GU4041: Intro to Modern Algebra I

Professor Michael Harris
Solutions by Iris Rosenblum-Sellers

Homework 6

1) Determine the *index* (that is, the number of right cosets) of the following subgroups in the corresponding groups

a) {0,2,4} in $(\mathbb{Z}/6\mathbb{Z}, +)$

Answer: $\{0, 2, 4\} := H$. We note that if we can place each element of $\mathbb{Z}/6\mathbb{Z}$ in a coset, we will have found them all. Then certainly $[0], [2]$, and $[4]$ are in H. If we consider $H[1]$, we note that $[0]+[1] = [1], [2]+[1] = [3], [4]+[1] = [5]$, so between H and $H[1]$, we've placed all the elements, so there are only those right cosets, so there are 2.

b) $\mathbb{R}$ in $\{\mathbb{C}, +\}$.

Answer: Suppose $z_1, z_2 \in \mathbb{C}$, such that $\Im(z_1) \neq \Im(z_2)$, where $\Im(z)$ is the imaginary component of z. Then if $a_1, b_1, a_2, b_2 \in \mathbb{R}$ such that $z_1 = a_1 + b_1 i, z_2 = a_2 + b_2 i$, we have $b_1 \neq b_2$. Then in particular, $z_1 \in \mathbb{R}b_1 i$, and $z_2 \in \mathbb{R}b_2 i$, since each are expressible as the sum of a real number and $b_1 i$ or $b_2 i$ respectively. Note that if $b_1 \neq b_2$, we have $\mathbb{R}b_1 i \neq \mathbb{R}b_2 i$; if we did, we could write $r_1 + b_1 i = r_2 + b_2 i$, for $r_1, r_2 \in \mathbb{R}$, and obtain $(r_1 - r_2) = (b_1 - b_2)i$, that a real number is equal to a nonzero purely imaginary number. Then there is a distinct right coset for each possible value of $\Im(z)$, of which there are a continuum, so there are a continuum of right cosets.

c) $3\mathbb{Z}$ in $\mathbb{Z}$

Answer: We employ the same method as in part a. It is clear that every integer is either a multiple of 3, or a multiple of 3 plus 1, or a multiple of 3 plus 2. Definitionally, all multiples of 3 belong to the coset $3\mathbb{Z}0 = \{x+0, x \in 3\mathbb{Z}\}$. Then we note that $3\mathbb{Z}1 = \{x + 1, x \in 3\mathbb{Z}\}$, and $3\mathbb{Z}2 = \{x + 2, x \in 3\mathbb{Z}\}$. With these three cosets, we've sorted every integer, so they're the only 3 cosets.

2)

1) Which is the following subsets of S_4 is (i) a normal subgroup; (ii) a subgroup, but not normal; (iii) not a subgroup at all? (Note that we use H to refer to the subset in each case for ease.)

a) $\{\text{id}, (134), (143)\}$

Answer: It is a subgroup; we note that $(134) = (143)^{-1}$, so it's closed under inversion. Also, their product is id, definitionally, so it's closed under multiplication. We note that $(12)H = \{(12), (12)(134), (12)(143)\} = \{(12), (1342), (1432)\}$, whereas $H(12) = \{(12), (134)(12), (143)(12)\} = \{(12), (1234), (1243)\} \neq (12)H$, so it's certainly not normal.

b) $\{\text{id}, (12)(34), (13)(24), (14)(23)\}$

Answer: This is certainly a subgroup; we can see that each element squares to id, and the product of any two nonidentity elements is the other one, in either order. It is also normal; we note that any $\sigma \in S_4$ can be written as the product of transpositions τ_i, such that the normality condition is $(\tau_1\tau_2 \dots \tau_n)H(\tau_1\tau_2 \dots \tau_n)^{-1} = H$ is equivalent to $(\tau_1\tau_2 \dots \tau_n)H(\tau_n^{-1}\tau_{n-1}^{-1} \dots \tau_1^{-1})$, which is implied by the statement that H commutes with any transposition. We note that each of the double transpositions commute with their own transpositions: i.e. $(ab)(cd)(ab) = (cd) = (ab)(ab)(cd)$. Then it suffices to note that $(ab)(cd)(ac) = (adcb) = (ac)(ad)(bc)$, since then each coset will contain transposition being multiplied, the transposition consisting of the other two numbers, and two versions of this form, each of which is also in the coset from the other side, though originating from the other element. As an example, consider $a = 1, b = 2, c = 3$, and $d = 4$. The coset of (13) consists of (13), (24), (1423), and (1234). In the right coset,

