Problem set #2 solutions

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1

(a) This is an equivalence relation.

- R is reflexive: $x\mathrm{R}x$ for any $x \in \mathbb{Z}$, since $x + x = 2x$ is even for any $x \in \mathbb{Z}$.
- R is symmetric: if $x\mathrm{R}y$, then $y\mathrm{R}x$, since addition is commutative.
- R is transitive: if $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x + y$ and $y + z$ are even, so $x + z = (x + y) + (y + z) - 2y$ is a sum of even numbers and is thus even, so $x\mathrm{R}z$.

The equivalence classes are {even integers} and {odd integers}.

(b) This is not an equivalence relation.

- R is not reflexive: in fact, there is no $x \in \mathbb{Z}$ for which $x\mathrm{R}x$, since $x + x = 2x$ is even (i.e., not odd) for any $x \in \mathbb{Z}$.
- R is symmetric: if $x\mathrm{R}y$, then $y\mathrm{R}x$, since addition is commutative.
- R is not transitive: if $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x + y$ and $y + z$ are odd, so $x + z = (x + y) + (y + z) - 2y$ is a sum of two odd numbers and an even number and is thus even, so $x \not\mathrm{R} z$.

(c) This is an equivalence relation. [Recall that a *rotation* in $\mathbb{R}^3$ can be represented by a 3×3 matrix M which satisfies $M^{\mathrm{T}} = M^{-1}$ and $\det M = 1$.]

- R is reflexive: $x\mathrm{R}x$ for any $x \in \mathbb{R}^3$, since I_3 is a rotation matrix which maps x to x.
- R is symmetric: if M is a rotation matrix which maps x to y, then M^{-1} is a matrix which maps y to x, and M^{-1} is a rotation because $(M^{-1})^{\mathrm{T}} = (M^{\mathrm{T}})^{-1} = (M^{-1})^{-1}$ and $\det(M^{-1}) = (\det M)^{-1} = 1$. Hence $x\mathrm{R}y$ implies $y\mathrm{R}x$ for any $x,\ y \in \mathbb{R}^3$.
- R is transitive: if M_1 is a rotation matrix which maps x to y and M_2 is a rotation matrix which maps y to z, then their product M_2M_1 is a matrix which maps x to z, and M_2M_1 is a rotation because $(M_2M_1)^{\mathrm{T}} = M_1^{\mathrm{T}}M_2^{\mathrm{T}} = M_1^{-1}M_2^{-1} = (M_2M_1)^{-1}$ and $\det(M_2M_1) = (\det M_2)(\det M_1) = 1$. Hence for any $x,\ y,\ z \in \mathbb{R}^3$, if $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x\mathrm{R}z$.

The equivalence classes are the spheres $\{x \in \mathbb{R}^3 : |x| = r\}$ for each non-negative real number r. [This follows from the fact that any unit vector u can be extended to an orthonormal basis $\{u,\ v,\ w\}$ for $\mathbb{R}^3$ (simply choose any unit vector v orthogonal to u and let w be the cross product $u \times v$); then the matrix with columns $u,\ v,\ w$ is a rotation which maps $(1,\ 0,\ 0)$ to u, and now a rotation which maps v_1 to v_2, where $|v_1| = |v_2| = r$, is found by composing the inverse of a rotation which maps $(1,\ 0,\ 0)$ to $\frac{1}{r}v_1$ with a rotation which maps $(1,\ 0,\ 0)$ to $\frac{1}{r}v_2$.]

(d) This is not an equivalence relation.

- R is reflexive: $x\mathrm{R}x$ for any $x \in \mathbb{R}$, since $x - x = 0 = 0^2$ is the square of a real number.
- R is not symmetric: we have $1\,\mathrm{R}\,0$ since $1 - 0 = 1 = 1^2$, but $0 \not\mathrm{R}\, 1$ since $0 - 1 = -1$ is not the square of a real number.
- R is transitive: if $x\mathrm{R}y$ and $y\mathrm{R}z$, then $x - y$ and $y - z$ are non-negative, so $x - z = (x - y) + (y - z)$ is non-negative, so $x\mathrm{R}z$.

2

For ease of notation define the function $I : X \to \mathbb{R}$ given by $f \mapsto \int_0^1 f(x)\,dx$, so that $f\mathrm{R}g$ iff $I(f) = I(g)$.

We see that R is an equivalence relation on X because '$=$' is an equivalence relation on $\mathbb{R}$:

- R is reflexive: $f\mathrm{R}f$ for any $f \in X$, since $I(f) = I(f)$.
- R is symmetric: if $f\mathrm{R}g$, then $I(f) = I(g)$, so $I(g) = I(f)$, and thus $g\mathrm{R}f$.
- R is transitive: if $f\mathrm{R}g$ and $g\mathrm{R}h$, then $I(f) = I(g)$ and $I(g) = I(h)$, so $I(f) = I(h)$, and thus $f\mathrm{R}h$.

