{\beta}_r[m_r]$, $\tilde{\pi}' = \tilde{\chi}_1\times\ldots\times\tilde{\chi}_n$, and let $\tilde{\pi}''$ denote $\tilde{\pi}'/f(\tilde{\pi})$. There is a commutative diagram

(**II**.1.6.1)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\pi}^{\Gamma_0} & \xrightarrow{f} & (\tilde{\pi}')^{\Gamma_0} & \longrightarrow & (\tilde{\pi}'')^{\Gamma_0} & \longrightarrow & 0 \\ & & {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow & & {\scriptstyle g''}\downarrow & & \\ 0 & \longrightarrow & r_{H,L}(\tilde{\pi})^{\Gamma_L} & \longrightarrow & r_{H,L}(\tilde{\pi}')^{\Gamma_L} & \longrightarrow & r_{H,L}(\tilde{\pi}'')^{\Gamma_L} & \longrightarrow & 0 \end{array}$$

Exactness of the bottom row follows from exactness of the Jacquet functors. Now we have already seen in Proposition **II**.1.4 that g' is an isomorphism. On the other hand, all the vertical arrows are surjective [V1,II.3.3]; hence they are isomorphisms. This proves the analogue of the first part of (iv) over $\overline{\mathbb{Q}}_\ell$. The proof over $\overline{\mathbb{F}}_\ell$ then proceeds as in the proof of Proposition **II**.1.4.

In what follows, we let $\pi = \beta_1[m_1]\times\ldots\times\beta_r[m_r]$. It follows from what we have shown thus far that the dimension in (iii) is equal to

$$dim_R Hom_{\Gamma_0}(1, \pi).$$

Using the fact that $H = \Gamma\cdot\mathcal{P}$, it follows by standard arguments that this dimension equals the cardinality of $\Gamma\cap\mathcal{P}\backslash\Gamma/\Gamma_0$. The Bruhat decomposition then shows that this is equal to

(**II**.1.6.2) $$|\mathfrak{S}_{m_1}\times\cdots\mathfrak{S}_{m_r}\backslash\mathfrak{S}_n/\mathfrak{S}_1\times\mathfrak{S}_{n-1}|,$$

where $\mathfrak{S}_m$ is the symmetric group on m letters. It is elementary that (**II**.1.6.2) equals r.

It remains to prove the assertions about the eigenvalues of U_L and U. Indeed, Proposition **II**.1.4 shows that the assertions for U and for U_L are equivalent. Following a suggestion of Vignéras, we first replace Γ_0 by the standard Iwahori subgroup I_0; this is the subgroup of Γ of matrices whose reductions $\pmod{m_F}$ are upper triangular. There is a commutative diagram analogous to (**II**.1.6.1) when Γ_0 is replaced by I_0 and L is replaced by the diagonal torus T; thus Γ_T is the maximal compact subgroup of T. In this case, the above arguments (cf. [V2, Lemma VI.2 (b)]) show that

$$\pi^{I_0} = |\mathfrak{S}_{m_1} \times \cdots \mathfrak{S}_{m_r} \backslash \mathfrak{S}_n| = \frac{n!}{m_1! \cdot \cdots \cdot m_r!}. \qquad (\mathbf{II}.1.6.3)$$

Let $H_R(n,1)$ denote the Iwahori-Hecke algebra for $GL(n)$ over $R = \overline{\mathbb{F}}_\ell$, with the parameter $q = 1$. Then π^{I_0} is an $H_R(n,1)$-module. Let $\{X_i | i = 1, \ldots, n\}$, $\{S_j | j = 1, \ldots, n-1\}$ be the generators of $H_R(n,1)$ defined in [V1,I.2.14]. Here X_i is the product $T_i \cdot (T_{i-1})^{-1}$, where T_j is the I_0-double coset of the diagonal matrix $\prod_{i \leq j} t(\varpi)_i$, with $t(\varpi)_i$ as above, multiplied by a power of q which under our hypotheses equals 1. Similarly S_j is the I_0-double coset of the standard transposition s_j in the Weyl group of $GL(n)$ which exchanges j and $j+1$. The S_j generate the Hecke algebra $H_R^0(n,1)$ of $GL(n, k(w))$ relative to its Borel subgroup. Since $q = 1$ in R, this is just the R-group algebra $R[\mathfrak{S}_n]$. The $(X_i)^{\pm}$ generate a commutative subalgebra $A \subset H_R(n,1)$, normalized by $H_R^0(n,1)$, and we let $\pi_{ss}^{I_0}$ denote the semi-simplification of π^{I_0} as A-module. Then $\pi_{ss}^{I_0}$ is a sum of characters of A, the *weights* of π, which we identify with unramified characters of the diagonal torus T.

By construction, the character χ occurs non-trivially in $\pi_{ss}^{I_0}$ as the character on the standard function in the induced representation $\beta_1[m_1] \times \ldots \times \beta_r[m_r]$ which equals 1 on Γ. Now the Weyl group $\mathfrak{S}_n$ acts on A by permuting the X_i, and it follows from Corollary 2.12 of [V3] that, if $s \in \mathfrak{S}_n$, then $s(\chi)$ is also a weight of π. More precisely, the reference cited only applies explicitly when s is a simple transposition, but we may induct on the number of simple transpositions occuring in a minimal factorization of an arbitrary s. The set of distinct $s(\chi)$, as s varies through $\mathfrak{S}_n$, has cardinality precisely equal to $\frac{n!}{m_1! \cdots m_r!}$, which is just the dimension of π^{I_0}. In particular, each character of A in the set $\{s(\chi)\}$ occurs with multiplicity one, and the representation π^{I_0} of A is already semi-simple.

Note that the set of $\{s(\chi)\}$ is just the set of n-tuples of characters in which β_i occurs m_i times, with arbitrary order. On the other hand, let $\mathcal{H}_T$ denote the Hecke algebra of T relative to its maximal compact subgroup. Then there is a homomorphism $t_{H,T}^- : \mathcal{H}_T \rightarrow H_R(n,1)$ satisfying the analogue of Proposition **II**.1.4 (ii) (cf. [V2, Theorem II.6]); in particular, we have

$$t_{H,T}^-(t(\varpi)_1) = X_1. \qquad (\mathbf{II}.1.6.4)$$

It follows that, for each $i = 1, \ldots, n$, there is at least one $s \in \mathfrak{S}_n$ for which $s(\chi)(X_1) = \beta_i(\varpi)$.

Now $\pi^{\Gamma_0} \subset \pi^{I_0}$ is the subspace of invariants under the subgroup generated by the transpositions $s_j, j = 2, \ldots, n-1$, hence of the double cosets $S_j, j = 2, \ldots, n-1$ (since $q = 1$ in R). On the other hand, the operator U is the Γ_0 double coset containing X_1; it is of the form $\sum S \cdot X_1 \cdot S'$, where S and S' run through finite sets of finite products of elements of $\{S_j | j > 1\}$. But all such S commute with X_1. The set of eigenvalues of U on π^{Γ_0} thus equals the set of eigenvalues of X_1 on π^{Γ_0}, hence is contained in the set of eigenvalues of X_1 on π^{I_0}. It follows from the above description in terms of the $\{s(\chi)\}$ that the latter set is precisely the set indicated in (iv).

Note that if we assume $\ell > n-1$ we can conclude immediately. Indeed, by projecting π^{I_0} on its $\mathfrak{S}_{n-1}$-invariant subspace π^{Γ_0}, where $\mathfrak{S}_{n-1}$ is generated by the s_j with $j > 1$, we see that X_1 already has r distinct eigenvalues on the latter space. But the above argument shows that every eigenvalue of X_1 on π^{Γ_0} is also an eigenvalue of U.

In general, we will show in the course of the proof of the following lemma – the reader can check that this involves no circularity – that

(**II**.1.6.5) $$U \text{ has at least } r \text{ eigenvalues on } \pi^{\Gamma_0}.$$

This suffices to complete the proof.

The above proof actually provides more information about the spherical vector.

Lemma II.1.7. *We retain the hypotheses of the previous proposition.*

(i) Let $\pi = \beta_1[m_1]\times \ldots \times\beta_r[m_r]$ and let $v_0 \in \pi^{\Gamma}$ be a non-zero spherical vector. Then v_0 generates the Iwahori-fixed subspace π^{I_0} as a module over the commutative algebra A.

(ii) The spherical vector v_0 generates the Γ_0-fixed subspace π^{Γ_0} over the algebra $R[U]$ of polynomials in the operator U.

Proof. It follows from the irreducibility of π that v_0 generates π^{I_0} as module over $H_R(n,1)$. We may characterize $R \cdot v_0 \subset \pi^{I_0}$ as the fixed subspace under $H_R^0(n,1) = R[\mathfrak{S}_n]$. But we have seen in the proof of Proposition **II**.1.6 that $\mathfrak{S}_n$ acts transitively on the weights of π. It then follows formally that every weight of π occurs in the A-module generated by v_0. Since the weights have multiplicity one, this implies (i).

Now let $V_1 \subset \pi^{I_0}$ denote the $R[X_1]$-module generated by v_0. Since the remaining X_i's commute with X_1, the above argument shows that the r distinct eigenvalues $\beta_i(X_1)$, $i = 1, \ldots, r$, all occur in V_1. Now there is a commutative diagram

(**II**.1.7.1)
$$\begin{tikzcd} \pi^{\Gamma_0} \arrow[r, "i"] \arrow[d, "p_{G,L}"'] & \pi^{I_0} \arrow[d, "p_{G,T}"'] \\ r_{H,L}\pi^{\Gamma_L} \arrow[r, "p_{L,T}"] & r_{H,T}\pi. \end{tikzcd}$$

Here i is the inclusion, $p_{G,L}$ is the isomorphism of Proposition **II**.1.4, and the other two maps are constructed analogously via the corresponding Jacquet functors. Applying the analogue of Proposition **II**.1.4 (ii) for the Levi subgroup T of L (cf. [V2, loc. cit.]), and recalling (**II**.1.6.4), we find that

(**II**.1.7.2)
$$\begin{aligned} p_{L,T} \circ p_{G,L}(v_0 U) &= p_{L,T} \circ p_{G,L}(v_0 t(\varpi)_1) \\ &= p_{G,T}(i(v_0) X_1). \end{aligned}$$

We now see that U has r distinct eigenvalues on the subspace $R[U]\cdot v_0 \subset \pi^{I_0}$, which implies **II**.1.6.5 and completes the proof of Proposition **II**.1.6. Since U fixes π^{Γ_0}, this also completes the proof of the lemma.

We now replace $\overline{\mathbb{F}}_\ell$ by the finite field k of characteristic ℓ, and let R be a finite local $\mathbb{Z}_\ell$-algebra with maximal ideal m_R and residue field k; let $x \mapsto \overline{x} : R \rightarrow k$ denote reduction mod m_R. Let $K_R = R \otimes \mathbb{Q}_\ell$; we assume K_R to be a finite direct sum $\oplus_i K^i$ of ℓ-adic fields with the same residue field, although there is probably no reason to exclude nilpotents. Let $\overline{m}_j$ denote the multiplicity of the character $\overline{\beta}_j := \overline{\bullet} \circ \beta_j$. Suppose $\overline{m}_1 = 1$, so that $\chi_1 = \beta_1$ is distinct from the other characters mod m_R, and let $\lambda = \chi_1(\varpi) \in R^\times$. Let π be an admissible $R[H]$-module generated by a spherical vector, say $\mathbf{v}$, corresponding to the same character of $\mathcal{H}$ as $I(\chi)$. We assume $\pi \otimes \mathbb{Q}_\ell$ is a direct sum $\oplus \pi^i$ where π^i is an irreducible spherical $K^i[H]$-module corresponding to the $K^{i,\times}$-valued character χ^i, say. Thus each χ^i takes values in the integers of K^i and reduces modulo the maximal ideal to $\overline{\chi}$.

It follows from Proposition **II**.1.6 that there is a surjective morphism

$$(\mathbf{II}.1.8) \qquad \phi : \overline{\pi} \rightarrow \overline{\beta}_1[m_1] \times \ldots \times \overline{\beta}_r[m_r]$$

of $k[H]$-modules, the right-hand side being the irreducible quotient of $\overline{\pi}$, generated by the image of $\mathbf{v}$. Let $\overline{\mathbf{v_0}}$ generate the 1-dimensional U-eigenspace of $[\overline{\beta}_1[m_1] \times \ldots \times \overline{\beta}_r[m_r]]^{\Gamma_0}$ with eigenvalue $\overline{\lambda}$. It follows from Lemma **II**.1.7 that $\overline{\mathbf{v_0}} \in \phi(k[U]\overline{\mathbf{v}})$. Then, letting

$$(\mathbf{II}.1.9) \qquad X = \prod_{i>1} \frac{U - \beta_i(\varpi)^{-1}}{\lambda - \beta_i(\varpi)^{-1}},$$

it is clear that

$$(\mathbf{II}.1.10) \qquad X(\mathbf{v}) \equiv \overline{\mathbf{v_0}} \pmod{m_R + \ker(\phi)}.$$

In particular, $X(\mathbf{v})$ defines a lifting of the $\overline{\lambda}$-eigenspace of U to $\overline{\pi}$.

Let $\mathbf{V_0} \subset \pi$ denote the R-submodule on which U acts with eigenvalue λ; i.e., $\mathbf{V_0}$ is the intersection of π with the corresponding U-eigenspaces in π^i.

Lemma II.1.11. . *Define π, χ, and β_i as above, and assume χ_1 has multiplicity one mod m_R.*

(i) The R-submodules π^Γ and $\mathbf{V_0}$ of π^{Γ_0} are free of rank 1.

(ii) The limit $X_\infty = lim_{m\rightarrow\infty} X^{\ell^m}$ exists as a continuous operator on π^{G_0} and defines a projection onto $\mathbf{v_0}$. The restriction of X_∞ to the spherical subspace π^G defines an isomorphism

$$X_\infty : \pi^G \rightarrow \mathbf{v_0}$$

of R-modules.

Proof. Part (i) follows from Nakayama's lemma and the corresponding facts for induced representations of H over k. Part (ii) follows from (**II**.1.10) by successive approximation modulo increasing powers of m_R.

II.*2. Notation for Hecke algebras.*

Let $\mathcal{CS}(\mathbb{Q})$ denote the set of all primes of $\mathbb{Q}$, and let $\mathcal{CS}^+(\mathbb{Q})$ be the subset of finite primes that split in $\mathcal{K}_0$ and are unramified in $\mathcal{K}$. For $v \in \mathcal{CS}^+(\mathbb{Q})$ of residue characteristic p we choose a place v_1 of $\mathcal{K}_0$ above v, and let $\Sigma = \Sigma(v_1)$ be the set of primes of $\mathcal{K}$ dividing v_1. Then we have

$$(\mathbf{II}.2.1) \qquad G_v \cong \prod_{w\in\Sigma} GL(n, \mathcal{K}_w)\times\mathbb{Q}_v^\times.$$

The Hecke algebra $\mathbf{T}_v$ of G_v relative to any maximal compact subgroup (conjugate to $\prod_{w\in\Sigma} GL(n, \mathcal{O}_{\mathcal{K}_w})\times\mathbb{Z}_v^\times$) is isomorphic to a polynomial algebra over $\mathbb{Z}[\frac{1}{p}]$ in the variables

$$\{T_{i,w}, i = 1, \ldots, n, T_{n,w}^{-1}, w \in \Sigma; T_{0,v}, T_{0,v}^{-1}\}.$$

Here $\mathbb{Z}[T_{0,v}, T_{0,v}^{-1}]$ is the Hecke algebra of the factor $\mathbb{Q}_v^\times$ in **II**.2.1; by abuse of language we refer to the $T_{i,w}$ as the Hecke operators at w, or at the prime of E below w. The Hecke operators at w are normalized so that

$$(\mathbf{II}.2.1) \qquad P_w(q^{-s}) = 1 + \sum_{i=1}^{n}(-1)^i T_{i,w} q^{-is}$$

is the local Euler factor at w of the motivically normalized standard L-function of $GL(n)$. Here q is the order of the residue field $k(w)$ and the inverse roots of $P_w(X)$ are the Satake parameters, multiplied by $q^{(n-1)/2}$. Up to canonical isomorphism the algebra $\mathbf{T}_v$ does not depend on the choice of v_1 above v.

The global Hecke algebra $\mathbf{T}$ is the tensor product over $v \in \mathcal{CS}^+(\mathbb{Q})$ of the $\mathbf{T}_v$. If S is a finite subset of $\mathcal{CS}^+(\mathbb{Q})$, we let $\mathbf{T}^S \subset \mathbf{T}$ be the subalgebra generated by the $\mathbf{T}_v$ for $v \notin S$.

