HW 13 Solutions

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1. Let $p > 3$ be a prime number. Show that every group of order $3p$ is solvable.

   Solution. Let G have order $3p$. The number of p-Sylow subgroups ($\iff$ cyclic subgroups of order p) is $\equiv 1 \pmod{p}$, and thus equal to either $1, 1+p$, or $1+2p$. Any two distinct ones intersect in the trivial subgroup (since there are no non-trivial proper subgroups of a group of order p). So if there are n p-Sylow subgroups, there are $(p-1)n$ elements of order p in G. If $n \geq 1+p$ then $(p-1)n \geq (p-1)(1+p) \geq 4p > |G|$ a contradiction. Thus there is only one p-Sylow subgroup H and it follows that $H \lhd G$. Then $G/H \cong \mathbf{Z}/3\mathbf{Z}$ and $H \cong \mathbf{Z}/p\mathbf{Z}$, so G is an extension of abelian groups, i.e., solvable.

2. 11. Up to a rotation, how many ways can the faces of a cube be colored with three different colors?

   12. Consider 12 straight wires of equal lengths with their ends soldered together to form the edges of a cube. Either silver or copper wire can be used for each edge. How many different ways can the cube be constructed?

   Solution. 11. Let X be the set of functions $\{\text{Faces of the Cube}\} \to \{\text{Red, Blue, Green}\}$. Thus X has $3^6 = 729$ elements. Let $G \cong S_4$ be the group of rigid motions of $\mathbf{R}^3$ preserving the cube. Then G acts on X by $\sigma \cdot f = f \circ \sigma^{-1}$. We are asked to find the number of orbits of this action. By Burnside's theorem, this is equal to the average size of a fixed point set X_σ, the average taken over all $\sigma \in G$. Since conjugate elements of G have the same number of fixed points, it suffices to count the size of X_σ for one σ from each conjugacy class: First $\sigma = e$, its fixed set is all of X. Next, $\sigma = (1234)$ corresponds to a 90° rotation fixing a face. For a coloring in X_σ, the two fixed faces can be any color but the moving faces all must be the same color. Thus there are $3^3 = 27$ elements of X_σ (also note that the conjugacy class of σ has 6 elements). If $\sigma = (13)(24) = (1234)^2$ then σ corresponds to a 180° rotation fixing a face. Here the fixed faces can be any color and of the four moving faces, the ones on opposite sides must be the same color. Thus we get $3^4 = 81$ elements of X_σ, and here the conjugacy class of σ has 3 elements. If $\sigma = (12)$ then σ corresponds to a 180° rotation whose axis is the line joining midpoints of a pair of opposite edges (cf. Theorem 5.27 in Judson's notes). This σ fixes no face so pairs up faces of the same color. There are 3 pairs so there are 3^3 elements of X_σ and there are 6 transpositions. Finally, if $\sigma = (123)$ then σ corresponds to a rotation whose axis is a line through opposite vertices. Again this fixes no face, but this time the three faces neighboring one fixed vertex all must have the same color, and likewise for the other. Thus there are $3^2 = 9$ fixed colorings. There are 8 3-cycles. Thus:

$$\text{Number of Orbits} = \frac{1}{24}(729 + 6 \cdot 27 + 3 \cdot 81 + 6 \cdot 27 + 8 \cdot 9) = 57.$$

12. This time the same G acts on the 2^{12} element set X of colorings of the edges, where there are two possible colors. Again e fixes all colorings. If $\sigma = (1234)$ then σ splits the edges into three sets of uniform color, so X_σ has $2^3 = 8$ elements. If $\sigma = (13)(24)$ then σ then instead there are 6 sets of uniform color, so X_σ has $2^6 = 64$ elements. If $\sigma = (12)$ then two edges are fixed and the other 10 are paired up so X_σ has $2^7 = 128$ elements. Finally, if $\sigma = (123)$ then no edges are fixed and the twelve edges are split into 4 sets of uniform color, so there are $2^4 = 16$ edges in the fixed set. Thus:

$$\text{Number of Orbits} = \frac{1}{24}(2^{12} + 6 \cdot 8 + 3 \cdot 64 + 6 \cdot 128 + 8 \cdot 16) = 218.$$

3. Show that no group of order 64 or 96 is simple. Construct two distinct non-abelian groups of each order.

   Solution. Suppose G has order 64. Then G is a 2-group of order > 2, hence not simple (for instance it has $Z(G) \neq 1$, so either $Z(G)$ itself gives a non-trivial normal subgroup or $Z(G) = G$ in which case any subgroup is normal). Two examples of such groups are D_{64} and $\mathbf{Z}/8\mathbf{Z} \times Q_8$. To show they are not isomorphic, we can note that $Z(D_{64})$ has 2 elements but $Z(\mathbf{Z}/8\mathbf{Z} \times Q_8) = Z(\mathbf{Z}/8\mathbf{Z}) \times Z(Q_8)$ has 16.

