ALGEBRAIC NUMBER THEORY W4043

Homework, week 4, due February 17

1. Let $d > 0$ be a square-free positive integer congruent to 2 (mod 4).
   (a) Every unit $u \in \mathbb{Z}[^d]$ is of the form $a - b \sqrt{d}$ where $a^2 - db^2 = \pm 1$, and
   the group $\Gamma$ of units is the product of an infinite cyclic group with \{±1\}.
   Consider the subset $\Sigma$ of $\Gamma$ consisting of $u_i = a_i - b_i \sqrt{d}$ with $a_i > 0, b_i > 0$,
   ordered so that $b_1 \leq b_2 \leq b_3 \ldots$. Show that $u_1$ and $-1$ are generators of $\Gamma$.
   The element $u_1$ is called the fundamental unit of $\mathbb{Z}[^d]$.
   (b) Show that the following algorithm finds $u_1$: Letting $b = 1, 2, 3, \ldots$, consider
   the quantities $q^\pm(b) = db^2 \pm 1$. Let $b_1$ be the smallest positive integer such that either
   $q^+(b_1)$ or $q^-(b_1)$ is a perfect square. Let $a_1$ be the positive square root of $q(b_1)$; then
   $u_1 = a_1 - b_1 \sqrt{d}$.
   (c) Use this algorithm to find the fundamental units $u_1$ of $\mathbb{Z}[^6], \mathbb{Z}[^{10}], \mathbb{Z}[^{14}]$.
   In each case determine $N_{K/\mathbb{Q}}(u_1)$, where $K = \mathbb{Q}(\sqrt{d})$ in each case.


3. As Hindry shows on p. 99, the ring $R = \mathbb{Z}[^{10}]$, which is equal to
   the ring of integers in $\mathbb{Q}(\sqrt{10})$, is not a principal ideal domain. Indeed, the
   integer 9 has two inequivalent factorizations:
   $9 = 3^2 = (\sqrt{10} - 1)(\sqrt{10} + 1)$.

   (a) Use the computation in 2 (c) to confirm that the two factorizations
   are indeed inequivalent.
   (b) The integer 10 is definitely a square modulo 3. What is the prime
   factorization of the ideal $(3) \subset R$?

4. Let $R$ be a Dedekind ring with only finitely many prime ideals. Show
   that $R$ is a PID. (Hint: say $p_1, p_2, \ldots, p_r$ are the prime ideals. Find an
   element $x_i \in p_i$ that is not in any of the $p_j$ with $j \neq i$, and factor the ideal
   $(x_i)$. Another piece of information is necessary.)

5. List the squares modulo 8.
   (1) Show by testing all possibilities that the equation $x^2 - 11y^2 = 7$ has
   no solution modulo 8.
   (2) Use this to show that every unit in the ring of integers of $\mathbb{Q}(\sqrt{11})$
   has norm 1.