Suppose $q \in \mathcal{CS}^+(\mathbb{Q})$, v_1 a divisor of q in $\mathcal{K}_0$ and let $\Sigma(v_1)$ be as above. For $\mathfrak{q} \in \Sigma(v_1)$, we consider the parabolic subgroup $P = LN$ of type $(1, n-1)$ in $H = GL(n, \mathcal{K}_w)$, as in §**II**.1, and let Γ_0 be the corresponding parahoric subgroup. Let $T'_{i,\mathfrak{q}}$, $1 \leq i \leq n-1$, denote the Hecke operator as above for the factor $GL(n-1, \mathcal{K}_\mathfrak{q})$ of L, and define

$$(\mathbf{II}.2.2) \qquad V_{i,\mathfrak{q}} = T_{H,L}^{-}(T_{n-1,\mathfrak{q}}^{\prime,-1} \cdot T'_{n-i,\mathfrak{q}}) \in \mathcal{H}_?,\ 1 \leq i \leq n-1,\ ? = 0, 1.$$

Here we have realized $T_{n-1,\mathfrak{q}}^{\prime,-1} \cdot T'_{n-i,\mathfrak{q}}$ as an element of $\mathcal{H}_L^-$ in the obvious way.

II.2.3. Let $\varpi \in \mathcal{K}_w$ be a uniformizer. Suppose $R = \overline{\mathbb{Q}}_\ell$ and $\chi \in X_R(T)$ are as in §**II**.1, and let $I(\chi)$ be the induced representation with coefficients in R. Let $a \in r_{H,L}I(\chi)$ be a non-zero eigenvector in the $\chi_1(\varpi^{-1})$-eigenspace for the operator U. Assume a is a spherical vector for Γ_L; then a is unique up to scalar multiples. Moreover, a is an eigenvector for the Hecke operators $T'_{i,\mathfrak{q}}$ defined above. We denote by $b^i(\chi;\chi_1)$ the eigenvalue of $T_{n-1,\mathfrak{q}}^{\prime,-1} \cdot T'_{n-i,\mathfrak{q}}$, acting on a.

II.2.4: The U-operator. Choose $\mathfrak{q} \in Q$, and define $\Gamma_{1,\mathfrak{q}} \subset GL(n, \mathbb{Z}_q)$ as in §**I**.2. We consider the double coset $U_\mathfrak{q} = \Gamma_{1,\mathfrak{q}} \cdot t(\varpi)_1 \Gamma_{1,\mathfrak{q}} \subset GL(n, \mathbb{Q}_q)$. Here and in what follows we are identifying $GL(n, \mathbb{Q}_q) = U(D^\#)_\mathfrak{q}$ with $D_{\mathfrak{q}^{(2)}}^{\#,\times}$. The identification with

$D^{\#,\times}_{\mathfrak{q}^{(1)}}$ identifies $U_{\mathfrak{q}}$ with the double coset $\Gamma'_{1,\mathfrak{q}} \cdot (t(\varpi)_1)^{-1}\Gamma'_{1,\mathfrak{q}}$, where $\Gamma'_{1,\mathfrak{q}}$ consists of invertible matrices over $\mathbb{Z}_q$ congruent to $\begin{pmatrix} 1 & 0 \\ * & *_{n-1} \end{pmatrix} \pmod{q}$.

One verifies immediately that

$$U_{\mathfrak{q}} = \coprod \Gamma_{1,\mathfrak{q}} \cdot t(\varpi)_1 \cdot \begin{pmatrix} 1 & \mathbf{b} \\ 0 & I_{n-1} \end{pmatrix}, \tag{\textbf{II}.2.4.1}$$

where $\mathbf{b} = (b_1, \ldots, b_{n-1})$ runs through $(\mathbb{Z}_q/q\mathbb{Z}_q)^{n-1}$. Write $W = \mathbb{Q}_q^n$. Let $\Lambda = \mathbb{Z}_q^n \subset W$ be the standard lattice, and let λ denote the row vector $(q^{-1}, 1, \ldots, 1)$, so that

$$\Gamma_{1,\mathfrak{q}} = \{g \in GL(W) \mid g(\Lambda) = \Lambda, g(\lambda) = \lambda \pmod{()\Lambda}\}.$$

Let Λ' be the lattice generated by Λ and λ. Then the description (**III**.2.2) shows that, in the standard right action on lattices, we have

$$(\Lambda, \lambda) \cdot U_{\mathfrak{q}} = \{(\Lambda', \lambda') \mid q\lambda' = \lambda \pmod{\Lambda})\}. \tag{\textbf{II}.2.4.2}$$

III. Unitary automorphic forms and λ-adic representations

III.1. Review of λ-adic representations.

We return to the situation of §**I**. Let SC, Q, $\mathfrak{r}$, $K_{1,Q}$, V, and σ be as in **I**.2. We assume throughout that ℓ is unramified in $\mathcal{K}$ and that $K_{1,Q}$ contains a hyperspecial maximal compact subgroup of $J(\mathbb{Q}_\ell)$. The primes in Q are denoted w rather than $\mathfrak{q}$, and the primes above them in $\mathcal{K}$ are denoted w_1 and w_2. The identification $\Gamma_w \simeq GL(n, \mathbb{Q}_q)$ for $w \in Q$ is made via w_2, so that $K_{0,Q}$ is upper triangular parahoric. Let $\pi \in Coh(J, V)$, in the notation of **I**.3, and assume π satisfies the hypothesis of Proposition **I**.2.10:

$$(\mathbf{III}.1.1) \qquad Hom_{K_{1,Q}}(\sigma, \pi) \neq 0.$$

At primes that ramify in $\mathcal{K}/\mathbb{Q}$ we assume π_v to be spherical with respect to a very special compact open subgroup K_v, cf. **I**.3.4.

We assume π satisfies either condition (i) or (ii) of Theorem **I**.3.3. Then π admits an automorphic base change. Let Π be the automorphic representation of $GL(n)_{\mathcal{K}}$ denoted $BC(\pi)_1$ in **I**.3. It follows as in [CL], *** that Π satisfies the following three hypotheses:

(i) Π_∞ is a cohomological representation
(ii) $\Pi \circ c \xrightarrow{\sim} \Pi^\vee$ (Π is θ stable, where θ is the involution of J'_E corresponding to the group J');
(iii) For some finite place v of $\mathcal{K}$, Π_v is supercuspidal

Let $\mathcal{E}(\pi)$ be a number field which is simultaneously a field of definition for π_f and V, and let λ be a prime of $\mathcal{E}(\pi)$ dividing ℓ. Let $\mathcal{O} = \mathcal{O}_\lambda$ be the ring of integers of E_λ, $\mathfrak{m}_\lambda$ its maximal ideal, and $k = k(\lambda)$ its residue field. Under these hypotheses, it is shown in [HT1], following Clozel and Kottwitz, how to associate a continuous representation[1]

$$r_\rho(\pi) : G_{\mathcal{K}} \rightarrow GL(n, \mathcal{O}).$$

where henceforth we write $G_{\mathcal{K}}$ for $Gal(\overline{\mathcal{K}}/\mathcal{K})$. In our applications the residual representation over k will be irreducible, so the lattice will be unique up to scalar multiplication.

Next, we let G_E denote $Gal(\overline{E}/E)$, and define

$$\rho(\pi) = Ind_{\mathcal{K}/E} r_\rho(\pi)$$

to be the induced $2n$-dimensional representation of G_E, acting on the $\mathcal{O}$-module $Ind_{\mathcal{K}/E} A_\lambda[\pi_f]$. Finally, let

$$\overline{\rho} = \overline{\rho}(\pi) : Gal(\overline{E}/E) \rightarrow Aut(Ind_{\mathcal{K}/E} A_\lambda[\pi_f] \otimes_{\mathcal{O}} k)$$

denote the residual representation, and let $r_{\overline{\rho}} = r_{\overline{\rho}}(\pi)$ denote the restriction of $\overline{\rho}$ to $G_{\mathcal{K}}$.

[1] More precisely, [HT1] defines a Galois representation over some finite extension of $\mathcal{E}(\pi)$. Since the Galois group is compact, it necessarily stabilizes an integer lattice. For the purposes of this paper the descent to $\mathcal{E}(\pi)$ is not strictly necessary, however COMPLETE???

Proposition III.1.3. *Suppose SC is non-empty and contains a place w such that π_w is either supercuspidal (if $J'_w \cong GL(n,E_w)$) or corresponds under the Jacquet-Langlands correspondence to a supercuspidal representation of $GL(n,E_w)$ (if J'_w is isomorphic to the multiplicative group of a division algebra). Then $r_\rho(\pi)$ is absolutely irreducible. Moreover, suppose $\ell \nmid |GL(n,k(w))|$ (i.e., ℓ is* **banal** *for $GL(n,E_w)$, in the sense of Vignéras [V1]). Then $r_{\overline{\rho}}(\pi)$ is absolutely irreducible.*

The proof will be given at the end of this section.

For any prime v of $\mathbb{Q}$ that splits in $\mathcal{K}_0$ and is unramified in $\mathcal{K}$, and such that π_v is unramified we let $\phi_{v,\pi} : \mathbf{T}_v \to E(\pi)$ denote the character by which the local unramified Hecke algebra acts on $\pi_v^{K_v}$. Let $S(\pi)$ be the set of all such unramified primes, let $\mathbf{T}^S$ be the corresponding global Hecke algebra, as in §**II**.2, and let $\phi_\pi : \mathbf{T}^S \to E(\pi)$ be the corresponding character; ϕ_π gives the natural action of $\mathbf{T}^{S(\pi)}$ on π^K. If w is a prime of E dividing some prime in $S(\pi)$, we let

$$\text{(III.1.4)} \qquad \phi_\pi(P_w)(X) = 1 + \sum_{i=1}^{n} (-1)^i \phi(\pi)(T_{i,w}) X^i$$

in the notation of (**II**.2.1).

Theorem III.1.5. *Suppose π contains a fixed vector for a hyperspecial maximal compact subgroup of $J_{q(\mathfrak{r})}$ (i.e., not only a fixed vector for $K_{q(\mathfrak{r})}$, as implied by (**III**.1.1). Let S^{bad} be the set of primes of $\mathcal{K}$ dividing primes in $Q \cup SC$ or of residue characteristic ℓ. Then the representation $r_\rho(\pi)$ is unramified outside S^{bad}. Moreover, for all but finitely many primes w of E dividing rational primes in $S(\pi)$, there is a prime w_1 of $\mathcal{K}$ dividing w and such that the arithmetic Frobenius $Frob_{w_1}$ satisfies*

$$\phi_\pi(P_w)(Frob_{w_1}) = 0.$$

Remark. The choice of w_1 in the above theorem is determined as in **II**.2 by the choice of identification of J'_w with a general linear group. The base change of π to $\mathcal{K}$ is conjugate self-dual, and one verifies that, if w_2 is the other prime dividing w, then $\phi_\pi(P_w)(q^{n-1}Frob_{w_2}^{-1}) = 0$. Bear in mind also that the natural action on the cohomology of the Shimura variety is that of the Galois group of $\overline{\mathbb{Q}}$ over the reflex field, and that there is an implicit identification of the reflex field with $\mathcal{K}$.

An alternative way of phrasing this theorem is in terms of partial L-functions: there is a finite set S of finite primes such that we have the equality of Euler products

$$\text{(III.1.5.1)} \qquad L^S(s, r_\rho(\pi)) = L^S(s - \frac{n-1}{2}, BC_{\mathcal{K}}(\pi)).$$

Here the right-hand side is the standard L-function with the unitary (Langlands) normalization; the superscript ^{S} indicates that factors at S have been removed. As mentioned above, we have normalized r_ρ to make (**III**.1.5.1) true; cf. the discussion on p. 100 ff. of [H1].

Proof. The proof of this theorem is mainly due to Kottwitz, and is contained in [K2]. Specifically, Kottwitz proves there that $r_\rho(\pi)$ is unramified outside S^{bad}, except possibly at primes of $\mathcal{K}$ ramified over $\mathbb{Q}$, and shows that the arithmetic Frobenius satisfies the Hecke polynomial for all but finitely many unramified places. The proof is completed in Theorem VII.1.9 of [HT1]; by (2) of that theorem, it suffices to verify that Π_v is unramified for all $v \notin Q \cup SC$, and this follows from Proposition **I**.3.4.

For each archimedean place τ of E, let

$$\underline{a}(\tau) = (a_1(\tau) \geq \cdots \geq a_n(\tau))$$

be the highest weight of the τ-component V_τ of the representation V of $J'(\mathbb{R})$, with respect to some maximal torus; let $\underline{a} = (\underline{a}(\tau))_{\tau \in \Sigma_E}$. For each τ, let

$$(\mathbf{III}.1.6) \qquad n(\tau, V) = inf\{n_1 + n_2 \mid V_\tau \subset St^{\otimes n_1} \otimes St^{\otimes n_2,*}\}$$

where St denotes the standard representation of $GL(n)$ and inclusion is as a direct summand. Let $n(V) = n - 1 + \sum_{\tau \in \Sigma_E} n(\tau, V)$.

Proposition III.1.7. *Under the hypotheses of Theorem* **III**.*1.5, the representation* $\rho(\pi)$ *is crystalline at all primes of* $\mathcal{K}$ *dividing* ℓ. *Suppose moreover that* $\ell > n(V)+1$. *Then the representation on* $A_\lambda[\pi_f]$ *is crystalline in the sense of belonging to the category* $\mathbf{Rep}_{\mathcal{O},cris,[0,\ell-1[}$ *defined in* **IV**.*4, below.*

Proof. Let w be a prime of $\mathcal{K}$ dividing ℓ, $\mathcal{O}_w$ the w-adic completion of $\mathcal{O}_{\mathcal{K}}$. By ℓ-adic Hodge theory [FM,F] it suffices to show that $A_\lambda[\pi_f]$ occurs as a Grothendieck submotive of the cohomology of a smooth scheme $\mathcal{A}_V$ of dimension $n(V)$ over $Spec\ \mathcal{O}_w$, where the projectors are given by correspondences with ℓ-integral coefficients. If V is the trivial representation then this is clear. The general case requires a standard argument to relate $M_\lambda[\pi_f]$ to the cohomology of a Kuga fiber variety. We omit this argument, since only the case of trivial V will be considered in the remainder of this paper.

Theorem III.1.8. *Let* $w \in Q$ *of residue characteristic* q. *Let* w_1 *be the prime of* $\mathcal{K}$ *above* w *and let* $Z_w \subset G_E$ *denote a decomposition group. Suppose* π_w *is in case (b) of Proposition* **I**.*2.10. Let* $(\alpha, \beta_1, \ldots, \beta_{n-1})$ *be the corresponding* n*-tuple of characters. Then* $r_\rho(\pi)|_{Z_w}$ *breaks up as a direct sum*

$$r_\rho(\pi)|_{Z_w} \xrightarrow{\sim} A \oplus B.$$

Here B *is an unramified representation and the inertia subgroup* I_w *of* Z_w *acts on* A *via the restriction to* I_w *of the character associated to* α *via local class field theory.*

This is a special case of Theorem VII.1.7 of [HT1]; an earlier proof has been published in [HT2].
Proof of Proposition **III**.*1.3* Now let $w \in SC$ be the place mentioned in the statement of the proposition. Let $\sigma(\pi_w)$ denote the ℓ-adic representation of the decomposition group Z_w associated to π_w by the local Langlands correspondence [HT1,He]. Proposition **III**.1.3 is a consequence of the following two theorems and the multiplicity one theorem (Theorem **I**.3.7).

Theorem III.1.9 [H1,§3; HT1]. *Under the hypotheses of this section, let $w \in SC$. Then $r_\rho(\pi)|_{Z_w} \cong \sigma(\pi_w)$.*

Theorem III.1.10. *Let F be a p-adic field, $p \neq \ell$, and let τ be a supercuspidal representation of $GL(n,F)$ with coefficients in the ℓ-adic field L, with continuous central character ω_τ.*

(a)([BHK], Theorem 3.3) Let $\sigma(\tau)$ be the ℓ-adic representation of W_F associated to τ by the correspondence of [H1]. Then $\sigma(\tau)$ is irreducible.

(b) ([V3], 1.20) Moreover, suppose ω_τ is ℓ-adically integral, and suppose ℓ is banal for $GL(n,F)$. Then the image of $\sigma(\tau)$ fixes a lattice A over the integer ring $\mathcal{O}_L \subset L$, and the residual representation $\overline{\sigma}(\tau)$ on $A \otimes k(L)$ is irreducible; here $k(L)$ is the residue field of $\mathcal{O}_L$.

Proof. The article [BHK], where part (a) is stated as Theorem 3.3, only treats the case of characteristic zero coefficients. The argument in 1.20 of [V3] shows that the semisimplified reduction $\pmod{\ell}$ decomposes as the sum of m irreducible representations of dimension $e = \frac{n}{m}$. Here if $m > 1$ then m is a divisor of n of the form $\ell^y \cdot \epsilon(|k(E)|^d)$, where y is a non-negative integer, necessarily equal to 0 since $\ell > n$ (ℓ is banal). Next, d is a divisor of e, E is a tamely ramified extension of F, with residue field $k(E)$, such that $[E : F] = \frac{e}{d}$, and $\epsilon(\bullet)$ is the order of the integer $\bullet$ in $\mathbb{F}_\ell^\times$. Thus $|k(F)|^e = |k(E)|^d$, hence $m = \epsilon(|k(E)|^d) = \epsilon(|k(F)|^{\frac{n}{m}})$. The hypothesis that ℓ is banal implies that $\epsilon(|k(F)|) > n$, thus that $\epsilon(|k(F)|^{\frac{n}{m}} > m$, which is a contradiction unless $m = 1$. In other words, $\overline{\sigma}(\tau)$ is irreducible.

