GU4041: Intro to Modern Algebra I

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Homework 12

1) Let p be an odd prime number. Show that any group of order $2p$ is either cyclic or isomorphic to D_{2p}.

Proof. Let G be a group of order $2p$. By Cauchy's theorem/Sylow's theorem, there exists a subgroup H of G such that $|H| = p$. H is index 2, so it's normal. Likewise, there exists a subgroup K of G of order 2. We note that $K \not\subset H$; this just follows from Lagrange's theorem; since $|H| = p, |K| = 2, |K| \not| \, |H|$. Then K and H have trivial intersection, so $KH = G$, so by the recognition principle for semidirect products, $G \cong \mathbb{Z}_2 \rtimes \mathbb{Z}_p$, for some isomorphism $\phi : K \to \mathrm{Aut}(H)$. Such a homomorphism must map $[1]$ to some order-2 automorphism of $\mathbb{Z}_p$; we know that all automorphisms of $\mathbb{Z}_p$ are of the form $[k] \to [ak]$, so we must have one such that $a \cong -1 \bmod p$ or $a \cong 1$; we note that if $a \cong b \bmod p$, then $[k] \to [ak]$ is the same automorphism as $[k] \to [bk]$, so as a result, we have that there are only two possible semidirect product structures on G; the usual direct product, in which case, by the chinese remainder theorem, we have $G \cong \mathbb{Z}_{2p}$, and otherwise, we have $G \cong \mathbb{Z}_2 \rtimes_\phi \mathbb{Z}_p$, where ϕ is the inversion isomorphism; this has the presentation $< r, s : s^2, r^p, srs = r^{-1} > = < r, s : s^2, r^p, rsrs >$, which is D_{2p}. □

2) Let A be a finite abelian group of order N. Let $p_1 < p_2 < ...p_n$ denote the distinct prime numbers dividing N.

a) Prove that A has a unique Sylow p-subgroup A_i of order p_i^α for $i = 1, \dots n$, for maximal α.

Proof. By Sylow's first theorem, there exist subgroups of order p_i^α for each i, and by the third, for a fixed i, each are conjugate. Then let $H_1, H_2 \subset G$ be subgroups of order p_i^α. Then $\exists g \in A$ such that $gH_1g^{-1} = H_2$. Then since A is abelian, $gH_1g^{-1} = H_2$, so $H_1 = H_2$, so there is only one subgroup of order p_i^α. □

b) Show that

$$A \simeq A_1 \times A_2 \times \cdots \times A_n$$

Proof. By the classification of finitely generated abelian groups, and since A is finite, $A \cong \prod_{k=1}^m \mathbb{Z}_{q_k^{\beta_k}}$, for β_k positive integers, and q_k not necessarily distinct primes. We note that for any i, we can define the subgroup $P_i = \{(x_1, x_2, \cdots x_m) : x_k = 0 \Leftarrow q_k \neq p_i\}$; we can also think of this as $\prod_{j=1}^{m_i} \mathbb{Z}_{p_i^{\beta_j}}$, where we canonically identify $\mathbb{Z}_{p_i^{\beta_j}}$ as a component of A with a subgroup of A. In any case, this is a maximal p_i-group, so it's A_i, and $A \cong \prod_{i=1}^n P_i$ still, so $A \cong \prod_{i=1}^n A_i$. □

3) Construct Sylow p-subgroups for the symmetric group S_5 and the alternating group A_5 for $p = 2, 3, 5$.

Note that $|S_5| = 2^3 \times 3 \times 5, |A_5| = 2^2 \times 3 \times 5$. Then a Sylow 2-subgroup of S_5 is a subgroup of order 8; an example of one is $< (1234), (12)(34) > = < r, s : r^4, s^2, rsrs >$, since $(1234)(12)(34) = (13)$ has order 2. With this presentation, it's clear that this is isomorphic to D_8; moreover, if we think of the elements of S_4 as arbitrary permutations of the vertices of a square, these two elements are a rotation by 90 degrees and a flip, so they ought to generate the dihedral group that way as well. For A_5, a Sylow 2-subgroup is order 4; a suitable example is $< (12)(34), (13)(24), (14)(23) >$; in A_4, this is the familiar normal subgroup; in A_5, it's not normal, but it's still a subgroup of order 4. Then for the others, if we can come up with Sylow 3-subgroups and 5-subgroups of A_5, they will obviously work for S_5 as well. Then these are easy; we can just pick cyclic subgroups generated by 3 and 5 cycles, since $< (123) >$ and $< (12345) >$, for example, are subgroups of the alternating group.

