Problem set #8 solutions

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(a) By definition $f(a) = 0$ iff n divides a; likewise $g(a) = 0$ iff m divides a. Therefore, $(f \times g)(a) = (0, 0)$ iff both n and m divide a; equivalently, iff nm divides a, since $\gcd(n, m) = 1$.

Hence $\ker(f \times g) = nm\mathbb{Z}$. //

(b) Recall the first isomorphism theorem, which states the following:

> *Suppose $f : G_1 \to G_2$ is a group homomorphism.*
> *Then $\operatorname{im} f \subseteq G_2$ is a subgroup and $\ker f \trianglelefteq G_1$ is a normal subgroup, with $G_1/\ker f \cong \operatorname{im} f$.*

Applying this to $(f \times g) : \mathbb{Z} \to \mathbb{Z}_n \times \mathbb{Z}_m$, we have that $\mathbb{Z}/\ker(f \times g) = \mathbb{Z}/nm\mathbb{Z} \cong \mathbb{Z}_{nm}$ is isomorphic to $\operatorname{im}(f \times g)$, which is a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_m$.

But $\mathbb{Z}_{nm}$ has nm elements, as has $\mathbb{Z}_n \times \mathbb{Z}_m$; so having already shown that $\mathbb{Z}_{nm}$ is isomorphic to a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_m$, we can in fact conclude that $\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m$. //

(c) Suppose $n = m$. Then $f(a) = 0$ and $g(a) = 0$ iff n divides a, so $\ker(f \times g) = n\mathbb{Z}$; and it is readily seen that $\operatorname{im}(f \times g) = \{(a, a) : a \in \mathbb{Z}_n\} \subset \mathbb{Z}_n \times \mathbb{Z}_n$.

An isomorphism from $\mathbb{Z}_n$ to the image $\{(a, a) : a \in \mathbb{Z}_n\}$ is given by $a \leftrightarrow (a, a)$.

Hence $\mathbb{Z}/\ker(f \times g) = \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$ is isomorphic to $\operatorname{im}(f \times g) \cong \mathbb{Z}_n$, as expected. // ■

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If (i) is true, then $H \subset N$, so $HN := \{hn : h \in H, n \in N\}$ must be equal to N, not G; thus, (ii) is false. //

Conversely, suppose (i) is false. Recall the second isomorphism theorem, which states the following:

> *Suppose $H \subseteq G$ is a subgroup and $N \trianglelefteq G$ is a normal subgroup.*
> *Then $HN \subseteq G$ is a subgroup and $(H \cap N) \trianglelefteq H$ is a normal subgroup, with $H/(H \cap N) \cong HN/N$.*

So $HN \subseteq G$ is a subgroup. Clearly $N \subseteq HN$, whereas $H \not\subset N$, so there exists $h \in H \subseteq HN$ with $h \notin N$; thus, $N \subsetneq HN$. Now consider $N \subsetneq HN \subseteq G$. Since $N \triangleleft G$, we can apply the correspondence theorem, which yields $\{1\} \subsetneq HN/N \subseteq G/N$; but $|HN/N|$ divides $|G/N| = [G : N] = p$ by Lagrange's theorem, and since $1 < |HN/N|$, we deduce that $|HN/N| = p$. Hence $HN/N = G/N$, and $HN = G$.

Finally, we have $H/(H \cap N) \cong HN/N$; hence $[H : (H \cap N)] = |H/(H \cap N)| = |HN/N| = p$.

Thus, (ii) is true. //

Having proven both "(i) is true implies (ii) is false" and "(i) is false implies (ii) is true", we conclude that exactly one of (i) and (ii) must be true. ■

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(a) Let $f : G \to G/N$ and $g : G \to G/M$ be the two natural maps, and define $(f \times g) : G \to (G/N) \times (G/M)$. Then, as in problem 1, we have $f(a) = 1$ iff $a \in N$, and $g(a) = 1$ iff $a \in M$. Therefore, $(f \times g)(a) = (1, 1)$ iff $a \in N$ and $a \in M$; that is, iff $a \in (N \cap M)$. Hence $\ker(f \times g) = (N \cap M)$.

Also, $(f \times g)$ is surjective: Let $(xN, yM) \in (G/N) \times (G/M)$ be any element. Since $yx^{-1} \in G = NM$, we can write $yx^{-1} = nm$ with $n \in N$ and $m \in M$. Let $a \coloneqq xn = ym^{-1}$; then $f(a) = aN = xnN = xN$ and $g(a) = aM = ym^{-1}M = yM$. Hence $(f \times g)(a) = (xN, yM)$, and $(f \times g)$ is surjective.

By the first isomorphism theorem, $G/\ker(f \times g) = G/(N \cap M)$ is isomorphic to $(G/N) \times (G/M)$. //

(b) By the second isomorphism theorem, we have $N/(N \cap M) \cong NM/M$ and $M/(M \cap N) \cong MN/N$.

Note, since $M \lhd G$, that any $x = nm \in NM$ where $n \in N$ and $m \in M$ can be written as $x = (nmn^{-1})n$ with $nmn^{-1} \in M$, so $x \in MN$; so $NM = G$ implies $MN = G$. Thus, if $NM = G$ and $(N \cap M) = \{1\}$, then $N \cong G/M$ and $M \cong G/N$. Hence $G = G/(N \cap M) \cong (G/N) \times (G/M) \cong M \times N$, as desired. // ■

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§11.3, exercise 14

We know that $\varphi : G \to G/N$ is a homomorphism, and that H is a subgroup. We must show that $\varphi^{-1}(H)$ is closed under multiplication, contains the identity, and contains inverses.

- If $g_1, g_2 \in \varphi^{-1}(H)$, then $\varphi(g_1), \varphi(g_2) \in H$; so $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) \in H$, which means $g_1 g_2 \in \varphi^{-1}(H)$.
- Clearly $\varphi(1)$ is the identity coset, which must be in H; so $1 \in \varphi^{-1}(H)$.
- If $g \in \varphi^{-1}(H)$, then $\varphi(g) \in H$, so $\varphi(g^{-1}) = \varphi(g)^{-1} \in H$, which means $g^{-1} \in \varphi^{-1}(H)$.

Hence $\varphi^{-1}(H)$ is a subgroup. //

Finally, each element $h \in H$ is a coset containing $|N|$ elements of G, and φ maps these $|N|$ elements to h. Since different cosets are disjoint, $\varphi^{-1}(H)$ contains $|H||N|$ elements of G. // ■

§11.3, exercise 17

False. Consider $G_1 = \mathbb{Z}_6$ and $G_2 = \mathbb{Z}_2$; these are abelian, so all subgroups are normal. Then let $\varphi : G_1 \to G_2$ be reduction mod 2; that is, $\varphi([0]) = \varphi([2]) = \varphi([4]) = [0]$ and $\varphi([1]) = \varphi([3]) = \varphi([5]) = [1]$.

Then let $H_1 \coloneqq \{[0], [3]\} \trianglelefteq G_1$ and $H_2 \coloneqq \mathbb{Z}_2 \trianglelefteq G_2$; we check that $\varphi(H_1) = H_2$.

However, $|G_1/H_1| = [G_1 : H_1] = 3 \neq 1 = [G_2 : H_2] = |G_2/H_2|$, so G_1/H_1 is not isomorphic to G_2/H_2. ■