III.2. Representations over the Hecke algebra

We now introduce the ℓ-adic Hecke algebras of operators on spaces of automorphic forms; these will be indicated by $\mathbb{T}$'s (as opposed to $\mathbf{T}$'s). Fix a $\pi \in Coh(J,V)$ as above, and let S be a finite set of rational primes including ℓ and all primes bad for π. We will allow S to grow as necessary; in particular, when we choose a finite set Q of auxiliary primes we will assume that S contains Q. Let $\mathbf{T}^S$ be the tensor product over all $\mathbf{T}_v$ as in §**II**.2, where $v \notin S$ runs through rational primes that split in $\mathcal{K}_0$. We define σ and Λ_{W_σ} for π as in §**I**.2.8. Then $\mathbf{T}^S$ acts on the modules $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K)$ of ℓ-adic automorphic forms, introduced in §**I**.2, when $K = K_{0,Q}$ or $K_{1,Q}$. We let $\mathbb{T}_{K,\sigma}$ denote the complete $\mathbb{Z}_\ell$-subalgebra of $End(\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K))$ generated by $\mathbf{T}^S$ and by the operators $U_\mathfrak{q}$ and $V_{i,\mathfrak{q}}$, defined as in §§**III**.2 and **II**.2, respectively, for $\mathfrak{q} \in Q$, $i = 1, \dots, n-1$.

Define $K_{1,Q}^{[\ell]} \subset K_{0,Q}$ as in §**I**.2. When $K = K_{0,Q}$, (resp. $K_{1,Q}^{[\ell]}$) we write $\mathbb{T}_0(Q)$ (resp. $\mathbb{T}_1(Q)$) for $\mathbb{T}_{K,\sigma}$. It follows from the characteristic zero theory that the algebras $\mathbb{T}_i(Q) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ are semisimple, $i = 0, 1$. The choice of a component $\mathfrak{m}$ of $\mathbb{T}_i(Q) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ determines the local Galois representation $r_\rho(\pi)$ locally at all but finitely many primes that split in $\mathcal{K}/E$, for any π that corresponding to $\mathfrak{m}$. The set of such primes has Dirichlet density one, hence determines $r_\rho(\pi)$, and therefore the partial L-function $L^S(s - \frac{n-1}{2}, BC_\mathcal{K}(\pi))$ via formula **III**.1.5.1, for some finite set S of bad primes.

We now fix $i = 0$ or 1 and let $\mathbb{T}$ denote the complete subalgebra of $\mathbb{T}_i(Q)$ for $i = 0, 1$ generated only by the unramified Hecke operators in $\mathbf{T}^S$. Let $\mathfrak{m}$ denote a maximal ideal of $\mathbb{T}_i(Q)$, and denote by $L_{\mathfrak{m}}$ the localization of $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K)$ at $\mathfrak{m}$; in §**V** this will be denoted $L_{i,Q}$. Let $M(\mathfrak{m})$ denote the set of distinct irreducible components of $L_{\mathfrak{m}} \otimes \mathbb{Q}_\ell$ and let $C(\mathfrak{m})$ denote the corresponding set of automorphic representations of J; we assume our chosen π belongs to $C(\mathfrak{m})$. For $\pi' \in C(\mathfrak{m})$ we let $E(\pi')$ denote the corresponding direct factor of $\mathbb{T} \otimes \mathbb{Q}_\ell$. Then the λ-adic representation $\rho(\pi')$ can be realized over $E(\pi')$.

Say there are r distinct elements in $M(\mathfrak{m})$, and let $\{\pi_1, \ldots, \pi_r\}$ be a corresponding set of elements of $C(\mathfrak{m})$. The sum of the ℓ-adic representations $r_\rho(\pi_i)$ for $i = 1, \ldots, r$ can be viewed ad hoc as a $E(\pi_1) \times \cdots \times E(\pi_r) = \mathbb{T} \otimes \mathbb{Q}_\ell$-module. Denote this Hecke module $V(\mathfrak{m})$, with action $r_\rho(\mathfrak{m})$ of $Gal(\overline{\mathbb{Q}}/\mathcal{K})$.

Proposition III.2.1. *Assume the residual representation $r_{\overline{\rho}}(\pi)$ is absolutely irreducible. Then the action of $Gal(\overline{\mathbb{Q}}/\mathcal{K})$ on $V(\mathfrak{m})$ can be realized over the localization $\mathbb{T}_{\mathfrak{m}}$ of $\mathbb{T}$ at $\mathfrak{m}$.*

Proof. This is a simple application of Carayol's theorem [Ca, Théorème 1.2]. By construction, the traces of $r_\rho(\mathfrak{m})(Frob_w)$ lie in $\mathbb{T}_{\mathfrak{m}}$ for almost all w that split in $\mathcal{K}/E$. This is a dense subset of $Gal(\overline{\mathbb{Q}}/\mathcal{K})$. It is now clear that $r_\rho(\mathfrak{m})$ satisfies the hypotheses of Carayol's theorem, the most important being absolute irreducibility of the residual representation.

III.3. Action of complex conjugation.

Let k be a field of characteristic $\neq 2$, V an n-dimensional vector space over k, and let $r_\phi : G_{\mathcal{K}} \to GL(V)$ be an irreducible representation. Let $c \in G_E$ be any complex conjugation, and let ϕ^c denote the representation $g \mapsto \phi(cgc^{-1})$. Assume there exists a character $\chi : G_E \to k^\times$ such that

$$r_\phi^c \cong \check{r}_\phi \otimes \chi \tag{**III**.3.1}$$

as representations of G_E. Choosing dual bases, we identify V and $\check{V}$ with k^n. Then for $g \in G_{\mathcal{K}}$,

$$\phi(g) = (r_\phi(g), \check{r}_\phi \cdot \chi(g)) \in GL(n,k) \times GL(n,k) \tag{**III**.3.2}$$

lies in the group $\tilde{G}(k)$ defined in (**I**.1.5).

There is an isomorphism $B : \check{V} \to V$ intertwining the representation $\check{r}_\phi \otimes \chi$ of $G_{\mathcal{K}}$ on $\check{V}$ with the representation r_ϕ^c on V:

$$B\check{r}_\phi \cdot \chi(g) B^{-1} = r_\phi^c(g). \tag{**III**.3.3}$$

By Schur's Lemma B is unique up to scalar multiples.

Lemma III.3.4. *The intertwining map B is either symmetric or skew-symmetric, and this is independent of the choice of B and of the complex conjugation c.*

Proof.. When χ is the trivial character, this follows from Lemma 15.1.1 of [R2], bearing in mind that $c^2 = 1$. The same proof works for general χ.

Let $\epsilon(r_\phi) = 1$ if B is symmetric, $= -1$ if B is skewsymmetric. Evidently $\epsilon(r_\phi) = 1$ if n is odd. In terms of the chosen bases, we identify B with an invertible $n \times n$ matrix. Then it follows from Lemma **III**.3.4 that the element

(**III**.3.5)
$$\phi(c) = (B, B^{-1}) \ltimes c = (B, {}^tB^{-1} \cdot \epsilon(r_\phi)) \ltimes c \in [GL(n,k) \times GL(n,k)] \ltimes Gal(\mathcal{K}/E)$$

actually belongs to $\tilde{G}(k)$. One verifies immediately that

$$\phi(c)^2 = 1$$

and (**III**.3.3) translates into the relation

$$\phi(c)\phi(g)\phi(c)^{-1} = \phi(cgc^{-1}).$$

Thus

Lemma III.3.6. *The map*

$$\phi : G_E \cong G_{\mathcal{K}} \ltimes \{1, c\} \to \tilde{G}(k)$$

is a homomorphism of groups.

We return to the notation of §**III**.1. It follows from (**III**.1.5.1) and Chebotarev density that $r_\phi = r_{\overline{\rho}}(\pi)$ satisfies (**III**.3.1) with $\chi = \omega^{1-n}$, where ω is the cyclotomic character. Let $r_\rho(\pi) \to GL(n, \mathcal{O})$ be as in §**III**.1. Then the argument above applies to yield

Corollary III.3.7. *Suppose SC is non-empty, so $r_\rho(\pi)$ is absolutely irreducible. Then $\rho(\pi)$ factors through a homomorphism to the L-group $\tilde{G}(\mathcal{O})$. More precisely, there is a sequence of maps*

$$Gal(\overline{E}/E) \to \tilde{G}(\mathcal{O}) \to Aut(Ind_{\mathcal{K}/E} A_\lambda[\pi_f])$$

whose composite is equivalent to $\rho(\pi)$. Moreover, the restriction to $G_{\mathcal{K}}$ of the composite $\nu \circ \rho(\pi) : Gal(\overline{E}/E) \to \mathcal{O}^\times$ equals the $(1-n)$th power of the cyclotomic character; here ν is the similitude character defined at the end of §**I**.*1.*

Let

(**III**.3.8)
$$\omega_\pi(1-n) = \nu \circ \rho(\pi)$$

be the character of G_E defined by Corollary **III**.3.7. It is determined by its restriction to $G_{\mathcal{K}}$ and by the formula

(**III**.3.9)
$$\omega_\pi(1-n)(c) = \epsilon(r_\rho(\pi)),$$

which is immediate from (**III**.3.5).

Let $ad\ r_{\overline{\rho}}(\pi)$ (resp. $ad\ r_\rho$) denote the adjoint representation G_E on the Lie algebra $M(n,k)$ of $\tilde{G}(k)$ (resp. on $M(n, \mathcal{O})$). It is immediate that

$$ad\ r_{\overline{\rho}}(\pi)(c)(X) = -B^t X B^{-1},\ X \in M(n,k);$$

the same formula holds for $ad\ r_\rho$.

Lemma III.3.10. *Let c denote any complex conjugation in G_E. Then the dimension of the $+1$-eigenspace of ad $r_{\overline{\rho}}(\pi)(c)$ equals $\frac{n(n-1)}{2}$ if $\epsilon(r_\phi) = 1$, $\frac{n^2}{2}$ if $\epsilon(r_\phi) = -1$. The same formula holds for ad r_ρ.*

In particular, the dimension is always $\geq \frac{n(n-1)}{2}$.

Proof. This is an elementary calculation.

The results of the present section apply to the representations over the Hecke algebras constructed in §**III**.2. In particular, the representation of $G_{\mathcal{K}}$ constructed in Proposition **III**.2.1 extends to a homomorphism from G_E to $\tilde{G}(\mathbb{T}_{\mathfrak{m}})$.

IV. Deformation of Galois representations

IV.1. *Definition of the deformation problem.* As in the previous sections, we let $G_E = Gal(\overline{\mathbb{Q}}/E)$, $G_{\mathcal{K}} = Gal(\overline{\mathbb{Q}}/\mathcal{K})$. A decomposition group at the prime v is denoted Z_v, the inertia subgroup by I_v. The L-group $\tilde{G}$, viewed as a $\mathbb{Z}$-group scheme, is as in (**I**.1.5). We fix a prime ℓ, *unramified* in $\mathcal{K}$, and a representation $\overline{\rho} : G_E \to \tilde{G}(\overline{\mathbb{F}}_l)$, and let $r_{\overline{\rho}}$ denote the restriction of $\overline{\rho}$ to $G_{\mathcal{K}}$. The representation $\overline{\rho}$ is assumed to satisfy the following conditions:

IV.1.1.0. *There is a finite subfield $k \subset \overline{\mathbb{F}}_l$ such that $\overline{\rho}$ takes values in $\tilde{G}(k)$.*

IV.1.1.1. *The composite $G_E \to \tilde{G}(\overline{\mathbb{F}}_l) \to \{1, c\}$ cuts out $\mathcal{K}/E$.*

IV.1.1.2. *$r_{\overline{\rho}}$ is unramified except at primes above ℓ and above a non-empty finite set of primes SC of E. At primes above ℓ, $r_{\overline{\rho}}$ is crystalline. If $\mathfrak{p} \in SC$ then $\mathfrak{p} = vv^c$ splits in $\mathcal{K}$ and $r_{\overline{\rho}}|_{Z_v}$ breaks up as a direct sum of irreducible representations $\mathfrak{r}_{i,v}$. Moreover, there is at least one $\mathfrak{p} \in SC$ such that $r_{\overline{\rho}}|_{Z_v}$ is irreducible, with v as above.*

IV.1.1.3. *Denote by c any lifting of c to a complex conjugation in G_E. In the adjoint representation $ad\,\overline{\rho}$ of G_E on $Lie(\tilde{G})$, the $+1$-eigenspace of c has dimension $\geq \frac{n(n-1)}{2}$.*

IV.1.1.4. *The composite $\omega_{\overline{\rho}} = \nu \circ \overline{\rho} : G_E \to k^\times$, restricted to $G_{\mathcal{K}}$, equals the $(1-n)$th power of the cyclotomic character, where $\nu : \tilde{G} \to GL(1)$ is the similitude character defined in* §**I**.*1.*

Here and in what follows the term "crystalline," applied to ℓ-torsion modules, is used to refer to Galois representations obtained by the Fontaine-Laffaille construction (see **IV**.4.3, below). The details of this theory are recalled in §**IV**.4.

Lemma IV.1.1.5. *In the situation of* **IV**.*1.1.2, suppose ℓ is banal for $GL(n, E_{\mathfrak{p}})$ (cf. Proposition* **III**.*1.3). Then $\ell \nmid \#\overline{\rho}(I_{\mathfrak{p}})$ and, for any lifting ρ of $\overline{\rho}$ to characteristic zero, $\ell \nmid \#\rho(I_{\mathfrak{p}})$.*

Proof. Let ρ be a lifting as in the statement of the lemma. It is irreducible, hence $\rho(I_{\mathfrak{p}})$ is a finite group. Let $q = Nv$. Let $I = I_v$, the inertia subgroup of Z_v, and let $P \subset I$ be the wild ramification subgroup (i.e., the p-Sylow subgroup, where p is the residue characteristic of $\mathfrak{p}$). By the banality assumption $\ell > n \geq 2$, so it suffices to show that ℓ is prime to the image of $r_{\overline{\rho}}(I)$. Let $g \in \mathbb{Z}_\ell(1) \subset I/P$ be a topological generator and assume some lift $\tilde{g} \in I$ has non-trivial image under ρ. Let $\sigma \in Z_v$ be a Frobenius element. It acts on I by conjugation, and $\tilde{g}^\sigma = \tilde{g}^q \cdot b$ for some $b \in P$. Then the set of eigenvalues of $\rho(\tilde{g})$ and of $\rho(\tilde{g}^q \cdot b)$ are the same. The closed subgroup $< \tilde{g}^q \cdot b > \subset I$ topologically generated by $\tilde{g}^q \cdot b$ maps bijectively to $\mathbb{Z}_\ell(1)$ under projection to I/P; let $g' \in < \tilde{g}^q \cdot b >$ be the lifting of g, so that $(g')^q = \tilde{g}^q \cdot b$. By conjugacy of Sylow ℓ-subgroups, g' is conjugate to g in I, hence has the same eigenvalues. It follows that, if $s \in \mu_{\ell^\infty}$ is an eigenvalue of $\rho(g)$, then so is s^q. The set of eigenvalues has at most n elements, hence there is $m \leq n$ such that $s^{q^m} = s$. Thus ℓ divides $q^m - 1$, which contradicts the banality hypothesis.

We note the following consequence of (**IV**.1.1.2):

IV.1.1.7. *The intersection $\mathcal{K} \cap \mathbb{Q}(\zeta_\ell) = \mathbb{Q}$.*

Let $\mathcal{O}$ denote the ring of integers in a totally ramified finite extension $\mathbb{K}$ of the fraction field of the Witt ring $W(k)$. Let $\mathcal{C}_{\mathcal{O}}$ denote the category of complete noetherian local $\mathcal{O}$-algebras with residue field k; morphisms in $\mathcal{C}_{\mathcal{O}}$ are assumed to be local (take maximal ideals to maximal ideals). If R is an object of $\mathcal{C}_{\mathcal{O}}$ we let m_R denote its maximal ideal. Since $\ell > 2$ by the banality hypothesis, the character $\omega_{\overline{\rho}}$ defined by **IV**.1.1.4 has a unique lift $\omega_{\overline{\rho},R} : G_E \to R^\times$ for any object R of $\mathcal{C}_{\mathcal{O}}$.

IV.1.2. *Let R be an object of $\mathcal{C}_{\mathcal{O}}$. A* **deformation** *of $\overline{\rho}$ to R is a homomorphism $\rho : G_E \to \tilde{G}(R)$ such that*

$$\overline{\rho} \equiv \rho \pmod{m_R}. \tag{IV.1.2.1}$$

$$\nu \circ \rho(g) = \omega_{\overline{\rho},R}. \tag{IV.1.2.2}$$

Here $\nu : \tilde{G}(R) \to R^\times$ is the similitude character.