$(1423) = (12)(34)(13)$, and $(1234) = (14)(23)(13)$; in the left, $(1423) = (13)(14)(23)$, and $(1234) = (13)(12)(34)$.

c) $\{\mathrm{id}, (1234), (1432), (13)(24)\}$

Answer: This is a subgroup; it is the subgroup generated by (1234). It is not normal; we just noted that $(13)(24)(12) = (12)(14)(23)$. So when we take (12)'s right coset of this subgroup, we get the element $(14)(23)(13)$, which, if this is normal, must appear in it's left coset. In order for it to appear in the left coset, we must have $(14)(23)$ as an element, which we do not.

d) $\{\mathrm{id}, (123), (132), (234), (243)\}$

Answer: This is not a subgroup; by Lagrange's theorem, the order of a subgroup divides the order of the group, and 5 does not divide 24. We can also check by hand that $(123)(234) = (12)(34) \notin H$

2) There is a homomorphism f from $(\mathbb{Z}/12\mathbb{Z}, +)$ to $(\mathbb{C} \setminus \{0\}, \cdot)$ defined by $f(k) = i^k$ for all $k \in \mathbb{Z}/12\mathbb{Z}$ (where $i = \sqrt{-1} \in \mathbb{C}$). List all the elements of

a) $\mathrm{im}(f)$, the image of f

Answer: We note that $i^4 = 1 = \mathrm{id}_{(\mathbb{C}\setminus\{0\},\cdot)}$. Then we only need to consider the image of $\{[1], [2], [3], [4]\} \subset \mathbb{Z}/12\mathbb{Z}$. $i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$, so the image is $\{i, -1, -i, 1\}$.

b) $\ker(f)$, the kernel of f

Answer: We've just established that $i^4 = 1$, so it's in the kernel. Also, since f is a group homomorphism, we know that $f([4]^n) = f([4])^n = 1^n = 1$, so any multiple of $[4]$ is in the kernel. We lastly note that no other elements are; since $[1], [2]$, and $[3]$ aren't, and any integer is a multiple of 4 plus 0,1,2, or 3, then a number $4k + j$ has $f([4k + j]) = f([4])^k f(j) = f(j)$, so anything that has remainder 1 when divided by 4 is taken to i, and so on. Then the kernel is the multiples of $[4]$ in $\mathbb{Z}/12\mathbb{Z}$, so $[0], [4]$, and $[8]$, or alternatively written as $\langle[4]\rangle$.

c) The quotient group $\mathbb{Z}_{12}/\ker(f)$.

Answer: as part of determining the kernel in part b, we determined that the elements with distinct cosets were $[0], [1], [2]$, and $[3]$. Then the quotient group contains $[0]\langle[4]\rangle \equiv \bar{0}, \bar{1}, \bar{2}$, and $\bar{3}$.

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{0}$	$\bar{1}$	$\bar{2}$

$\cdot$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

3. Choose a subgroup H of order 2 in S_3 (we choose $\langle(12)\rangle$)

a) Find $g \in S_3$ such that $gHg^{-1} \neq H$, thus demonstrating that H is not a normal subgroup.

Answer: $(13)\{\mathrm{id}, (12)\} = \{(13), (123)\}, \{\mathrm{id}, (12)\}(13) = \{(13), (132)\}$

b) Write down representatives of the sets of left cosets S_3/H and right cosets $H\backslash S_3$

As written: left cosets: id, (13), (23). Right cosets: id, (13), and (23).

Alternate: left cosets: $\{\mathrm{id}, (12)\}, \{(13), (123)\}, \{(23), (132)\}$. Right cosets: $\{\mathrm{id}, (12)\}, \{(13), (132)\}, \{(23), (123)\}$.