The equivalence classes are the sets $I^{-1}(r) := \{f \in X : I(f) = r\}$ for each $r \in \mathbb{R}$. Note that $I^{-1}(r)$ is non-empty for each $r \in \mathbb{R}$ because the constant function $f_r : [0, 1] \to \mathbb{R}$, given by $f_r(x) = r$, is in $I^{-1}(r)$. The set of equivalence classes is thus $X/\mathrm{R} = \{I^{-1}(r) : r \in \mathbb{R}\}$.

The natural bijection F from $\mathbb{R}$ to $X/\mathrm{R} = \{I^{-1}(r) : r \in \mathbb{R}\}$ is given by $r \mapsto I^{-1}(r)$.

[The inverse bijection F^{-1} from X/R to $\mathbb{R}$ is given by $S \mapsto I(s)$, where s is any function in S, which is well-defined by the definition of X/R. Observe that, analogously to problem 5(b), we can write $I = F^{-1} \circ p$, where $p : X \to X/\mathrm{R}$ maps a function to its equivalence class.]

3

(i) If $a, b \in \mathbb{Z}$ are such that $[a]_{nm} = [b]_{nm}$, then by definition $a - b$ is a multiple of nm. Any multiple of nm is a multiple of m, so $a - b$ is a multiple of m, and thus $[a]_m = [b]_m$. Hence f is well-defined.

(ii) For each residue class $[a]_m \in \mathbb{Z}_m$ we have $f([a]_{nm}) = [a]_m$, so f is surjective.

(iii) As in problem 2, '$\sim_f$' is an equivalence relation on $\mathbb{Z}_{nm}$ because '$=$' is an equivalence relation on $\mathbb{Z}_m$:

- $\sim_f$ is reflexive: $x \sim_f x$ for any $x \in \mathbb{Z}_{nm}$, since $f(x) = f(x)$.
- $\sim_f$ is symmetric: if $x \sim_f y$, then $f(x) = f(y)$, so $f(y) = f(x)$, and thus $y \sim_f x$.
- $\sim_f$ is transitive: if $x \sim_f y$ and $y \sim_f z$, then $f(x) = f(y)$ and $f(y) = f(z)$, so $f(x) = f(z)$, so $x \sim_f z$.

For each $0 \le a < m$, there is an equivalence class $\{[a+km]_{nm} : 0 \le k < n\} \subset \mathbb{Z}_{nm}$ containing n elements.

4

(a) In $\mathbb{Z}_7$: $[0]^2 = [0]$; $[1]^2 = [6]^2 = [1]$; $[2]^2 = [5]^2 = [4]$; $[3]^2 = [4]^2 = [2]$, so the squares are $[0]$, $[1]$, $[2]$, $[4]$.

(b) In $\mathbb{Z}_{41}$: $[14] + [33] = [47] = [6]$ and $[7] \cdot [8] = [56] = [15]$.

(c) In $\mathbb{Z}_{10}$: $12 \cdot 12 \equiv 2 \cdot 2 \equiv 4 \pmod{10}$ and $107 + 413 \equiv 7 + 3 \equiv 10 \equiv 0 \pmod{10}$.

5

(a) Recall (from the law of cosines) that two triangles are congruent iff they have the same three side lengths. Thus, define the function $g : X \to \mathbb{R}^3$, which maps a triangle $T \in X$ to its three side lengths (a, b, c) (say, in non-descending order). Then $T_1 \cong T_2$ iff $g(T_1) = g(T_2)$.

Thus, as before, '$\cong$' is an equivalence relation on X because '$=$' is an equivalence relation on $\mathbb{R}^3$.

(b) We know that congruent triangles have equal areas; that is, if $T_1 \cong T_2$, then $f(T_1) = f(T_2)$.

Therefore, f takes the same value on every triangle in a given equivalence class, which is to say that $\tilde{f} : (X/\cong) \to \mathbb{R}$, which maps the equivalence class S to $f(T)$ where T is any triangle in S, is well-defined.

Hence $f = \tilde{f} \circ p$, where $p : X \to (X/\cong)$ maps a triangle T to its equivalence class.

6

(i) We have

$$\begin{aligned}\gcd(104, 950) &= \gcd(950 - 9 \cdot 104, 104) = \gcd(14, 104) = \gcd(104 - 7 \cdot 14, 14) = \gcd(6, 14) \\ &= \gcd(14 - 2 \cdot 6, 6) = \gcd(2, 6) = \gcd(6 - 3 \cdot 2, 2) = \gcd(0, 2) = 2\end{aligned}$$

and $\operatorname{lcm}(104, 950) = 104 \cdot 950/2 = 49400$.

(ii) We have

$$\gcd(18, 207) = \gcd(207 - 11 \cdot 18, 18) = \gcd(9, 18) = \gcd(18 - 2 \cdot 9, 9) = \gcd(0, 9) = 9$$

and $\operatorname{lcm}(18, 207) = 18 \cdot 207/9 = 414$.