Problem set #5 solutions

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§5.3, exercise 1

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix} = (12453)$

(b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix} = (14)(53)$

(c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix} = (13)(25)$

(d) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix} = (24)$

§5.3, exercise 2

(a) $(1345)(234) = (135)(24)$

(b) $(12)(1253) = (253)$

(c) $(143)(23)(24) = (14)(23)$

(d) $(1423)(34)(56)(1324) = (12)(56)$

(e) $(1254)(13)(25) = (1324)$

(f) $(1254)(13)(25)^2 = (13254)$

(j) $(1254)^{100} = [(1254)^4]^{25} = \mathrm{id}^{25} = \mathrm{id}$, since the kth power of a k-cycle is the identity.

2

The permutation (1234) has order 4 in Σ_4. Note that the elements of Σ_4, when decomposed into cycles, are:

- the identity (which has order 1)
- 2-cycles (all of which have order 2)
- 3-cycles (all of which have order 3)
- 4-cycles (all of which have order 4)
- products of disjoint 2-cycles (all of which have order 2)

so the maximal order in Σ_4 is 4. Finally, (1234) is already a cycle, so it is the product of (one) disjoint cycle. [*Note*: The maximal order in Σ_n is not n, in general. For instance, $(123)(45) \in \Sigma_5$ has order 6.]

3

Consider $\sigma = (12) \in \Sigma_4$ and $\tau = (23) \in \Sigma_4$.
Since σ and τ are both 2-cycles, we have $\sigma^2 = \tau^2 = \mathrm{id}$. But $\sigma\tau = (12)(23) = (123) \neq (132) = (23)(12) = \tau\sigma$.

4

Observe that if σ is a permutation in Σ_4 which 'preserves the square', then (since vertices 1 and 2 are adjacent) the vertices $\sigma(1)$ and $\sigma(2)$ must be adjacent, whereupon the vertices $\sigma(3)$ and $\sigma(4)$ are uniquely determined. Therefore, either $\sigma(1+i) - \sigma(1) \equiv i \pmod 4$ for all i, or $\sigma(1+i) - \sigma(1) \equiv -i \pmod 4$ for all i.

Now we will show that the subset $D := \{\sigma : \sigma(1+i) - \sigma(1) \equiv \pm i\} \subset \Sigma_4$ is a subgroup. [In this problem, the symbol '$\equiv$' will denote congruence modulo 4.]

- D is closed under multiplication: if $\sigma, \tau \in D$, with $\sigma(1+i) - \sigma(1) \equiv u_\sigma i$ and $\tau(1+i) - \tau(1) \equiv u_\tau i$, where $u_\sigma, u_\tau = \pm 1$, then

$$\begin{aligned} \tau(\sigma(1+i)) - \tau(\sigma(1)) &= (\tau(\sigma(1+i)) - \tau(1)) - (\tau(\sigma(1)) - \tau(1)) \\ &\equiv u_\tau(\sigma(1+i) - 1) - u_\tau(\sigma(1) - 1) \\ &= u_\tau(\sigma(1+i) - \sigma(1)) \\ &\equiv u_\tau u_\sigma i \end{aligned}$$

  so $(\tau\sigma)(1+i) - (\tau\sigma)(1) \equiv u_\tau u_\sigma i$, which (since $u_\tau u_\sigma = \pm 1$) means $\tau\sigma \in D$.
- D contains the identity: we have $\mathrm{id}(1+i) - \mathrm{id}(1) = (1+i) - 1 = +i$ for all i, so $\mathrm{id} \in D$.
- D contains inverses: if $\sigma \in D$, with $\sigma(1+i) - \sigma(1) \equiv u_\sigma i$ where $u_\sigma = \pm 1$, then

$$\begin{aligned} i = (1+i) - 1 &= \sigma(\sigma^{-1}(1+i)) - \sigma(\sigma^{-1}(1)) \\ &= (\sigma(\sigma^{-1}(1+i)) - \sigma(1)) - (\sigma(\sigma^{-1}(1)) - \sigma(1)) \\ &\equiv u_\sigma(\sigma^{-1}(1+i) - 1) - u_\sigma(\sigma^{-1}(1) - 1) \\ &= u_\sigma(\sigma^{-1}(1+i) - \sigma^{-1}(1)) \end{aligned}$$

  so $\sigma^{-1}(1+i) - \sigma^{-1}(1) \equiv u_\sigma^{-1} i$, which (since $u_\sigma^{-1} = u_\sigma = \pm 1$) means $\sigma^{-1} \in D$.

So $D \subset \Sigma_4$ is a subgroup.

Finally, we see that choosing an element $\sigma \in D$ entails choosing one of four possibilities for $\sigma(1)$ and choosing whether σ is order-preserving or order-reversing (i.e., whether $\sigma(1+i) - \sigma(1) \equiv i$ or $\sigma(1+i) - \sigma(1) \equiv -i$). Hence there are $4 \cdot 2 = 8$ different possibilities for $\sigma \in D$, so the order of $D \subset \Sigma_4$ is 8.

Indeed, one verifies that $D = \{\mathrm{id}, (1234), (13)(24), (1432), (13), (12)(34), (24), (14)(23)\} \subset \Sigma_4$.

[*Note*: This proof can be easily modified to show that the subset $D_n \subset \Sigma_n$ of permutations which preserve the regular n-gon is a subgroup of order $2n$. The geometric view is that the order-preserving permutations in D_n are rotations of the n-gon, while the order-reversing permutations in D_n are reflections of the n-gon.]

5

(a) The 3-cycle $(132) \in \Sigma_3$ has order 3.

(b) The 6-cycle $(125364) \in \Sigma_6$ generates a cyclic subgroup of order 6, which means $(125364)^2 = (156)(234)$ has order $\frac{6}{\gcd(2, 6)} = 3$ in Σ_6.

[Alternatively, we know the product of disjoint cycles in Σ_n of lengths $\ell_1, \ldots, \ell_m$ has order $\mathrm{lcm}(\ell_1, \ldots, \ell_m)$, so $(156)(234)$ has order $\mathrm{lcm}(3, 3) = 3$ in Σ_6.]

(c) The 5-cycle $(14235) \in \Sigma_5$ generates a cyclic subgroup of order 5, so $(14235)^2$ has order $\frac{5}{\gcd(2, 5)} = 5$ in Σ_5.

[Alternatively, $(14235)^2 = (12543)$ is also a 5-cycle and thus has order 5.]