4) Let H be a subset of a group G and let $g_1, g_2 \in G$. Show that the following are equivalent:

- $g_1H = g_2H$
- $Hg_1^{-1} = Hg_2^{-1}$
- $g_1H \subset g_2H$
- $g_1 \in g_2H$
- $g_1^{-1}g_2 \in H$

Answer: We show that the first is equivalent to each.

Proof.

$$g_1H = g_2H \Leftrightarrow \exists h_1, h_2 \in H : g_1h_1 = g_2h_2 \Leftrightarrow g_2^{-1}g_1 = h_2h_1^{-1} \Leftrightarrow h_2^{-1}g_2^{-1} = h_1^{-1}g_1^{-1} \Leftrightarrow Hg_2^{-1} = Hg_1^{-1}$$

□

Proof. Certainly if two sets are equal, one is a subset of the other. Additionally, since all cosets of a single normal subgroup are the same size, if one is a subset of the other, it's in fact the same set. □

Proof.

$$g_1H = g_2H \Leftrightarrow \exists h_1, h_2 \in H : g_1h_1 = g_2h_2 \Leftrightarrow g_1 = g_2h_2h_1^{-1} \Leftrightarrow (h_2h_1^{-1} \in H \Rightarrow g_2h_2h_1^{-1} \in g_2H \Rightarrow) g_1 \in g_2H$$

□

Proof.

$$g_1H = g_2H \Leftrightarrow \exists h_1, h_2 \in H : g_1h_1 = g_2h_2 \Leftrightarrow h_1h_2^{-1} = g_1^{-1}g_2 \Leftrightarrow g_1^{-1}g_2 \in H$$

□

5) Let G denote the set of 3×3 matrices with entries in $\mathbb{R}$, of the form

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}$$

That satisfy the relation $(ad - bc)\lambda = 1$.

a) Show that G is a group (under multiplication)

Proof. We note that for any matrix of the given form, $(ad-bc)\lambda$ is the determinant, so since $\det(AB) = \det(A)\det(B)$, and $\det(A^{-1}) = \det(A)^{-1}$, if the set of matrices of that form is closed under multiplication, the subset that satisfy the relation are also. Then we notice that the lower left indices, the ones that must remain 0, will always be 0 in the product of two such matrices, since their entries will be the product of a row of two zeroes in the first two indices, and a column of a zero in the final index, so the set is closed under multiplication. It remains to show that it's closed under inversion, since we know matrix multiplication is associative, and that the identity is of this form. To see this, we calculate A^{-1}:

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b & \frac{bf-ed}{\lambda} \\ -c & a & \frac{ec-af}{\lambda} \\ 0 & 0 & \frac{ad-bc}{\lambda} \end{pmatrix}\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} ad-bc & bd-bd & ed-bf+\lambda(\frac{bf-ed}{\lambda}) \\ -ac+ac & -bc+ad & -ec+af+\lambda(\frac{ec-af}{\lambda}) \\ 0 & 0 & \lambda(\frac{ad-bc}{\lambda}) = I \end{pmatrix}$$

We see that this is in G, so G is a group under multiplication. □

b) Show that the subset $H \subset G$ for which $a = d = 1$ and $b = c = 0$ is a subgroup.

Proof. It suffices to show closure under multiplication and inversion. Note that elements of H must have $\lambda = 1$, since $\lambda(ad - bc) = 1 = \lambda(1 - 0) = \lambda$. For multiplication, we note that the product of two such matrices will be of the correct form:

$$\begin{pmatrix} 1 & 0 & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & e' \\ 0 & 1 & f' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & e'+e \\ 0 & a & f'+f \\ 0 & 0 & 1 \end{pmatrix}$$

For inversion, we return to our inverse form in the previous part, and observe

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b & \frac{bf-ed}{\lambda} \\ -c & a & \frac{ec-af}{\lambda} \\ 0 & 0 & \frac{ad-bc}{\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -e \\ 0 & 1 & -f \\ 0 & 0 & 1 \end{pmatrix}$$

So it's a subgroup. □

c) Show that H is a *normal* subgroup of G.