We assume

IV.1.3. *$\overline{\rho}$ has a deformation ρ_0 to $\mathcal{O}$ such that for each prime λ of $\mathcal{K}$ dividing ℓ $r_{\rho_0}|_{G_\lambda}$ is crystalline and the filtered module has n graded pieces, each free of rank one over $\mathcal{O}$, and of weights* $0, 1, \ldots, n-1$.

IV.1.4. We will be considering deformations of $\overline{\rho}$ with conditions at certain auxiliary sets of primes. Let Q denote a finite set of height one primes $\mathfrak{q}$ of E disjoint from $SC \cup \{\ell\}$ [divisors of ℓ] which satisfy

IV.1.4.1. *$\mathfrak{q}$ splits in $\mathcal{K}$ and the division algebras D and $D^\#$ are split above $\mathfrak{q}$;*

IV.1.4.2. *The residue characteristic q of $\mathfrak{q}$ satisfies $q \equiv 1 \pmod{\ell}$;*

IV.1.4.3. *$\overline{\rho}(Frob_{\mathfrak{q}})$ has a distinguished eigenvalue $\alpha_{\mathfrak{q}}$ of multiplicity one.*

As representations of $Z_{\mathfrak{q}}$, we write

$$\overline{\rho} = \overline{\rho}_\alpha \oplus \overline{\rho}_\beta, \tag{IV.1.4.4}$$

where $\overline{\rho}_\alpha$ is the $\alpha_{\mathfrak{q}}$-eigenspace of $\overline{\rho}(Frob_{\mathfrak{q}})$ and $\overline{\rho}_\beta$ is the direct sum of the remaining eigenspaces. Let $\Delta_{\mathfrak{q}}$ denote the maximal ℓ-power quotient of $(\mathbb{Z}/q\mathbb{Z})^\times$ and $\Delta_Q = \prod_{\mathfrak{q} \in Q} \Delta_{\mathfrak{q}}$.

By a deformation of $\overline{\rho}$ of type Q we shall mean a pair (R, ρ) as in Definition **IV**.1.2 such that:

IV.1.5.1. *For each prime λ of $\mathcal{K}$ dividing ℓ, $r_\rho|_{Z_\lambda}$ is crystalline and the filtered module has n graded pieces, each free of rank one over R, and of weights* $0, 1, \ldots, n-1$.

IV.1.5.2. *If $\mathfrak{q} \in Q$ then $r_\rho|_{Z_{\mathfrak{q}}} = \chi \oplus r'$ where $r' = r'_{\mathfrak{q}}$ is unramified and $\chi = \chi_{\mathfrak{q}} : Z_{\mathfrak{q}} \to R^\times$ is a character whose reduction modulo m_R is unramified and takes $Frob_{\mathfrak{q}}$ to $\alpha_{\mathfrak{q}}$.*

IV.1.5.3. *If $v \notin Q \cup \{\ell\}$ then $\rho(I_v) \xrightarrow{\sim} \overline{\rho}(I_v)$.*

Proposition IV.1.6. *There exists a universal deformation (R_Q, ρ_Q) of $\overline{\rho}$ of type Q.*

Proof. When $\overline{\rho}$ is replaced by the absolutely irreducible representation $r_{\overline{\rho}}$, the existence of a universal deformation follows from Theorem 1.1 of [R], as in [DDT, Lemma 2.37]; cf. the proof of Proposition **IV**.2.3 below. The extension to the disconnected group $\tilde{G}$ can be seen easily from the approach (using "well-placed liftings") described in [loc. cit.]. The crucial point is that the lifting of $\overline{\rho}(c)$ to $\tilde{G}(R)$ is unique, up to an element of the center of $GL(n, R)$. Indeed, if c_1 and c_2 are any two liftings, then $(c_2)^{-1}c_1$ intertwines r_ρ with itself, hence is a scalar matrix congruent to 1 (mod m_R).

IV.1.7 For $\mathfrak{q} \in Q$ we let $\chi_{\mathfrak{q}} : Z_{\mathfrak{q}} \rightarrow R_Q^\times$ be the character defined in (**IV**.1.5.2). Then $\chi_{\mathfrak{q}}$ necessarily factors through a natural map $\Delta_{\mathfrak{q}} \rightarrow R_Q^\times$. Thus R_Q is tautologically an $\mathcal{O}[\Delta_Q]$-module.

IV.2. *Bounding the Selmer group.*

Henceforward, we assume $\ell > n$. We fix a finite set Q of primes of E as in **IV**.1.4. Let $ad\ r_{\overline{\rho}}$ denote the composition of $\overline{\rho}$ with the adjoint representation $ad : \tilde{G} \rightarrow Aut(\mathfrak{gl}(n))$, where $\mathfrak{gl}(n) \subset Lie(\tilde{G})$ is viewed as the kernel of the similitude map. For each place v of E we fix a k-subspace $L_{Q,v} \subset H^1(Z_v, ad\ r_{\overline{\rho}})$. The $L_{Q,v}$ are chosen as follows:

IV.2.1.1. *For v dividing ℓ, $L_{Q,v}$ is the Bloch-Kato group $H^1_f(Z_v, ad\ r_{\overline{\rho}})$.*

In [BlK], Bloch and Kato work with characteristic zero coefficients. The ℓ-torsion group $H^1_f(Z_v, ad\ r_{\overline{\rho}})$ will be defined in **IV**.4, below.

IV.2.1.2. *For $v = \mathfrak{q} \in Q$, write*

$$ad\ r_{\overline{\rho}} = ad\ \overline{\rho}_\alpha \oplus ad\ \overline{\rho}'_\alpha,$$

where

$$ad\ \overline{\rho}'_\alpha = ad\ \overline{\rho}_\beta \oplus Hom(\overline{\rho}_\alpha, \overline{\rho}_\beta) \oplus Hom(\overline{\rho}_\beta, \overline{\rho}_\alpha),$$

*(notation **IV**.1.4.4). We set*

$$L_{Q,\mathfrak{q}} = H^1(Z_{\mathfrak{q}}, ad\ \overline{\rho}_\alpha) \oplus H^1(Z_{\mathfrak{q}}/I_{\mathfrak{q}}, ad\ \overline{\rho}'_\alpha).$$

IV.2.1.3. *At all other finite primes v $L_{Q,v} = H^1(Z_v/I_v, ad\ r_{\overline{\rho}}^{I_v})$.*

IV.2.1.4. *At archimedean primes we take $L_{Q,v} = 0$.*

There is a natural isomorphism (Poincaré duality)

$$ad\ r_{\overline{\rho}} \xrightarrow{\sim} ad\ r^*_{\overline{\rho}},$$

hence natural non-degenerate pairings for each place v

$$(\textbf{IV}.2.1.5) \qquad H^i(Z_v, ad\ r_{\overline{\rho}}) \times H^{2-i}(Z_v, ad\ r_{\overline{\rho}}(1)) \rightarrow \mathbb{Q}/\mathbb{Z}$$

(Tate's local duality), where (1) denotes Tate twist. For each v we let $L_{Q,v}^{\perp} \subset H^1(Z_v, ad\ r_{\overline{\rho}}(1))$ be the annihilator of $L_{Q,v}$ with respect to (**IV**.2.1.5), and define the Selmer group of $ad\ r_{\overline{\rho}}(1)$, relative to the data $L_{Q,v}^{\perp}$:

(**IV**.2.1.6) $$H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1)) = \{h \in H^1(E, ad\ r_{\overline{\rho}}(1)) \mid \forall\ v\ r_v(h) \in L_{Q,v}^{\perp}\}$$

We write $\mathfrak{M}_Q$ for m_{R_Q}. The objective of this section is to prove the following theorem.

Theorem IV.2.2. *The Selmer group $H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1))$ is finite and we have the inequality*

$$\dim_k \mathfrak{M}_Q/({\mathfrak{M}_Q}^2, \ell) \leq \#Q + \dim_k H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1)).$$

In particular, if $\dim H^1_{Q^*}(E, adr_{\overline{\rho}}) = 0$ then the $\mathcal{O}$-algebra R_Q can be topologically generated by $\#Q$ elements.

This theorem generalizes Lemma 5 of [TW]. Henceforward we write dim instead of $\dim_k$. We begin by translating the theorem into a statement purely in terms of Galois cohomology.

Proposition IV.2.3. *Define the Selmer group of $ad\ r_{\overline{\rho}}$, relative to the data $L_{Q,v}$:*

$$H^1_Q(E, ad\ r_{\overline{\rho}}) = \{h \in H^1(E, ad\ r_{\overline{\rho}}) \mid \forall\ v\ r_v(h) \in L_{Q,v}\}.$$

Then

$$\dim \mathfrak{M}_Q/({\mathfrak{M}_Q}^2, \ell) = \dim H^1_Q(E, ad\ r_{\overline{\rho}}).$$

Proof. This is proved as in [DDT,Theorem 2.41]. Let $\mathcal{D}$ denote the category of $k[G_E]$-modules M finite over k with dimension divisible by n, satisfying the analogues of properties **IV**.1.5.1-3:

IV.2.3.1. *As a module over Z_v, v above ℓ, M is a Fontaine-Laffaille representation (cf. §**IV**.4, below).*

IV.2.3.2. *As a module over $Z_{\mathfrak{q}}$, $\mathfrak{q}$ in Q, M is a the sum of an unramified module B and a module A whose semisimplification is isotypic for the unramified character $\alpha_{\mathfrak{q}}$.*

IV.2.3.3. *If $v \notin Q \cup \{\ell\}$ then the action of I_v on M is a direct sum of copies of irreducible direct summands of $\overline{\rho}(I_v)$.*

The category $\mathcal{D}$ is closed under products and taking subobjects and quotient objects. Obviously it contains $\overline{\rho}$. Thus Lemma 2.39 of [DDT] applies and yields

$$\dim \mathfrak{M}_Q/({\mathfrak{M}_Q}^2, \ell) = \dim H^1_{\mathcal{D}}(E, ad\ r_{\overline{\rho}}),$$

where $H^1_{\mathcal{D}}(E, ad\, r_{\overline{\rho}}) \subset H^1(E, ad\, r_{\overline{\rho}}) \simeq Ext^1_{G_E}(\overline{\rho}, \overline{\rho})$ is the subspace of classes whose corresponding extensions lie in $\mathcal{D}$.

Now we have to verify that conditions **IV**.2.3.1-3 for extensions translate into the cohomological conditions **IV**.2.1.1-3. Specifically, the equivalence of **IV**.2.3.1 and **IV**.2.1.1 is proved below in **IV**.4.7. The equivalence of **IV**.2.3.2 with **IV**.2.1.2

is easy to verify. At finite places $v \notin Q \cup \ell \cup SC$, and such that v is unramified in $\mathcal{K}$, **IV**.2.3.3 says the action of I_v is trivial, which is obviously equivalent to **IV**.2.1.3. Now suppose v in SC. The compatibility of **IV**.2.3.3 and **IV**.2.1.3 is equivalent to the condition

$$H^1(Z_v/I_v, Hom_{I_v}(\overline{\rho}, \overline{\rho})) \simeq Ker[H^1(Z_v, Hom(\overline{\rho}, \overline{\rho})) \to H^1(I_v, Hom(\overline{\rho}, \overline{\rho}))],$$

and this is just the inflation-restriction sequence. For v ramified in $\mathcal{K}$, the argument is similar.

We thus need to prove the inequality

$$(\mathbf{IV}.2.4) \qquad \dim H^1_Q(E, ad\ r_{\overline{\rho}}) - \dim H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1)) \leq \#Q.$$

Following Wiles [W,Prop. 1.6], the left hand side of (**IV**.2.4) can be expressed as a sum of local terms. We write the formula as in [DDT, Theorem 2.19], where it is stated for a general number field:

Proposition IV.2.5. *Let* $h^0 = \dim H^0(E, ad\ r_{\overline{\rho}})$, $h^{0,*} = \dim H^0(E, ad\ r_{\overline{\rho}}(1))$. *For any place* v *of* E *let* $h^0_v = \dim H^0(Z_v, ad\ r_{\overline{\rho}})$. *Then we have the formula*

$$\dim H^1_Q(E, ad\ r_{\overline{\rho}}) - \dim H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1)) = h^0 - h^{0,*} + \sum_v (\dim L_{Q,v} - h^0_v).$$

Lemma IV.2.6. *Under the hypotheses of Proposition* **IV**.*2.5, the local terms are computed as follows:*

(a) For v *real,* $h^0_v \geq \frac{n(n-1)}{2}$, $\dim L_{Q,v} = 0$.
(b) For $v \in Q$, $\dim L_{Q,v} - h^0_v = 1$.
(c) For v *above* ℓ, $\dim L_{Q,v} - h^0_v = [k(v) : \mathbb{F}_\ell] \cdot \frac{n(n-1)}{2}$.
(d) For all other places v, $\dim L_{Q,v} - h^0_v = 0$.
Finally, the global terms are given by $h^0 = h^{0,*} = 0$.

Admit this lemma for the moment. Comparing Proposition **IV**.2.5 with Lemma **IV**.2.6, we find

$$\begin{aligned}(\mathbf{IV}.2.7) \quad \dim H^1_Q(E, ad\ r_{\overline{\rho}}) &- \dim H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1)) \\ &\leq \#Q - \sum_{v \text{ real}} \frac{n(n-1)}{2} + \sum_{v|\ell} [k(v) : \mathbb{F}_\ell] \cdot \frac{n(n-1)}{2} \\ &\leq \#Q - [E : \mathbb{Q}] \frac{n(n-1)}{2} + [E : \mathbb{Q}] \frac{n(n-1)}{2} \leq \#Q\end{aligned}$$

Theorem **IV**.2.2 now follows by comparing (**IV**.2.7) with Proposition **IV**.2.3.

IV.2.8. We begin by calculating the global terms in Lemma **IV**.2.6. The hypothesis that SC is non-empty implies that $\overline{\rho}$ is already irreducible when restricted to a decomposition group of $G_{\mathcal{K}}$ above a prime in SC. Thus $H^0(\mathcal{K}, ad\ r_{\overline{\rho}})$ is one-dimensional and given by the trace of $r_{\overline{\rho}}$. But complex conjugation c acts as -1 on the center of the $GL(n)$-component of the L-group, so $H^0(E, ad\ r_{\overline{\rho}})$ is trivial. In the same way, and using **IV**.1.1.5, we see that $h^{0,*} = 0$.

The local terms will be computed in the next two sections.

IV.3. *Local calculations, char $v \neq \ell$.*

In this section we carry out the calculations summarized in Lemma **IV**.2.6. For any place v and any finite $\mathbb{F}_\ell[Z_v]$-module M we set

$$h^i(M) = \dim H^i(G_v, M); \quad h^{i,unr}(M) = \dim H^i(Z_v/I_v, M^{I_v}),$$

$i = 0, 1, 2$.

IV.3.1. If M is an unramified Z_v-module then of course $h^{0,unr}(M) = h^0(M)$. On the other hand, M is always assumed to be $Frob_v$-semi-simple when ℓ is not equal to the residue characteristic of v. Then M is the sum of characters of Z_v/I_v and

$$h^{1,unr}(M) = h^0(M) = \dim M^{Z_v}. \tag{\textbf{IV}.3.1}$$

It follows that, for v unramified, $v \notin Q$, we have

$$\dim L_{Q,v} - h^0_v = 0.$$

This verifies **IV**.2.6 (d) at unramified places.

IV.3.2. Now take $v \in Q$. We have

$$\dim L_{Q,v} - h^0_v = h^1(ad\ \overline{\rho}_\alpha) - h^0(ad\ \overline{\rho}_\alpha) + h^{1,unr}(ad\ \overline{\rho}_\alpha)' - h^0(ad\ \overline{\rho}_\alpha)'.$$

Since $ad(\overline{\rho}_\alpha)'$ is unramified the last two terms cancel, by (**IV**.3.1). On the other hand, the first two terms give

$$h^0(ad\ \overline{\rho}_\alpha(1))$$

by the local Euler characteristic formula and local duality (cf. [W,p. 473]). But $\overline{\rho}_\alpha$ is one-dimensional, so $ad(\overline{\rho}_\alpha)$ is the trivial Z_v module. Since $q \equiv 1 \pmod{\ell}$ the Tate twist is also trivial, and we find

$$\dim L_{Q,v} - h^0_v = 1,$$

which verifies **IV**.2.6 (b).

IV.3.3. For v real, we have $\dim L_{Q,v} = 0$, by hypothesis. On the other hand,

$$h^0_v = \dim[ad\ r_{\overline{\rho}}]^{c=1},$$

independently of v. Then (a) follows immediately from hypothesis **IV**.1.1.3.