4)

Judson 1)
$18 = 2 \times 3^2$, so Sylow 2-subgroups are order 2, and Sylow 3-subgroups are order 9.
$24 = 2^3 \times 3$, so Sylow 2-subgroups are order 8, and Sylow 3-subgroups are order 3.
$54 = 2 \times 3^3$, so Sylow 2-subgroups are order 2, and Sylow 3-subgroups are order 27.
$72 = 2^3 \times 3^2$, so Sylow 2-subgroups are order 8, and Sylow 3-subgroups are order 9.
$80 = 2^4 \times 5$, so Sylow 2-subgroups are order 16, and Sylow 5-subgroups are order 5.

Judson 3)

Proof. Let G be a group of order 45, and let H be a Sylow 3-subgroup; it has order 9 and exists by Sylow's first theorem. By Sylow's third theorem, its normal iff there is only one such Sylow 3-subgroup, and the number of Sylow 3-subgroups n_3 divides 5 and is equivalent to 1 mod 3, so it's 1, so H is normal. □

Judson 6)

Proof. $160 = 2^5 \times 5$. Then by Sylow's first theorem, it has a subgroup of order 5, and by Sylow's third theorem, the number of such subgroups divides 32 and is equivalent to 1 mod 5, so it's in $\{1, 2, 4, 8, 16, 32\} \cap \{5\mathbb{Z} + 1\} = \{1, 16\}$. If there is only one, then we're good, since it's normal; otherwise, there are 16. Denote the set of such subsets by X (sometimes the notation $\mathrm{Syl}_p(G)$ is used). Then there is a transitive action by G on X given by conjugation by elements of G on the Sylow 5-groups. This takes the form of a surjective homomorphism Φ from G to S_{16}; it's surjective, since by transitivity, for any transposition t in S_{16}, there exists an element $g \in G$ such that $\Phi(g) = t$; since the transpositions generate S_{16}, this means $\mathrm{im}(\Phi) = S_{16}$. However, this implies $|S_{16}| \leq |G|$, which is clearly false. Therefore, G has a normal subgroup of order 5. □

Judson 7)

Proof. Let α be the multiplicity of p in the order of G We first show that $H \subset S$ for some Sylow p-group S. If $k = \alpha$ then H clearly is a Sylow p-subgroup of G. Otherwise, consider ${}^{G}\!/_{H}$. This is a group of order $|G|/p^k$; in particular, since H is *not* a Sylow p-subgroup of G, we have that ${}^{G}\!/_{H}$ has order a multiple of $p^{\alpha-k}$. Then we take a Sylow p-subgroup of ${}^{G}\!/_{H}$, which we call $\bar{S} : |\bar{S}| = p^{\alpha-k}$. Let $\Phi : G \to {}^{G}\!/_{H}$ be the canonical projection homomorphism $g \to gH$. Then consider $\Phi^{-1}(\bar{S})$. This is a set of order $|\ker(\Phi)||\bar{S}| = p^k p^{\alpha-k} = p^{\alpha}$, and it's a subgroup of G, since it's the preimage of a subgroup under a homomorphism. Moreover, it contains $\Phi^{-1}(e_{G/H})$ as a subgroup, which is of course H. Then H is a subset of one Sylow p-group, S. We now use that each Sylow p-subgroup of G is conjugate. Then for any Sylow p-subgroup S', there exists a $g \in G$ such that $gSg^{-1} = S'$. In particular, since $H \subset S$, $gHg^{-1} \subset S'$. However, since H is normal, $gHg^{-1} = H$, so $H \subset S'$. □

Judson 9)

Proof. By Sylow's third theorem, we know that n_3 divides 11 and is equivalent to 1 mod 3. Then $n_3 \in \{1, 11\} \cap 3\mathbb{Z} + 1 = \{1\}$. □