Proof. To show normality, we want to show that for any matrix $A \in G$, $B \in H$, we have $ABA^{-1} \in H$. We calculate

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 & e' \\ 0 & 1 & f' \\ 0 & 0 & 1 \end{pmatrix}\frac{1}{ad-bc}\begin{pmatrix} d & -b & \frac{bf-ed}{\lambda} \\ -c & a & \frac{ec-af}{\lambda} \\ 0 & 0 & \frac{ad-bc}{\lambda} \end{pmatrix} = \begin{pmatrix} a & b & ae'+bf'+e \\ c & d & ce'+df'+f \\ 0 & 0 & \lambda \end{pmatrix}\frac{1}{ad-bc}\begin{pmatrix} d & -b & \frac{bf-ed}{\lambda} \\ -c & a & \frac{ec-af}{\lambda} \\ 0 & 0 & \frac{ad-bc}{\lambda} \end{pmatrix}$$

$$= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & -ab+ab & \frac{a(bf-ed)+b(ec-af)+(ad-bc)(ae'+bf'+e)}{\lambda} \\ -ab+ab & ad-bc & \frac{c(bf-ed)+d(ec-af)+(ce'+df'+f)(ad-bc)}{\lambda} \\ 0 & 0 & ad-bc \end{pmatrix} \in H$$

□

d)

Proof. We calculate

$$\phi\left(\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}\begin{pmatrix} a' & b' & e' \\ c' & d' & f' \\ 0 & 0 & \lambda' \end{pmatrix}\right) = \phi\left(\begin{pmatrix} aa'+bc' & ab'+bd' & ae'+bf'+e\lambda' \\ ca'+dc' & cb'+dd' & ce'+df'+f\lambda' \\ 0 & 0 & \lambda\lambda' \end{pmatrix}\right) = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc & cb'+dd' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

$$= \phi\left(\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & \lambda \end{pmatrix}\right)\phi\left(\begin{pmatrix} a' & b' & e' \\ c' & d' & f' \\ 0 & 0 & \lambda' \end{pmatrix}\right)$$

Clearly $\phi(A) = I$ iff $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I$, which is true iff $a = d = 1, b = c = 0$, i.e. $A \in H$.

□

A note about this problem: intuitively, many of these steps work because the 0s in the bottom row let us kind of only care about the top left section. Though I've gone through the actual calculations, a sufficiently convincing argument in words about why we can neglect the right column would be sufficient. Also, it's worth noting that the punchline is that H, a normal subgroup, is the kernel of some homomorphism.

6) Let $n > 2$ be an integer. Show that the group of rotations of the n-gon is a normal subgroup of the dihedral group D_{2n}, and identify the quotient group.

Lemma: index 2 subgroups are normal.

Proof. Let $H \leq G$, $2|H| = |G|$. Then suppose for some $g \in G, gHg^{-1} \neq H$. Then, equivalently, $gH \neq Hg$. We note that there are exactly two distinct cosets of H, since cosets constitute a partition of G, and H is index 2. Then if $gH \neq Hg$, either gH or Hg is H, since that's one of the cosets. Then that one is just H, which means that $g \in H$, so the other one is also H, which is a contradiction, so H is normal. □

Proof. Recall $D_{2n} = \langle r, s | r^n, s^2, rsrs\rangle$. We recall that D_{2n} has $2n$ elements, and that the group of rotations $\langle r\rangle$ has n elements, since it's identical to the group of vertices of the n-gon under multiplication, and also because n is the order of r, which we know from the presentation. We also know that $\langle r\rangle$ is a subgroup; it's a cyclic subgroup generated by r. Then since any index-2 subgroup is normal, $\langle r\rangle$ is normal. We know that the order of the quotient group will be the quotient of the order of the original group by the normal subgroup, i.e. $\frac{|D_{2n}|}{|\langle r\rangle|} = \frac{2n}{n} = 2$. The only order 2 group is $\mathbb{Z}/2\mathbb{Z}$, so the quotient group is $\mathbb{Z}/2\mathbb{Z}$. □