IV.3.4. Now suppose v is ramified, but of residue characteristic $\neq \ell$. By hypothesis, either $v \in SC$, or v ramifies in $\mathcal{K}/E$ and $r_{\overline{\rho}}$ is unramified at the prime above v. We need to calculate

$$\dim L_{Q,v} - h^0_v = h^{1,unr}(ad\ r_{\overline{\rho}}) - h^0(ad\ r_{\overline{\rho}}).$$

First, suppose $v \in SC$, and $r_{\overline{\rho}} = \oplus_{i=1}^{r}(\mathfrak{r}_i)^{a_i}$, where the $\mathfrak{r}_i$ are irreducible and distinct. Returning to **IV**.1.1.2, we find that

$$\dim[ad\ r_{\overline{\rho}}]^{Z_v} = \sum_i (a_i)^2.$$

Let $L_{ij} = H^1(Z_v/I_v, \mathfrak{r}_i \otimes \mathfrak{r}_j^*)$, where $*$ denotes dual. It suffices to show that $\dim L_{ij} = \delta_{ij}$. Suppose $\mathfrak{r}_i|_{I_v}$ breaks up as the sum of d irreducible representations τ_{ik}. Then

$$(\mathbf{IV}.3.5) \qquad (\mathfrak{r}_i \otimes \mathfrak{r}_i^*)^{I_v} = \oplus_{k=1}^{d}[ad\ \tau_k]^{I_v}$$

has dimension d. As a representation of the cyclic group Z_v/I_v, the right-hand side of **IV**.3.5 is isomorphic to the sum $\oplus\chi$ of the distinct characters of Z_v/H, where $H \supset I_v$ is the stabilizer in Z_v of τ_1, say. Thus

$$\dim L_{ii} = \sum_{\chi} \dim H^1(Z_v/H, \chi) = 1,$$

since only the trivial character has non-trivial cohomology. The verification for L_{ij} with $i \neq j$ breaks up into two cases. If $\mathfrak{r}_j$ is not an unramified twist of r_i, then $(\mathfrak{r}_i \otimes \mathfrak{r}_j^*)^{I_v} = 0$. If $\mathfrak{r}_i = \mathfrak{r}_j \otimes \xi$, with ξ an unramified character, then we find

$$(\mathfrak{r}_i \otimes \mathfrak{r}_j^*)^{I_v} = \oplus_{k=1}^{d}\chi \cdot \xi$$

where χ runs through the characters of Z_v/H, as above. We conclude that $\dim L_{ij} = 0$ by observing that the non-isomorphy of $\mathfrak{r}_i$ and $\mathfrak{r}_j$ implies that ξ does not factor through Z_v/H.

Now suppose v ramifies in $\mathcal{K}/E$. Let w denote the prime above v. In this case Z_v acts via the abelian group $Gal(\mathcal{K}/E) \times Z_w/I_w$. Let M denote the subspace of $ad\ \overline{\rho}$ fixed by $Gal(\mathcal{K}/E)$. Then $\dim L_{Q,v} - h_v^0 = h^{1,unr}(M) - h^0(M) = 0$ as in **IV**.3.1. This completes the verification of (d).

To complete the proof of Lemma **IV**.2.6, it remains to estimate the local terms at primes dividing ℓ. This is the subject of the next section.

IV.4 *Local calculations, crystalline case.*

In this section we complete the proof of Lemma **IV**.2.6 by all the facts we need about comparison theorems between crystalline and étale cohomology. As above, we work in residue characteristic ℓ and restrict our attention to the situation of good reduction. We begin by defining crystalline representations of $Gal(\overline{E_v}/E_v)$, when v is of residue characteristic ℓ, with coefficients in $\overline{\mathbb{F}}_l$, in terms of the Fontaine-Laffaille construction [FL]. Recall that ℓ is assumed unramified in E. Next we define crystalline deformations of mod ℓ crystalline representations and show that the crystalline condition is captured, in terms of local Galois cohomology, by an ℓ-torsion analogue of the Bloch-Kato group H_f^1. The inequality of Lemma **IV**.2.6 (c) is then proved by a calculation in the Fontaine-Laffaille category. We conclude by showing that under a certain regularity hypothesis (satisfied in our applications) the image of tame inertia contains an element whose eigenvalues have multiplicity one.

Our base ring will be the Witt ring $W(k(v))$, where $k(v)$ is the residue field of v; thus $W(k(v)) = \mathcal{O}_v$ in our previous notation. We let $\mathcal{O}_{\mathbb{C}_\ell}$ denote the ring of integers in the completion $\mathbb{C}_\ell$ of the algebraic closure of the fraction field of $W(k(v))$. Let $A_{cris} = A_{cris}(\mathcal{O}_{\mathbb{C}_\ell})$ be the ring defined in [Fo, Asterisque, 2.3], an ℓ-integral form of B_{cris}. The $W(k(v))$-algebra A_{cris} is endowed with a decreasing filtration $Fil^i A_{cris}$, $i \geq 0$ and with a σ-linear operator ϕ such that

$$\phi(Fil^i A_{cris}) \subset \ell^i A_{cris}, 0 \leq i \leq \ell - 1. \tag{IV.4.1}$$

Here σ is absolute Frobenius.

Recall the ℓ-adic coefficient ring $\mathcal{O}$ from **IV**.1. Let $\mathbf{Rep}_{W(k(v))\otimes\mathcal{O}}$ denote the category of $\mathcal{O}[Gal(\overline{E_v}/E_v)]$-modules of finite type. By analogy with [FPR, p. 638], we define $\mathbf{Rep}_{W(k(v))\otimes\mathcal{O},cris,[0,\ell-1[}$ to be the full subcategory of $\mathbf{Rep}_{W(k(v))\otimes\mathcal{O}}$ whose objects are isomorphic to subquotients of crystalline $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathcal{O}[Gal(\overline{E_v}/E_v)]$-modules. For any object Λ of $\mathbf{Rep}_{W(k(v))\otimes\mathcal{O},cris,[0,\ell-1[}$ we can define the *filtered module* (Fontaine-Laffaille module) $M_{cris}(\Lambda)$. It is an object of the category $MF_{W(k(v))\otimes\mathcal{O},[0,\ell-1[}$ consisting of

(i) a $W(k(v)) \otimes \mathcal{O}$-module M of finite type, a decreasing filtration $Fil^i(M)$ by $W(k(v)) \otimes \mathcal{O}$-submodules which are direct factors, with $Fil^0 M = M$, $Fil^\ell M = 0$; and

(ii) a family $\phi^i : Fil^i(M) \to M$ of σ-linear maps such that $\phi^i(x) = \ell\phi^{i+1}(x)$ for $x \in Fil^{i+1}M$, and such that M is the sum of the images of the ϕ^i, as i ranges over $\mathbb{Z}$.

It is further assumed that M contains no non-trivial subobject M' with $Fil^{\ell-1}M' = M'$.

The definition is

(**IV**.4.2)

$$M_{cris}(\Lambda) = \bigcup M \mid M \subset A_{cris} \otimes_{W(k)} \Lambda)^{Gal(\overline{E_v}/E_v)};\ M \in MF_{W(k(v)),[0,\ell-1[}$$

(cf. [Niz, p. 750]; [Wa,Remarque 2.4.4]). The filtration is inherited from the filtration on A_{cris}, the $\mathcal{O}$-action on Λ is left undisturbed, and ϕ^i is inherited from $\ell^{-i}\phi$ on $Fil^i A_{cris}$, which makes sense by (**IV**.4.1). Then $M_{cris} : \mathbf{Rep}_{W(k),cris,[0,\ell-1[} \to MF_{W(k),[0,\ell-1[}$ is an equivalence of categories. An inverse equivalence [FL] is given by the formula

$$\Lambda(M) = Fil^0(A_{cris} \otimes_{W(k)} M)^{\phi=1}. \tag{IV.4.3}$$

For our purposes, crystalline Galois representations are those of the form $\Lambda(M)$, for some $M \in MF_{W(k(v)),[0,\ell-1[}$. More generally, let A be a $W(k(v))$ algebra of finite type with a σ-linear automorphism σ_A. Then we can define the categories $MF_{A,[0,\ell-1[}$ and $\mathbf{Rep}_{A,cris,[0,\ell-1[}$ by analogy with (i), (ii) above, taking A as coefficient ring. The functors M_{cris} and Λ can be defined as inverse equivalences between these two categories. In the applications, A will be a $W(k(v)) \otimes \mathcal{O}$-algebra, where $\sigma_{W(k(v))\otimes\mathcal{O}}$ is defined to be $\sigma \otimes 1 : W(k(v)) \otimes \mathcal{O} \to W(k(v)) \otimes \mathcal{O}$.

Let $MF_{tor,[0,\ell-1[}$ denote the subcategory of $MF_{W(k(v)),[0,\ell-1[}$ of objects of finite length (as $W(k(v))$-modules). By [FL,Prop. 1.8], $MF_{tor,[0,\ell-1[}$ is an abelian category. Let

$$\mathbf{Rep}_{tor,cris,[0,\ell-1[} \subset \mathbf{Rep}_{W(k(v)),cris,[0,\ell-1[}$$

denote the essential image of the functor Λ, restricted to $MF_{tor,[0,\ell-1[}$. Then $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$ is a full subcategory of $\mathbf{Rep}_{W(k(v)),cris,[0,\ell-1[}$, itself a full subcategory of $\mathbf{Rep}_{W(k(v))}$. The functors M_{cris} and Λ define inverse equivalences between the abelian categories $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$ and $MF_{tor,[0,\ell-1[}$. Thus, given two objects $M, N \in MF_{tor,[0,\ell-1[}$, there is a natural isomorphism

$$(\mathbf{IV}.4.4) \qquad Ext^1_{MF}(M,N) \xrightarrow{\sim} Ext^1_{cris}(\Lambda(M),\Lambda(N)),$$

where Ext^1_{MF} (resp. Ext^1_{cris}) is shorthand for extensions in $MF_{tor,[0,\ell-1[}$ (resp. $\mathbf{Rep}_{tor,cris,[0,\ell-1[}$). Composing the isomorphism ($\mathbf{IV}$.4.4) with the forgetful functor, we obtain a homomorphism

$$(\mathbf{IV}.4.5) \qquad Ext^1_{MF}(M,N) \to Ext^1_{\mathbf{Rep}_{W(k(v))}}(\Lambda(M),\Lambda(N)).$$

The isomorphism ($\mathbf{IV}$.4.4) and the homomorphism ($\mathbf{IV}$.4.5) respect $\mathcal{O}$-structures; i.e., take extensions with compatible $\mathcal{O}$-structure to extensions with $\mathcal{O}$-structures.

Write $k(v)\otimes k = k(v)\otimes_{\mathbb{Z}_\ell} k$, and suppose now that M and N are free $k(v)\otimes k$-modules. In particular, $\ell\cdot M = 0$, $\ell\cdot N = 0$. Let $Ext^1_{MF,k(v)\otimes k}(M,N) \subset Ext^1_{MF,k(v)\otimes k}(M,N)$ denote the subgroup of extensions in the category of Fontaine-Laffaille modules which are free $k(v)\otimes k$-modules. Likewise, let $\mathbf{Rep}_{k(v)\otimes k}$ denote the category of $k(v)\otimes k[Gal(\overline{E_v}/E_v)]$-modules free over $k(v)\otimes k$. Then

$$\begin{aligned} Ext^1_{\mathbf{Rep}_{k(v)\otimes k}}(\Lambda(M),\Lambda(N)) &\xrightarrow{\sim} Ext^1_{\mathbf{Rep}_{k(v)\otimes k}}(k(v)\otimes k, Hom_{k(v)\otimes k}(\Lambda(M),\Lambda(N))) \\ &\xrightarrow{\sim} H^1(Gal(\overline{E_v}/E_v), Hom_{k(v)\otimes k}(\Lambda(M),\Lambda(N))). \end{aligned}$$

With respect to this isomorphism, we let

$$\begin{aligned} &H^1_f(Gal(\overline{E_v}/E_v), Hom_{k(v)\otimes k}(\Lambda(M),\Lambda(N)) \\ &\qquad\qquad \subset H^1(Gal(\overline{E_v}/E_v), Hom_{k(v)\otimes k}(\Lambda(M),\Lambda(N))) \end{aligned}$$

denote the image of $Ext^1_{MF,k(v)\otimes k}(M,N)$.

Say $M \in MF_{W(k(v))\otimes\mathcal{O},[0,\ell-1[}$ is *regular* if $gr^i(M) = Fil^i(M)/Fil^{i+1}(M)$ is a free $k(v)\otimes k$-module of rank ≤ 1 for all i.

Lemma IV.4.6. *Let Λ be a crystalline $k(v)\otimes k[Gal(\overline{E_v}/E_v)]$-module, and let $Ad(\Lambda)$ denote the $k(v)\otimes k[Gal(\overline{E_v}/E_v)]$-module $Hom_{k(v)\otimes k}(\Lambda,\Lambda)$. Suppose $\Lambda = \Lambda(M)$, with M a regular Fontaine-Laffaille module of rank n over $k(v)\otimes k$. Then*

$$rank_{k(v)\otimes k} H^1_f(Gal(\overline{E_v}/E_v), Ad(\Lambda)) - \dim H^0(Gal(\overline{E_v}/E_v), Ad(\Lambda)) = \frac{1}{2}n(n-1).$$

Proof. By definition, the left-hand side equals

$$rank_{k(v)\otimes k} Ext^1_{MF,rank_{k(v)\otimes k}}(M,M) - rank_{k(v)\otimes k} Hom_{MF,rank_{k(v)\otimes k}}(M,M).$$

This is unchanged when we extend scalars from k to a finite extension k' and replace M by $M_{k'}$. We thus may assume $k \supset k(v)$, and then by projecting on irreducible components we may replace $k(v) \otimes k$ by k.

Since M is a k-module, its structure as Fontaine-Laffaille module reduces to a triple consisting of a k-vector space V, a decreasing filtration $Fil^\bullet V$, and an isomorphism $\phi_V : gr^\bullet V \xrightarrow{\sim} V$. Let $gr^\bullet V = \oplus_{i=1}^n gr^{a_i} V$, where $0 \leq a_1 < a_2 \cdots < a_n \leq \ell - 1$ is an n-tuple of positive integers. Fix a basis $e_1, \ldots, e_n$ of V such that $Fil^i V$ is the span of $e_{i+1}, \ldots, e_n$. Suppose $\mathcal{E}$ is an extension of M by itself in $Ext^1_{MF,k}(M,M)$ and let E be the underlying k-module. There is a short exact sequence

$$0 \to V \xrightarrow{i} E \xrightarrow{\pi} V \to 0$$

compatible with filtrations and the morphisms ϕ_E and ϕ_V. Let $s : V \to E$ denote any splitting of π as filtered module; s is determined uniquely up to an element $\alpha \in F^0 End(V)$, where $F^0 End(V) \subset End_k(V)$ denotes the subspace of filtration-preserving endomorphisms. Then $gr^\bullet E = gr(i)(gr^\bullet V) \oplus gr(\pi)(gr^\bullet V)$. In terms of this basis, ϕ_E can be written $\begin{pmatrix} \phi_V & \mu \\ 0 & \phi_V \end{pmatrix}$, for some $\mu \in Hom(gr^\bullet V, V)$. We have

$$\mu = \phi_E \circ gr(s) - s \circ \phi_V.$$

Moreover, replacing s by $s + \alpha$ changes μ to $\mu + \phi_V \circ gr(\alpha) - \alpha \circ \phi_V$. Thus the map $M \mapsto \mu$ (mod equivalence) defines an isomorphism

$$Ext^1_{MF,k}(M,M) \to \{\phi \in Hom(V,V)\}/\{\phi \circ gr(\alpha) - \alpha \circ \phi | \alpha \in F^0 End(V))\}.$$

Moreover,

$$Hom_{MF,k}(M,M) \to \{\alpha \in F^0 End(V) | \phi \circ gr(\alpha) = \alpha \circ \phi\}.$$

This yields an exact sequence

$$0 \to Hom_{MF,k}(M,M) \to F^0 End(V) \xrightarrow{j} Hom_k(gr^\bullet V, V) \to Ext^1_{MF,k}(M,M) \to 0,$$

where the map j takes α to $\phi \circ gr(\alpha) - \alpha \circ \phi$.

It follows that

$$\begin{aligned} \dim\ Ext^1_{MF,k}(M,M) - \dim\ Hom_{MF,k}(M,M) &= \dim\ Hom_k(gr^\bullet V, V) - \dim\ F^0 End(V) \\ &= n^2 - \frac{1}{2} n(n+1) = \frac{1}{2} n(n-1). \end{aligned}$$

IV.4.7. Let $A = k[\varepsilon]/(\varepsilon^2)$, let $\Lambda \in \mathbf{Rep}_{k,cris,[0,\ell-1[}$, and let $\tilde{\Lambda}$ be a deformation of Λ to A. Multiplication by ε defines an isomorphism

$$\Lambda \cong \tilde{\Lambda}/\varepsilon\tilde{\Lambda} \xrightarrow{\sim} \varepsilon\tilde{\Lambda}$$

of $k[Gal(\overline{E_v}/E_v)]$-modules. Thus $\tilde{\Lambda}$ defines an extension

$$0 \to \Lambda \to \tilde{\Lambda} \to \Lambda \to 0, \qquad (\mathbf{IV}.4.7.1)$$

and this correspondence defines a bijection between the equivalence classes of deformations of Λ to A and $Ext^1_{\mathbf{Rep}_k}(\Lambda,\Lambda)$ (cf. [DDT], p. 67). By definition, the deformation $\tilde{\Lambda}$ is crystalline if and only if (**IV**.4.7.1) is an extension in the category of crystalline representations; in other words, crystalline deformations are in bijection with $Ext^1_{cris}(\Lambda,\Lambda) \cong H^1_f(Gal(\overline{E_v}/E_v), Ad(\Lambda)$. We have thus verified the equivalence of **IV**.2.3.1 and **IV**.2.1.1 in the definition of $H^1_{\mathcal{D}}(E, adr_{\overline{\rho}})$.

