HW 7 Solutions

April 1, 2020

1. If S_n has a subgroup isomorphic to $\mathbf{Z}_7 \times \mathbf{Z}_7$, then by Lagrange's theorem, we must have

$$49 = \#(\mathbf{Z}_7 \times \mathbf{Z}_7) | \#S_n = n!$$

This happens only if $n \geq 14$ (14! contains 7 and 14 as factors, but 13! only has one factor of 7). Conversely, we must show that $\mathbf{Z}_7 \times \mathbf{Z}_7$ is a subgroup of S_n for $n \geq 14$. Let $\sigma = (12 \ldots 7), \tau = (8 \ldots 14)$. Let $H = \langle \sigma, \tau \rangle \subset S_n$. This is $\{\sigma^i \tau^j | i, j \in \mathbf{Z}\}$ since σ and τ commute. It is enough to define an injective homomorphism $f : \mathbf{Z}_7 \times \mathbf{Z}_7 \to S_n$ with image H. For this, we set $f([i], [j]) = \sigma^i \tau^j$. This is:

Well-defined: If $i' = i + 7m$ and $j' = j + 7n$, then $\sigma^{i'} \tau^{j'} = \sigma^i \sigma^j$ since $\sigma^7 = \tau^7 = e$.

A homomorphism:

$$\begin{aligned} f([i_1] + [i_2], [j_1] + [j_2]) = f([i_1 + i_2], [j_1 + j_2]) &= \sigma^{i_1 + i_2} \tau^{j_1 + j_2} = \sigma^{i_1} \tau^{j_1} \sigma^{i_2} \tau^{j_2} \\ &= f([i_1], [j_1]) \cdot f([i_2], [j_2]), \end{aligned}$$

where we use that σ and τ commute as they are disjoint cycles.

Injective: If some $\sigma^i \tau^j = e$, then $\sigma^i = \tau^{-j} \in \langle \sigma \rangle \cap \langle \tau \rangle = \{e\}$. Thus $\sigma^i = \tau^j = e$, which implies $7|i$ and $7|j$, as needed.

2. (a) (14356) = (14)(43)(35)(56) is even.

   (b) (156)(234) = (15)(56)(23)(34) is even.

   (c) (1426)(142) = (14)(42)(26)(14)(42) is odd.

3. 8. (12345)(678) has order 15 and is in the alternating group since any cycle having odd length is in the alternating group. (Proof: $(n_1 \ldots n_k) = (n_1 n_2)(n_2 n_3)...(n_{k-1} n_k)$ is a product of $k - 1$ transpositions, so a cycle of odd length is even and a cycle of even length is odd.)

   9. A_8 has no element of order 26, in fact, neither does S_8, since $26 \nmid 8!$.

4. 22. We are allowed to use Judson, Lemma 5.14. So, suppose $\sigma_1 \cdots \sigma_m = \sigma = \tau_1 \cdots \tau_n$ with σ_i and τ_i transpositions. Then we can write $\sigma^{-1} = \sigma_m \cdots \sigma_1$, and so

$$e = \sigma^{-1} \sigma = \sigma_m \cdots \sigma_1 \tau_1 \cdots \tau_n.$$

Then by Lemma 5.14, $m+n$ is even. This is true iff $m \equiv n \mod 2$, as needed.

23. If $\sigma = (n_1 \cdots n_{2k+1})$, then

$$\sigma^2 = (n_1 n_3 n_5 \cdots n_{2k+1} n_2 n_4 \cdots n_{2k}).$$

24. See solution to problem 8. $(abc) = (ab)(bc)$.

25. An element $\sigma \in A_n$ is by definition the product $\sigma = \sigma_1 \cdots \sigma_{2k} = (\sigma_1\sigma_2)(\sigma_3\sigma_4)\cdots(\sigma_{2k-1}\sigma_{2k})$. It suffices to show that each $\sigma_{2i-1}\sigma_{2i}$ can be written as a product of 3-cycles. If $\sigma_{2i-1} = \sigma_{2i}$ we may simply remove this term from the product above. Otherwise we have two cases.

Case 1. $\sigma_{2i-1} = (ab)$ and $\sigma_{2i} = (cd)$ are disjoint. Then we have

$$\sigma_{2i-1}\sigma_{2i} = (ab)(cd) = (abc)(bcd).$$

Case 2. $\sigma_{2i-1} = (ab)$ and $\sigma_{2i} = (bc)$ share one element in common. Then

$$(ab)(bc) = (abc).$$

26. (a) It suffices to show that every transposition (ij), $1 < i < j$ can be written as a product of transpositions of the form $(1k)$. For this use

$$(ij) = (1i)(1j)(1i).$$

(b) It suffices by (a) to show that every $(1i)$ can be written as a product of transpositions of the form $(k \quad k+1)$. We prove this by induction on i. For the base case $i = 2$ there is nothing to prove. Assume we know it for $(1i)$. Then

$$(1 \quad i+1) = (i \quad i+1)(1i)(i \quad i+1),$$

and using the inductive hypothesis to express $(1i)$ as a product of $(k \quad k+1)$'s, we are done.

(c) It suffices by (b) to show every $(i \quad i+1)$ can be written as a product of the elements $(12), (12\ldots n)$. We prove this by induction on i. For $i = 1$, there is nothing to prove. Assume we know it for i. Then

$$(i+1 \quad i+2) = (12\ldots n)(i \quad i+1)(12\cdots n)^{-1},$$

and using the inductive hypothesis and the fact that $(12\ldots n)^{-1} = (12\ldots n)^{n-1}$, we are done.