   Now suppose G has order $96 = 2^5 \cdot 3$. Let H be a 2-Sylow subgroup. Then G acts on G/H by $g(g'H) = gg'H$. Since the action is non-trivial (transitive even), it determines a non-trivial homomorphism $G \to S_3$ ($3 = |G/H|$). Then the kernel of $G \to S_3$ is a normal subgroup of index < 6, so G is not simple. Two such groups are $Q_8 \times \mathbf{Z}/12\mathbf{Z}$ and D_{96}. They are not isomorphic since the center of the first has 24 elements but the center of the second has only 2.

4. 20. What is the smallest possible order of a group G such that G is nonabelian and $|G|$ is odd? Can you find such a group?

   22. Show that if the order of G is $p^n q$ where p and q are primes and $p > q$, then G contains a normal subgroup.

   23. Prove that the number of distinct conjugates of a subgroup H of a finite group G is $[G : N(H)]$.

Solution. 20. Every group of order $1, 3, 5, 7, 9, 11, 13, 17, 19$ is abelian because these numbers are either 1, prime, or a prime squared. Every group of order 15 is abelian because $3 \nmid 5-1$. There is a non-abelian group of order 21 however, we may take $G = \mathbf{Z}/7\mathbf{Z} \ltimes_\phi \mathbf{Z}/3\mathbf{Z}$ where ϕ is any non-trivial homomorphism $\mathbf{Z}/3\mathbf{Z} \to \mathrm{Aut}(\mathbf{Z}/7\mathbf{Z}) \cong \mathbf{Z}/6\mathbf{Z}$.

22. Let H be a p-Sylow subgroup. Then the index of H in G is equal to the smallest prime dividing $|G|$. This implies that H is normal in G: Let G act on G/H by translation. This induces a homomorphism $G \to S_p$ where $p = [G : H]$ is the smallest prime dividing the order of G. The kernel K is contained in H, and since $G/K \hookrightarrow S_p$ we get that $[G : K]|p!$. But $[G : K] = [G : H][H : K] = p[H : K]$, so $[H : K]$ must be a product of primes less than p. But this is only possible if $[H : K] = 1$ by our assumption on $|G|$. Thus $H = K \lhd G$.

23. Let G act on the set of subgroups of G by conjugation. The orbit of H is the set of subgroups conjugate to H, and the stabilizer is the set of $g \in G$ such that $gHg^{-1} = H$. This is $N_G(H)$. By the orbit stabilizer formula, the number of subgroups conjugate to H is $[G : N_G(H)]$.

5. Show that no group of order 112 is simple. (Hint: if the group G is simple then it admits an injective homomorphism to the symmetric group S_r, where r is the number of 2-Sylow subgroups.)

   Solution. Suppose G is simple of order $112 = 2^4 \cdot 7$. Let G act on the set of 2-Sylow subgroups by conjugation. This action determines a permutation homomorphism $G \to S_r$ where r is the number of 2-Sylow subgroups. Let $K \lhd G$ be the kernel. Then $K = 1$ or $K = G$ by simplicity. If $K = G$ then the permutation homomorphism is trivial, so the action is trivial. But we know the action is transitive by the Sylow theorems, hence it would follow that $r = 1$ and so the unique 2-Sylow subgroup is normal. This is a contradiction since G is assumed simple. It follows that $K = 1$ and we have an injective homomorphism $G \to S_r$. Thus we know: (a) $r \equiv 1 \pmod 2$, (b) $r|112$ and (c) $112|r!$. For (c) to occur we must have $r \geq 7$. Then using (a) and (b) it follows that the only such number is $r = 7$. To get a contradiction, we will show that in fact $G \hookrightarrow S_7$ has image contained in A_7: $G \cap A_7 \lhd G$ since $A_7 \lhd S_7$. Thus either $G \cap A_7 = 1$ or $G \cap A_7 = G$. If the former holds, then $G = G/G \cap A_7 \cong G \cdot A_7/A_7 = S_7/A_7 = \{\pm 1\}$, a contradiction. Thus $G \cap A_7 = A_7$ so $G \subset A_7$. But this implies $112|7!/2$ which is false!