IV.4.8. It remains to verify Lemma **IV**.2.6 (c). Thus let v be a prime of E dividing ℓ. Set $h^1_f = \dim_{\mathbb{F}_\ell}\ H^1_f(Z_v, ad(r_{\overline{\rho}}))$. By hypothesis **IV**.1.3 $r_{\overline{\rho}}$ is the crystalline representation over $k(v)$ associated to a regular Fontaine-Laffaille module. Thus Lemma **IV**.2.6 (c) follows directly from Lemma **IV**.4.6. This completes the proof of Lemma **IV**.2.6.

For $M \in MF_{W(\overline{\mathbb{F}}_l),[0,\ell-1[}$ the action of tame inertia on $\Lambda(M)$ is calculated explicitly in [FL]. As a consequence of that calculation, we can prove

Lemma IV.4.9. *Let $M \in MF_{W(k),[0,\ell-1[}$ and suppose M is regular. Then the eigenspaces of tame inertia on $\Lambda(M)$ have dimension 1. More precisely, let $\overline{\Lambda}(M)_{ss}$ denote the semi-simplification of the $k[Gal(\overline{E_v}/E_v)]$-module $\Lambda/\ell\Lambda(M)$. Then the action of the inertia subgroup on $\overline{\Lambda}(M)_{ss}$ factors through the tame quotient, and the latter has $\dim M/\ell M$-distinct eigenvalues.*

Proof. The proof is a simple combinatorial exercise, using the results of [FL,§§4-5]. Without loss of generality we may assume M to be a semi-simple ℓ-torsion module, so that $\Lambda(M) = \overline{\Lambda}(M)_{ss}$. Let $n = dim_{\overline{\mathbb{F}}_l} M$ and let b_i, $i = 1,\ldots,n$ be the integers such that $\dim gr^{b_i}(M) = 1$; let $B = \{b_1,\ldots,b_n\} \subset \mathbb{Z}$. Let $q = |k| = \ell^r$, for some r. Then M is determined, up to isomorphism, by the following set of data:

(a) A partition $n = h_1 + \cdots + h_s$, with $h_j \geq 1$ for all j;

(b) For each j a map $\iota_j : \mathbb{Z}/h_j\mathbb{Z} \to B$, with image B_j, so that $B = \cup_j B_j$;

(c) The period of the map ι_j is exactly h_j; i.e. for any $h < h_j$ the map $\iota_j(a) \mapsto \iota_j(a+h)$ is a non-trivial permutation of B_j.

Indeed, we can write $M = \oplus_j M(j)$ as a sum of simple objects, and the object $M(j)$ is determined up to isomorphism by a pair (h_j, ι_j) as above. Let $\Lambda(j) = \Lambda(M(j))$. For any positive integer h, let $\alpha \in \overline{\mathbb{Q}}_\ell$ satisfy

$$\alpha^{q^h-1} = \ell$$

and set $\chi_h(g) = g(\alpha)/\alpha \pmod{\ell} \in \overline{\mathbb{F}}_l^{\times}$, for g in the tame inertia group I^t. Let

$$C(j) = \{\sum_{a=0}^{h_j-1} \iota_j(a+t)q^a \mid t = 0,1,\ldots,h_j-1\}. \qquad (\mathbf{IV}.4.9.1)$$

It follows from (c) above, and from the fact that $B \subset \{0,\ldots,p-1\}$, that $C(j)$ has h_j distinct elements. The calculation in [FL] shows that the action of I^t on $\Lambda(j)$ factors through the character χ_{h_j}, and $g \in I^t$ has eigenvalues

$$\chi_{h_j}(g)^c,\ c \in C(j).$$

The exponents in $C(j)$ being distinct, the action of I^t on $\Lambda(j)$ is multiplicity-free. On the other hand, it is easy to see, that I^t has no common eigenvalues on $\Lambda(j_1)$ and $\Lambda(j_2)$, first if $h_{j_1} \neq h_{j_2}$, then in the general case.

IV.5. *Capturing ramification by tame classes.*

In order to make Theorem **IV**.2.2 effective, we need to find sets Q for which $\dim H^1_{Q^*}(E, ad\,\overline{\rho}) = 0$. We follow the strategy of [TW]. For this additional hypotheses are needed. Unfortunately, we have not found an optimal set of hypotheses. In the coordinates of (**I**.1.4) the map

(**IV**.5.1) $$\tilde{G}^0 \to GL(n) \times GL(1);\ g \mapsto (g_1, a = \nu(g))$$

is an isomorphism. Let $r^i_{\overline{\rho}}$, $i = 1, 2$, denote the composition of $r_{\overline{\rho}}$ with projection on the i-th factor in (**IV**.5.1.1). Thus $Ker(r^1_{\overline{\rho}})$ determines an extension $\mathcal{K}^1$ of $\mathcal{K}$ with Galois group naturally a subgroup of $GL(n,k)$; $Ker(r^2(\overline{\rho}))$ determines the extension $\mathcal{K}(\zeta_\ell^{n-1})$ of $\mathcal{K}$, of degree $[\mathbb{Q}(\zeta_\ell^{n-1}) : \mathbb{Q}]$ (cf. (**IV**.1.1.4) and (**IV**.1.1.7)). We consider the following conditions.

Hypotheses IV.5.2.

(a) $\mathcal{K}^1 \cap \mathcal{K}(\zeta_\ell) = \mathcal{K}$.

(b) The group $Im(\overline{\rho})$ *has no quotient of order* ℓ.

(c) Let $V \subset ad\,\overline{\rho}$ *be an irreducible subrepresentation. Then there is* $s \in G_{\mathcal{K}}$ *such that* $r_{\overline{\rho}}(s)$ *has* n *distinct eigenvalues and such that* $ad\ (\overline{\rho})(s)$ *has eigenvalue* 1 *on* V.

Theorem IV.5.3. *Assume Hypotheses* **IV**.*5.2. Then there is an integer* r *such that, for any* $m \geq 1$ *there is a set* Q_m *satisfying the hypotheses of* **IV**.*1.4, and such that moreoever*

(a) $\#Q_m = r$;

(b) For all $\mathfrak{q} \in Q_m$ *we have* $q = N\mathfrak{q} \equiv 1 \pmod{\ell^m}$;

(c) $H^1_{Q^*_m}(E, ad\,\overline{\rho}(1)) = 0$.

(d) $r_{\overline{\rho}}(Frob_{\mathfrak{q}})$ *has* n *distinct eigenvalues, and in particular a distinguished eigenvalue* $\alpha_{\mathfrak{q}}$ *of multiplicity one.*

Proof. We begin by recalling that, for any Q as in **IV**.1.4, and any $\mathfrak{q} \in Q$, the subspace $L^\perp_{Q,\mathfrak{q}} \subset H^1(Z_{\mathfrak{q}}, ad\,\overline{\rho}(1))$ is defined by

$$H^1(Z_{\mathfrak{q}}/I_{\mathfrak{q}}, ad\,\overline{\rho}'_\alpha(1))$$

in the notation of **IV**.2.1.2. In other words, $L^\perp_{Q,\mathfrak{q}}$ consists of unramified classes with trivial $ad\ (\overline{\rho}_\alpha)(1)$-component. Thus

(**IV**.5.3.1)
$$H^1_{Q^*}(E, ad\ r_{\overline{\rho}}(1)) = Ker[H^1_\emptyset(E, ad\,\overline{\rho}(1)) \to \oplus_{\mathfrak{q} \in Q_m} H^1(Z_{\mathfrak{q}}/I_{\mathfrak{q}}, ad\,\overline{\rho}_\alpha(1))].$$

For r we take the dimension of $H^1_\emptyset(E, ad^0\overline{\rho}(1))$. As in [TW,p. 567] we need to find sets Q_m satisfying conditions (a), (b), (d), and the hypotheses of **IV**.1.4, and such that the natural map

(**IV**.5.3.2) $$H^1_\emptyset(E, ad\overline{\rho}(1)) \to \oplus_{\mathfrak{q} \in Q_m} H^1(Z_{\mathfrak{q}}/I_{\mathfrak{q}}, ad(\overline{\rho}_\alpha)(1))$$

is injective, hence an isomorphism for dimension reasons. Condition (b) asserts that $\mathfrak{q}$ splits completely in $\mathcal{K}(\zeta_{\ell^m})$.

Let $[\psi] \in H^1_{\emptyset}(E, ad\overline{\rho}(1))$ be a non-zero class. The objective is to find $\mathfrak{q}$ as above satisfying condition (b), (d), and **IV**.1.4 and such that

$$(\mathbf{IV}.5.3.3) \qquad res_{\mathfrak{q}}[\psi] \in H^1(Z_{\mathfrak{q}}/I_{\mathfrak{q}}, ad\,\overline{\rho}_{\alpha}(1)) \text{ is nontrivial.}$$

By Chebotarev density it thus suffices to find $\sigma \in G_E$ such that

IV.5.3.4. *(i) σ fixes $E(\zeta_{\ell^m})$;*
(ii) $\overline{\rho}(\sigma)$ has n distinct eigenvalues;
(iii) There is a distinguished eigenvalue α of $\overline{\rho}(\sigma)$ such that $\psi(\sigma) \notin ad\,\overline{\rho}'_{\alpha}(1)$

where $ad\,\overline{\rho}'_{\alpha} \subset ad\,\overline{\rho}$ is the codimension one subspace defined with respect to α by analogy with **IV**.2.1.2.

Let $E_m = E(\zeta_{\ell^m})$, and let F_m denote the extension of E_m fixed by the kernel of $ad\,\overline{\rho}$. We claim ψ restricts to non-trivially to $H^1_{\emptyset^*}(F_m, ad\,\overline{\rho}(1))$. The kernel of the restriction map is $H^1(Gal(F_m/E), ad\,\overline{\rho}(1))$. It suffices to show

$$(\mathbf{IV}.5.3.5) \qquad H^1(Gal(F_m/E), ad\,\overline{\rho}(1)) = 0.$$

We argue as in [DDT], p. 84. The inflation-restriction sequence for $F_m \supset F_1 \supset E$ is an exact sequence

$$H^1(Gal(F_1/E), ad\,\overline{\rho}(1)^{G_{F_1}}) \hookrightarrow H^1(Gal(F_m/E), ad\,\overline{\rho}(1)) \rightarrow [H^1(Gal(F_m/F_1), ad\,\overline{\rho}(1))]^{G_E}.$$

Now G_{F_1} acts trivially on $ad\,\overline{\rho}(1)$. Hence

$$[H^1(Gal(F_m/F_1), ad\,\overline{\rho}(1))]^{G_E} \cong Hom(Gal(F_m/F_1), [ad\,\overline{\rho}(1))]^{G_E}).$$

Moreover, it follows from Condition **IV**.5.2 (a) that $Gal(F_1/E)$ breaks up as the direct product $Gal(F_1/F_0) \times Gal(F_0/E)$. Thus

$$(\mathbf{IV}.5.3.6) \qquad [ad\,\overline{\rho}(1))]^{G_E} \subset [ad\,\overline{\rho}(1))]^{Gal(F_1/F_0)} = \{0\}.$$

Indeed, $Gal(F_1/F_0)$ acts on $ad\,\overline{\rho}(1))$ as a direct sum of copies of the natural action on the ℓth roots of unity. But $Gal(F_1/F_0)$ can be identified with the subgroup of $Aut(\mu_\ell)$ that acts trivially on $\mu_\ell^{\otimes(n-1)}$. The hypothesis $\ell > n$ implies that this subgroup is non-trivial.

Thus the above exact sequence simplifies to yield

$$(\mathbf{IV}.5.3.7) \qquad H^1(Gal(F_1/E), ad\,\overline{\rho}(1)) \xrightarrow{\sim} H^1(Gal(F_m/E), ad\,\overline{\rho}(1)).$$

On the other hand, applying the inflation restriction sequence for $F_1 \supset F_0 \supset E$ to the left-hand side of (**IV**.5.3.7), we find

$$H^1(Gal(F_0/E), ad\,\overline{\rho}(1)^{Gal(F_1/F_0)}) \hookrightarrow H^1(Gal(F_1/E), ad\,\overline{\rho}(1)) \rightarrow [H^1(Gal(F_1/F_0), ad\,\overline{\rho}(1))]^{Gal(F_0/E)}.$$

Here the right-hand side vanishes because $[F_1 : F_0]$ is prime to ℓ, while the left-hand side vanishes as in (**IV**.5.3.6). This completes the verification of (**IV**.5.3.5).

Now it follows from **IV**.5.2 (a) and (b) that $\overline{\rho}$ remains absolutely irreducible upon restriction to G_{E_m} for all m. Thus, to verify (**IV**.5.3.2), it suffices to find sets of height one primes of E_m satisfying conditions (b), (d), **IV**.1.4, and (**IV**.5.3.3), with E replaced by E_m. Conditions **IV**.1.4.1-2 are already satisfied, and **IV**.1.4.3 concerns only a finite set of primes, which we can avoid. We have

$$H^1_\emptyset(F_m, ad\ r_{\overline{\rho}}(1)) \subset Hom(G_{F_m}, ad\ r_{\overline{\rho}}(1))$$

is the subset satisfying various ramification conditions. Thus let $\psi \in H^1_\emptyset(E_m, ad\, r_{\overline{\rho}}(1))$. Its restriction to F_m is a homomorphism from G_{F_m} to $ad\ r_{\overline{\rho}}$ whose image is a $Gal(F_m/E_m)$-submodule, say V_ψ. Moreover, $Gal(F_m/E_m) = Gal(F_0/E)$ by **IV**.5.2 (a). Let $s \in Gal(F_m/E_m)$ satisfy the conditions of **IV**.5.2 (c), and let σ_0 be a lifting of s to G_{E_m}. It already satisfies conditions (i) and (ii) of **IV**.5.3.4, and so does $\sigma = \tau\sigma_0$ for any $\tau \in G_{F_m}$. It remains to show that we can choose α and τ so that σ satisfies condition (iii). Now the eigenvalues of $ad\ r_{\overline{\rho}}(s)$ are of the form $\alpha_i \cdot \alpha_j^{-1}$, where α_i, $i = 1, \ldots, n$ are the n distinct eigenvalues of $r_{\overline{\rho}}(s)$. Let v_{ij} be the corresponding eigenvectors. By hypothesis **IV**.5.2 (c) the fixed subspace V_ψ^s is non-trivial and is spanned by r non-trivial linear combinations $v_k = \sum_i a_{ik}v_{ii}$, $1 \leq k \leq r$. Now $\psi(\sigma) = \psi(\tau) + \psi(\sigma_0)$. Write $\psi(\sigma_0) = \sum b_{ij}v_{ij}$, $\psi(\tau) = \sum c_k(\tau)v_k + v'$, where v' is a linear combination of the v_{ij} with $i \neq j$. Thus the coefficient of v_{ii} in $\psi(\sigma)$ is

$$b_i(\tau) = \sum c_k(\tau)a_{ik} + b_{ii}.$$

But we may vary the $c_k(\tau)$ freely, and it is clear that by doing so we can arrange that at least one $b_i(\tau)$ is non-zero. Taking $\alpha = \alpha_i$, we then see that σ satisfies condition (iii). This completes the proof.

IV.6. *Eliminating tame deformations*

Let q be a rational prime, $q \neq \ell$, and let v be a prime of E dividing q. The maximal ℓ-power quotient $I_{v,\ell}$ of the inertia group I_v is isomorphic to $\mathbb{Z}_\ell(1)$ as a module over Z_v/I_v, where the (1) denotes Tate twist. Let $P^\ell \subset I_v$ be the kernel of the canonical map to $I_{v,\ell}$; it is a profinite group with pro-order prime to ℓ. Thus, for any Z_v-module M, the canonical inflation map $H^1(Z_v/P^\ell, M) \to H^1(Z_v, M)$ is an isomorphism.

Now let $(\overline{\rho}, V)$ be an n-dimensional semi-simple unramified representation of Z_v with coefficients in a finite field k of characteristic ℓ, and let $M = ad\ \overline{\rho}$.

Lemma IV.6.1. *Suppose $\overline{\rho}$ is trivial and $Nv \neq 1 \pmod{\ell}$. Then the inflation map*

(**IV**.6.2) $$H^1(Z_v/I_v, M) \to H^1(Z_v/P^\ell, M)$$

is an isomorphism.

Proof. We use the inflation-restriction sequence for the inclusion of $I_{v,\ell}$ in Z_v/P^ℓ:

(**IV**.6.3) $$\begin{aligned} 0 \to H^1(Z_v/I_v, M) \to H^1(Z_v/P^\ell, M) \to\ & Hom(I_{v,\ell}, M)^{Z_v/I_v} \\ &= Hom_{Z_v/I_v}(\mathbb{F}_\ell(1), M) \end{aligned}$$

By our hypothesis, Z_v/I_v acts non-trivially on $\mathbb{F}_\ell(1)$ but trivially on M. Thus the right-hand term in (**IV**.6.3) vanishes.

V. The main theorem

V.1. *Representations on global Hecke algebras.*

We now return to the language of automorphic forms on the group J. Fix ℓ, σ, $\mathfrak{r}$, Q and Λ_{W_σ} as in §**I**; ℓ is assumed unramified in $\mathcal{K}$. Let $K = \prod_p K_p$ be a level subgroup satisfying the hypotheses of **I**.2.8. We assume Q satisfies the hypotheses of **IV**.1.4 and Theorem **IV**.5.3. Denote by $D_{\mathcal{K}}$ the set of primes of $\mathbb{Q}$ divisible by primes of E that ramify in $\mathcal{K}$. Let $S_{bad} = SC \cup Q \cup \ D_{\mathcal{K}} \cup \{q(\mathfrak{r}), \ell\}$. We define $\mathbf{T}^{S_{bad}}$ and the completed Hecke algebras $\mathbb{T}_1(Q)$ and $\mathbb{T}_0(Q)$ as in §**III**.2. Recall that $\mathbb{T}_1(Q)$ is attached to the level group $K_{1,Q}^{[\ell]}$ rather than $K_{1,Q}$.

For $\mathfrak{q} \in Q$, we choose a lifting $\tilde{\alpha}_{\mathfrak{q}}$ of the Frobenius eigenvalue $\alpha_{\mathfrak{q}}$ to the ring $\mathcal{O}$. Similarly, let $\beta_{i,\mathfrak{q}}$, $i = 1, \ldots, n-1$, be the eigenvalues of Frobenius in $\overline{\rho}_\beta$ at $\mathfrak{q}$, and let $\tilde{\beta}_{i,\mathfrak{q}}$ be liftings to characteristic zero. We define unramified characters χ_i, $i = i, \ldots, n$ of $\mathcal{K}_{\mathfrak{q}}^\times$ by setting

$$\chi_1(\varpi) = \alpha_{\mathfrak{q}};\ \chi_i(\varpi) = \beta_{i-1,\mathfrak{q}},\ i = 1, \ldots, n-1.$$

Let $\overline{\rho}$ be a residual representation as in §**IV**.1.1. Let $\mathfrak{m} = \mathfrak{m}(\overline{\rho})$ be the maximal ideal of $\mathbb{T}_1(Q)$ generated by ℓ, by $U_{\mathfrak{q}} - \alpha_{\mathfrak{q}}$ and by

$$V_{i,\mathfrak{q}} - b^i(\alpha) \quad i = 1, \ldots, n-1$$

for each $\mathfrak{q} \in Q$, and by

$$trace\ \wedge^i r_{\overline{\rho}}(Frob_{\mathfrak{p}}) - T_{i,\mathfrak{p}} \quad i = 1, \ldots, n$$

for all divisors $\mathfrak{p}$ of p, as p runs through the set of primes not in S_{bad} that split in $\mathcal{K}_0$. Here $Frob_{\mathfrak{p}}$ denotes geometric Frobenius, $T_{i,\mathfrak{p}}$ is the standard Hecke operator defined in §**II**.2, $V_{i,\mathfrak{q}}$ is defined by (**II**.2.2), and $b^i(\chi;\chi_1)$ is the eigenvalue defined in **II**.2.3.

Note that $\mathbb{T}_1(Q)$ stabilizes the subspace $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{0,Q}) \subset \mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]})$. As in §**III**.2, we let $L_{0,Q}$ (resp. $L_{1,Q}$) denote the localization of $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{0,Q})$ (resp. $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q}^{[\ell]})$) at $\mathfrak{m}$. Let $\mathbb{T}_{0,Q}$ (resp. $\mathbb{T}_{1,Q}$) denote the Hecke algebra acting on $L_{0,Q}$ (resp. $L_{1,Q}$) generated only by the unramified Hecke operators at primes $p \notin S_{bad}$ that split in $\mathcal{K}_0$.

Hypothesis V.1.1. *The space $L_{0,\emptyset}$ is non-trivial.*

Hypothesis **V**.1.1 can be regarded as a strong form of Serre's conjecture for the group J. We are assuming that the residual representation $r_{\overline{\rho}}$ is modular of minimal level. In particular we are completely setting aside the problem of lowering the level solved by Ribet in the case of $GL(2)$. We let π denote an irreducible automorphic representation of J – with characteristic 0 coefficients – contained in the space $\mathcal{A}_{\mathfrak{m}}$ of automorphic forms on $J(\mathbf{A})$ generated by $L_{0,Q}$. By the *Hecke eigenvalues* of π we will mean the eigenvalues of the standard Hecke operators $T_{\mathfrak{p}}^{(i)}$, for $\mathfrak{p}$ dividing p as above, acting on the space of K_p-fixed vectors. By definition of $L_{0,Q}$, if π' is another irreducible automorphic representation contained in $\mathcal{A}_{\mathfrak{m}}$ then the Hecke eigenvalues of π and those of π' are congruent modulo ℓ.

Hypothesis V.1.2.

(i) The automorphic representation π is unramified at $Q \cup q(\mathfrak{r})$.

(ii) We assume $\mathfrak{r}$ has been chosen so that $q(\mathfrak{r})$ splits completely in $\mathcal{K}$ and is not congruent to $1 \pmod{\ell}$*, and so that the Frobenius at $\mathfrak{r}$ acts trivially on the extension $\mathcal{K}^1$ of $\mathcal{K}$ (notation as in §**IV**.5).*

(iii) The local component of π at $\mathfrak{q}$ is isomorphic to the induced representation $I(\chi)$.

The assumption that π is unramified at $q(\mathfrak{r})$ implies that $r_{\overline{\rho}}$ is unramified at primes dividing $q(\mathfrak{r})$, so the last condition of (ii) makes sense. Moreover, the characters χ_i are distinct $\pmod{\ell}$, by condition (d) of Theorem **IV**.5.3, so (iii) also makes sense.

Proposition V.1.3. *For an appropriate choice of $\mathfrak{r}$:*

1. There is a representation $\rho_{0,Q} : G_E \to \tilde{G}(\mathbb{T}_{0,Q})$ which is a deformation of $\overline{\rho}$ of type $\emptyset$ and satisfies trace $\wedge^i r_{\rho_{0,Q}}(Frob_{\mathfrak{p}}) = T_{\mathfrak{p}}^{(i)}$ for all divisors $\mathfrak{p}$ of rational primes not in S_{bad} that split in $\mathcal{K}_0$, and for $i = 1, \ldots, n$.

2. There is a representation $\rho_{1,Q} : G_E \to \tilde{G}(\mathbb{T}_{1,Q})$ which is a deformation of $\overline{\rho}$ of type Q and satisfies

$$trace \wedge^i r_{\rho_{1,Q}}(Frob_{\mathfrak{p}}) = T_{\mathfrak{p}}^{(i)}$$

for all divisors $\mathfrak{p}$ of rational primes not in S_{bad} that split in $\mathcal{K}_0$ and for $i = 1, \ldots, n$.

*Moreover, the residual representations $\overline{\rho}_{0,Q}$ and $\overline{\rho}_{1,Q}$ mod $\mathfrak{m}$ are isomorphic and satisfy conditions **IV**.1.1.0-4.*

Proof. The existence of the representations $\rho_{?,Q}$, with $? = 0$ or 1, is a consequence of Proposition **III**.2.1. That the residual representations are isomorphic follows from the relation between Frobenius elements and Hecke operators at unramified split places (Theorem **III**.1.5) and Chebotarev density. Conditions **IV**.1.1.0 and **IV**.1.1.1 are obvious, **IV**.1.1.2 follows from Propositions **III**.1.7 and **III**.1.3 and Theorem **III**.1.5, fe.1.1.3 is Lemma **III**.3.10, and **IV**.1.1.4 is Corollary **III**.3.7.

It remains to prove that $\rho_{0,Q}$ is a deformation of type $\emptyset$ and that $\rho_{1,Q}$ is a deformation of type Q. First, $\rho_{?,Q}$ is crystalline because each of its components is crystalline and because ℓ is assumed greater than n and unramified in $\mathcal{K}$. The second condition of **IV**.1.5.1 follows from the corresponding fact for the residual representation, and from Nakayama's lemma. Condition **IV**.1.5.3 for primes in SC is an immediate consequence of the last assertion of Proposition **III**.1.3. The only other possible ramification is at primes of E, and at primes above $\mathfrak{r}$. For the former set of primes, **IV**.1.5.3 follows easily from Theorem **III**.1.5 (since $\ell > n > 2$ and the ramification group at such primes is of order 2). The non-ramification at $\mathfrak{r}$ follows from Lemma **IV**.6.1, in view of Hypothesis **V**.1.2 (ii).

This leaves **IV**.1.5.2. Let π be an automorphic representation contributing to $\mathbb{T}_{?,Q}$. For $\mathfrak{q} \in Q$, $\pi_{\mathfrak{q}}$ is in (a) or (c) of Proposition **I**.2.10 for $? = 0$; for $\mathbb{T}_{1,Q}$ we also have case (b). We claim

(**V**.1.3.1) *Under hypothesis **IV**.5.3 (d), case (c) cannot occur.*

Admit (**V**.1.3.1) for the moment. Theorem **III**.1.5 then implies that $r_{\rho_{0,Q}}$ is unramified outside $SC \cup \ell$, hence is a deformation of type $\emptyset$. Now let $M_{1,Q}$ denote the free rank n $T_{1,Q}$-module on which $r_{\rho_{1,Q}}$ acts. Fix $\mathfrak{q} \in Q$ and a Frobenius element $F_{\mathfrak{q}}$ in the decomposition group $Z_{\mathfrak{q}}$, and let $M' \subset M_{1,Q}$ be the submodule of eigenvectors for $F_{\mathfrak{q}}$ with eigenvalue congruent to $\alpha_{\mathfrak{q}} \pmod{\mathfrak{m}}$. Then it follows from Corollary **IV**.5.10.3 and Nakayama's lemma, and our hypothesis that $\alpha_{\mathfrak{q}}$ has multiplicity one in the residual representation, that M' is a free rank one direct summand of $M_{1,Q}$ and that the quotient $M_{1,Q}/M'$ is unramified. This is what we need to verify **IV**.1.5.1.

The claim (**V**.1.3.1) is a consequence of the following Lemma.

Lemma V.1.4. *Fix $\mathfrak{q} \in Q$ and suppose π' is a cohomological automorphic representation of $J(\mathbf{A})$ such that $\pi'_{\mathfrak{q}}$ has a $K_{0,\mathfrak{q}}$-fixed vector but no $GL(n, \mathcal{O}_{\mathfrak{q}})$-fixed vector. Then π' does not contribute to $L_{0,Q}$.*

Proof. Assume π' contributes to $L_{0,Q}$, and let $\pi'_{\mathfrak{q}}$ be its local component at $\mathfrak{q}$. It is a subquotient of the representation $I(\chi')$ induced from some unramified n-tuple of characters χ'. Now $(\pi'_{\mathfrak{q}})^{K_{0,\mathfrak{q}}}$ has a non-trivial $U_{\mathfrak{q}}$-eigenspace V' with eigenvalue α', where $\alpha' \equiv \alpha \pmod{\mathfrak{m}_{\mathcal{O}}}$. Moreover, the Hecke operators $V_{i,\mathfrak{q}}$ act on V' with eigenvalues congruent to $b^i(\chi;\chi_1) \pmod{\mathfrak{m}_{\mathcal{O}}}$. It follows that the reductions $\pmod{\mathfrak{m}_{\mathcal{O}}}$ of $I(\chi')$ and $I(\chi)$ have a common subquotient. But the characters χ_i are distinct $\pmod{\mathfrak{m}_{\mathcal{O}}}$, hence the reduction $\pmod{\mathfrak{m}_{\mathcal{O}}}$ of $I(\chi)$ is irreducible by Proposition **II**.1.6 (i). It follows that $I(\chi')$ is irreducible, hence $\pi'_{\mathfrak{q}}$ is spherical.

Corollary V.1.5. *For any set Q as in* **IV***.1.4, there are surjective homomorphisms*

$$\phi_{0,Q} : R_{\emptyset} \rightarrow \mathbb{T}_{0,Q};$$

$$\phi_{1,Q} : R_Q \rightarrow \mathbb{T}_{1,Q}.$$

Proof. The existence of the homomorphisms $\phi_{0,Q}$ and $\phi_{1,Q}$ is a consequence of Proposition **V**.1.2 and the universal property of R_Q. To prove surjectivity it suffices to prove surjectivity mod $\mathfrak{m}$, by Nakayama's lemma. But mod $\mathfrak{m}$ the generator $T_{\mathfrak{p}}^{(i)}$ of $\mathbb{T}_{?,Q}$ is given by *trace* $\wedge^i r_{\overline{\rho}}(\gamma)$ for some $\gamma \in Gal(\overline{E}/E)$. To show that $T_{\mathfrak{p}}^{(i)}$ is in the image of $\phi_{?,Q}$ mod $\mathfrak{m}$, it therefore suffices to recall that the coefficients of the characteristic polynomial of $\rho_Q(\gamma)$ belong to R_Q. Indeed, it follows as in [W, p. 510] from [M, §1.8] and **IV**.2.8 that R_Q is generated by traces, but we do not need this fact.

We also have to show that $U_{\mathfrak{q}}$ and the $V_{i,\mathfrak{q}}$ are in the image of $\phi_{?,Q}$ mod $\mathfrak{m}$. But these are given respectively by $\chi_{\mathfrak{q}}(Frob_{\mathfrak{q}})$ and by the coefficients of the characteristic polynomial of $r'_{\mathfrak{q}}(Frob_{\mathfrak{q}})$, in the notation of **IV**.1.5.2.

Via $\phi_{0,Q}$ (resp. $\phi_{1,Q}$) the module $L_{0,Q}$ (resp. $L_{1,Q}$) becomes an $R_{\emptyset}$-module (resp. an R_Q-module). Write $\mathbb{T} = \mathbb{T}_{0,Q}$, $R = R_{\emptyset}$. We can finally state our main theorem.

Theorem V.1.6. *Let $\overline{\rho} : G_E \rightarrow \tilde{G}(\overline{\mathbb{F}}_l)$ be a representation satisfying Hypotheses* **V***.1.1 and* **IV***.5.2. We assume $\ell > n$ and, for all $v \in SC$, we assume ℓ is banal for $GL(n, E_v)$. Finally, assume $\mathfrak{r}$ has been chosen to satisfy Hypothesis* **V***.1.2 (ii). Then the homomorphism*

$$\phi_{0,\emptyset} : R \rightarrow \mathbb{T}$$

is an isomorphism. Moreover, R (and therefore $\mathbb{T}$) is a complete intersection of dimension zero, and $L_{0,\emptyset}$ is a free R-module.

The proof of this theorem, which occupies the folllowing section, is based on a recent improvement of the method of [TW], found independently and nearly simultaneously by Diamond and Fujiwara [D,Fu]. The interest of Theorem **V**.1.6, of course, lies in the following corollary:

Theorem V.1.7. *Let $\overline{\rho} : G_E \to \tilde{G}(\overline{\mathbb{F}}_l)$ be a representation satisfying conditions the hypotheses of Theorem* **V***.1.6. Then any deformation of $\overline{\rho}$ of type $\emptyset$, as in* **IV***.1.5.1-3, is modular. Specifically, suppose $\rho : G_E \to \tilde{G}(\mathcal{O}')$ is a deformation of $\overline{\rho}$ of type $\emptyset$, where $\mathcal{O}'$ is the ring of integers in a finite extension $\mathbb{K}'$ of the fraction field $\mathbb{K}$ of $\mathcal{O}$. Then there is an automorphic representation π' of G such that $\rho' = \rho(\pi')$.*

Other corollaries are derived in §**V**.4.

V.2. *Application of the theorem of Diamond and Fujiwara.*

For each integer $m > 0$ we now choose a set Q_m as in Theorem **IV**.5.3. In particular, there is an integer r such that each Q_m consists of r elements. For each such $Q = Q_m$ we have an R_Q module $L_{1,Q}$ which is free and finite over $\mathbb{Z}_\ell$. Define Δ_Q as in **IV**.1.4, and for any $\mathbb{Z}_\ell[\Delta_Q]$-module N let N_{Δ_Q} denote the module of coinvariants. Then we have:

Theorem V.2.1. *([D, Lemma 2.1;Fu, Theorem 1.2]) Suppose that*
(i) $L_{0,\emptyset} \neq \{0\}$;
(ii) For each $Q = Q_m$,
(a) R_Q can be topologically generated by r elements,
(b) $L_{1,Q}$ is a free $\mathcal{O}[\Delta_Q]$-module and
(c) there are isomorphisms

$$(L_{1,Q})_{\Delta_Q} \xrightarrow{\sim} L_{0,\emptyset}$$

as R_Q-modules, via the natural map $R_Q \to R = R_\emptyset$.

Then R is a complete intersection and $L_{0,\emptyset}$ is a free R-module. In particular R is isomorphic to its image $\mathbb{T}$ in the endomorphism algebra of $L_{0,\emptyset}$.

The theorem as stated here is closer to Fujiwara's formulation. In order to apply Fujiwara's axioms, we need

Lemma V.2.2. *For each $\mathfrak{q} \in Q$ let $\delta_{\mathfrak{q}}$ be a generator of $\Delta_{\mathfrak{q}}$. Let $m_Q \subset \mathcal{O}[\Delta_Q]$ be the ideal generated by $\{\delta_{\mathfrak{q}}, \mathfrak{q} \in Q\}$. Then the projection $R_Q \to R$ defines a natural isomorphism*

$$R_Q/m_Q \xrightarrow{\sim} R.$$

Proof. The quotient R_Q/m_Q is tautologically the universal ring for ℓ-adic deformations of $\overline{\rho}$ of type Q on which the inertia groups at primes in Q act trivially. This is just R.

We turn to the proof of Theorem **V**.1.6. Condition (i) of Theorem **V**.2.1 is Hypothesis **V**.1.1. Condition (ii) (a) is a consequence of Theorems **IV**.2.2 and **IV**.5.3, combined with Proposition **V**.1.3. Theorem **V**.1.6 is now a consequence of Theorem **V**.2.1 and the following two propositions.

Proposition V.2.3. *For each $Q = Q_m$, $L_{1,Q}$ is a free $\mathcal{O}[\Delta_Q]$-module.*

Proposition V.2.4. *For each $Q = Q_m$,*

$$(L_{1,Q})_{\Delta_Q} \xrightarrow{\sim} L_\emptyset$$

as R_Q-modules, via the natural map $R_Q \to R$.

Proof of Proposition **V***.2.3.* We return to the notation of §**I**.2. The hypotheses of Theorem **V**.1.6 imply that ℓ does not divide the order of $K_{1,Q}/\mathbb{K}$, hence the hypotheses of Corollary **I**.2.15 are verified in our situation. It follows from Corollary **I**.2.17 that $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q})$ is a free $\mathcal{O}[\Delta_Q]$-module. But the localization $L_{1,Q}$ at $\mathfrak{m}$ is a direct summand in $\mathcal{A}_\ell(J, \Lambda_{W_\sigma}, K_{1,Q})$ (cf. [DDT, p. 98]), hence $L_{1,Q}$ is also free over $\mathcal{O}[\Delta_Q]$.

Proof of Proposition **V***.2.4.* Corollary **I**.2.17 and Proposition **V**.2.3 imply that $(L_{1,Q})_{\Delta_Q} \xrightarrow{\sim} L_{0,Q}$ as R_Q-modules. So we need to construct an R_Q-isomorphism $L_\emptyset \to L_{0,Q}$.

For each $\mathfrak{q} \in Q$, let $X_{\infty,\mathfrak{q}}$ be the operator constructed in Lemma **II**.1.11, and let $X_{\infty,Q} = \prod \mathfrak{q} \in Q X_{\infty,\mathfrak{q}}$. This defines a homomorphism

$$(\mathbf{V}.2.3.1) \qquad X_{\infty,Q} : L_\emptyset \to L_{0,Q}$$

that commutes with all unramified Hecke operators outside S_{bad}, hence with $\mathbb{T}_{0,Q}$ and, via $\phi_{0,Q}$, with $R_\emptyset$. Each $X_{\infty,\mathfrak{q}}$ is injective $\pmod{\ell}$ by construction, since it just takes the spherical vector to the selected $U_\mathfrak{q}$-eigenvector. So to prove $X_{\infty,Q}$ defines an isomorphism it suffices to check that it becomes an isomorphism after tensoring with $\overline{\mathbb{Q}}_\ell$.

Now, Lemma **V**.1.3 implies that the $J(\mathbf{A}^f)$-representation generated by $L_{0,Q} \otimes \overline{\mathbb{Q}}_\ell$ is generated by the vectors spherical at $\mathfrak{q}$ for all $\mathfrak{q} \in Q$. So it suffices to prove that, if π' is an irreducible cohomological automorphic representation of $J(\mathbf{A})$ contained in $\mathcal{A}_\mathfrak{m}$ (cf. the discussion following Hypothesis **V**.1.1) and spherical at all $\mathfrak{q} \in Q$, then the the operator $X_{\infty,\mathfrak{q}}$ maps the spherical subspace of $\pi'_\mathfrak{q}$ onto the space of $U_\mathfrak{q}$-eigenvectors with eigenvalue congruent $\pmod{\ell}$ to $\alpha_\mathfrak{q}$. But for any $\pi' \subset \mathcal{A}_\mathfrak{m}$, it follows from **IV**.1.4.3 that the space of $U_\mathfrak{q}$-eigenvectors of $\pi'_\mathfrak{q}$ with eigenvalue congruent $\pmod{\ell}$ to $\alpha_\mathfrak{q}$, is one-dimensional. Thus it suffices to show that the operator $X_{\infty,\mathfrak{q}}$ is non-trivial on the spherical subspace, and this follows from the injectivity of (**V**.2.3.1).

Remark V.2.5. Looking more closely at the results in [D] and [Fu], it follows from Proposition **V**.2.3 that, with our choice of $Q = Q_m$, equality holds in Theorem **IV**.2.2. Tracing through the proof of Theorem **IV**.2.2, it follows that the $+1$-eigenspace of c in $ad\ r_{\overline{\rho}}$ must have dimension precisely equal to $\frac{n(n-1)}{2}$ in **IV**.1.1.3. It follows that in Lemma **III**.3.4 the matrix B must be symmetric. We can thus improve Corollary **III**.3.7, and conclude that the composite $\nu \circ \rho(\pi) : Gal(\overline{E}/E) \to \tilde{G}(\mathcal{O})$ equals ω^{1-n}.

This provides an unexpected proof of the fact that the representations $\rho(\pi)$ considered here correspond to Langlands parameters of the unitary group. Indeed, let Φ_n be the $n \times n$ anti-diagonal matrix with entries

$$(\Phi_n)_{ij} = (-1)^{1+i} \delta_{i,n+1-j}.$$

Define the L-group ${}^LG$ by analogy with the definition of $\tilde{G}$ in (**I**.1.4-6); we replace $\tilde{G}^0$ by

$$\hat{G} = \{g = (g_1, g_2) \in GL(n) \times GL(n) \mid \exists a \in GL(1) \text{ such that } g_2 = a \cdot \Phi_n \cdot {}^t g_1^{-1} \Phi_n^{-1}\}.$$

Let $\hat{r}_\rho(\pi) : G_{\mathcal{K}} \to \hat{G}$ be defined by

$$\hat{r}_\rho(\pi)(g) = (r_\rho(\pi)(g), \Phi_n \cdot (r_\rho(\pi)^c(g)\Phi_n^{-1} \cdot \omega^{1-n}(g)).$$

It follows from Lemma 15.1.2 of [R1], as corrected on p. 419 of [R2], that the existence of an L-homomorphism ${}^L\rho(\pi) : G_E \to {}^LG$, extending $\hat{r}_\rho(\pi)$, is equivalent to the symmetry of the matrix B.

V.3. *Examples for Hypotheses* **IV**.*5.2.*

Our hypotheses **IV**.5.2 on the image of the residual representation are certainly stronger than necessary. We chose them for convenience, as the shortest list of conditions we found that guarantee existence of the sets Q_m and the prime $\mathfrak{r}$.

Hypothesis **IV**.5.2 (a) can in some instances be checked purely locally. For example, this is the case if the local Galois extension generated by $r_{\overline{\rho}}$ at some $v \in SC$ is disjoint from $E_v(\zeta_\ell)$. Hypothesis **IV**.5.2 (b) seems the most unnatural, and ought to be a consequence of a more general condition.

If the complement $Ad(\overline{\rho})^0$ of the trace subspace of $Ad\,\overline{\rho}$ is absolutely irreducible, then **IV**.5.2 (c) follows immediately from Lemma **IV**.4.9. We do not have a good criterion for irreducibility of $Ad(\overline{\rho})^0$. On the other hand, for applications, we are interested in cases when $Ad(\overline{\rho})^0$ is highly reducible. The following Lemma will be applied to the problem of constructing automorphic tensor products.

Lemma V.3.1. *Let G be a group, $H \subset G$ a normal subgroup of index b, with G/H cyclic of order b, and let $\tau : G \to GL(a,k)$, $\chi : H \to GL(1,k)$ be representations of degree a and 1, respectively. Suppose $r_{\overline{\rho}} = \tau \otimes Ind_H^G \chi$ is absolutely irreducible and that the map $G \to Im(\tau) \times Im(Ind_H^G \chi)$ is surjective. Suppose $Ad\,\tau$ satisfies the analogue of Hypothesis* **IV**.*5.2(c), and suppose the elements s with distinct eigenvalues in τ have eigenvalues of order prime to b. Then $Ad\ r_{\overline{\rho}}$ satisfies Hypothesis* **IV**.*5.2 (c).*

Proof. Let D be a set of coset representatives for H in G, and for $\delta \in D$ let χ^δ be the δ-conjugate of χ. Our hypotheses imply that the χ^δ are all distinct. Then we see easily that

$$Ad\ r_{\overline{\rho}} \xrightarrow{\sim} Ad\,\tau \otimes \bigoplus_{\delta \in D} Ind_H^G \chi^{1-\delta}, \qquad (\mathbf{V}.3.2)$$

Say $Ad\,\tau = \oplus_{i \in I} V_i$. For any $\delta \in D$, let $G(\delta) \supset H$ be the stabilizer of $\chi^{1-\delta}$, and let $X(\delta)$ denote the set of extensions of $\chi^{1-\delta}$ to a character of $G(\delta)$, so that

$$Ind_H^G \chi^{1-\delta} = \oplus_{x \in X(\delta)} Ind_{G(\delta)}^G x.$$

Then the irreducible summands of $Ad\ r_{\overline{\rho}}$ are indexed by $\{(i, \delta, x) | i \in I, \delta \in D, x \in X(\delta)\}$. Let $G_1 = Im(\tau)$, $G_2 = Im(Ind_H^G \chi)$. For each triple (i, δ, x), it suffices to find pairs $(s_1, s_2) \in G_1 \times G_2$ where s_1 satisfies Hypothesis **IV**.5.2 (c) for V_i with eigenvalues of order prime to b, and s_2 satisfies Hypothesis **IV**.5.2 (c) for $Ind_{G(\delta)}^{G} x$. But the existence of s_1 is given, and we can take always take s_2 to be a generator of G/H.

The awkward hypothesis about the orders of the eigenvalues is easy to verify in examples.

V.4. *Examples of automorphic tensor products, and other applications*

We now introduce two positive integers, n and n', and define unitary similitude groups G_n and $G_{n'}$ attached to division algebras D_n and $D_{n'}$ of dimension n^2 and $(n')^2$, respectively, over $\mathcal{K}$, as in **I**.1. Let π and π' be cohomological automorphic representations of G_n and $G_{n'}$, respectively. Let Π and Π' denote the base changes to $GL(n, \mathcal{K})$ and $GL(n', \mathcal{K})$, respectively; these are the representations denoted $BC(\pi)_1$ and $BC(\pi')_1$ in **I**.3.

We suppose $\ell > n \cdot n'$. Define the modular representations $r_{\overline{\rho}}(\pi)$ and $r_{\overline{\rho}}(\pi')$ over an appropriate finite field k of characteristic ℓ. Our hypothesis on ℓ ensures that both $r_{\overline{\rho}}(\pi)$ and $r_{\overline{\rho}}(\pi')$ are Fontaine-Laffaille representations, with (Hodge-Tate) weights $(a_1 < \cdots < a_n)$ and $(b_1 < \cdots < b_{n'})$, respectively. Suppose

$$\{a_i + b_j \mid 1 \leq i \leq n;\ 1 \leq j \leq n'\} = \{0, 1, \ldots, n \cdot n' - 1\}. \qquad (\mathbf{V}.4.1)$$

For example, this is the case if π has cohomology with respect to the trivial representation, so $\{a_1, \ldots, a_n\} = \{0, \ldots, n-1\}$, and if $\mathfrak{b}_j = (n-1)j + 1$ for all j. Next, we suppose there is a cyclic extension $\mathcal{K}'$ of $\mathcal{K}$ of degree n', a complete local $W(k)$-algebra $\mathcal{O}$ with residue field k, and an algebraic Hecke character χ of $\mathcal{K}'$, with values in $\mathcal{O}$, such that

$$r_{\overline{\rho}}(\pi') \cong Ind_{\mathcal{K}'/\mathcal{K}} \overline{\chi}. \qquad (\mathbf{V}.4.2)$$

Here $\overline{\chi}$ is the reduction of (the λ-adic representation associated to) χ modulo the maximal ideal of $\mathcal{O}$.

For any cyclic extension K'/K of number fields and any integer n, we let $BC_{K'/K}$ (resp. $AI_{K'/K}$) denote the base change (resp. automorphic induction) map of Arthur-Clozel [AC] from automorphic representations of $GL(n, K)$ to automorphic representations of $GL(n, K')$ (resp. from automorphic representations of $GL(n, K')$ to automorphic representations of $GL(n[K' : K], K')$. The same notation is used for cyclic extensions of local fields. We make the following additional hypotheses.

Hypotheses V.4.2. *(a) Let $\pi(\chi)$ denote the automorphic induction of χ from $\mathcal{K}'$ to $\mathcal{K}$, as an automorphic representation of $GL(n', \mathcal{K})$. Then $\pi(\chi)$ is cuspidal and satisfies $\pi(\chi)^c \xrightarrow{\sim} \check{\pi}(\chi))$.*

(b) For every finite place v of $\mathcal{K}$, Π_v is either unramified or supercuspidal. If Π_v is supercuspidal, then v is split over E, and $\pi(\chi)_v$ is also supercuspidal. At such places, ℓ is prime to $|GL(n \cdot n', k(v))|$.

(c) The set of places v at which Π_v is supercuspidal is non-empty and contains at least two elements non-conjugate over E if $[E : \mathbb{Q}]$ is odd and $n \cdot n'$ is congruent to $2 \pmod 4$.

(d) If Π_v is unramified but v is ramified over E, let v_0 denote the restriction of v to E. Then the automorphic induction $AI_{\mathcal{K}'/\mathcal{K}}(BC_{\mathcal{K}'/\mathcal{K}}\pi_v) \otimes \chi_v$ descends to a representation π_0 of the quasi-split unitary group G_{nn',v_0} of degree $n \cdot n'$ over E_{v_0} such that π_0 is spherical with respect to the conjugacy class of very special maximal compact subgroups.

Theorem V.4.3. *Suppose π, π', χ, and ℓ satisfy Hypotheses* **V***.4.2. Suppose $r_{\overline{\rho}}(\pi)$ and $r_{\overline{\rho}}(\pi')$ satisfy (***V***.4.1), $r_{\overline{\rho}}(\pi)$ satisfies Hypothesis* **IV***.5.2, and the tensor product $r_{\overline{\rho}}(\pi) \otimes r_{\overline{\rho}}(\pi')$ satisfies Hypotheses* **IV***.5.2 (a) and (b) and the hypotheses of Lemma* **V***.3.1. Then there exists a cohomological cuspidal automorphic representation $\Pi \boxtimes \Pi'$ of $GL(n \cdot n', \mathcal{K})$ such that, for almost all places v of $\mathcal{K}$, the local Euler factor of $\Pi \boxtimes \Pi'$ satisfies*

$$L(s, (\Pi \boxtimes \Pi')_v) = L(s, \Pi_v \times \Pi'_v),$$

where the right-hand side is the local Rankin-Selberg convolution.

Proof. Let $\rho = r_\rho(\pi) \otimes r_\rho(\pi')$, and let r_ρ denote the residual representation on $GL(n \cdot n', k)$. We have set things up so that $\Pi \boxtimes \Pi(\chi) \overset{\mathrm{def}}{=} AI_{\mathcal{K}'/\mathcal{K}}(BC_{\mathcal{K}'/\mathcal{K}}\pi) \otimes \chi$ is a cuspidal automorphic representation of $GL(n \cdot n', \mathcal{K})$, cohomological with respect to the trivial representation. Our hypotheses **V**.4.2 (a) and (b) guarantee that $(\Pi \boxtimes \Pi(\chi))_v$ is either unramified or supercuspidal at all places v, and **V**.4.2 (c) guarantees the existence of a unitary similitude group $G_{n \cdot n'}$ of degree $n \cdot n'$, as in §**I**.1, such that $\Pi \boxtimes \Pi(\chi)$ descends to an automorphic representation Π'' of $G_{n \cdot n'}$. Lemma **V**.3.1 then implies that $r_\rho(\Pi'')$ satisfies Hypothesis **IV**.5.2. Moreover,

$$r_\rho \xrightarrow{\sim} r_\rho(\Pi'').$$

The theorem is thus a consequence of Theorem **V**.1.7.

In a later draft, we hope to include numerical examples of Theorem **V**.4.3. We will also indicate applications to non-solvable base change and automorphic induction, generalizing the results of Clozel and Hida for $GL(2)$.

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