Problem set #11 solutions

Anton Wu

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1

First we claim the following:

<u>Claim</u>: Let H be a group of order p^k, where $k > 1$. Then H is not simple.

Proof: Consider $\mathcal{Z}(H)$, the center of H; we know $\mathcal{Z}(H)$ is a normal subgroup of H. By the class equation,

$$|H| = |\mathcal{Z}(H)| + \textstyle\sum_A [H : \mathcal{C}_H(A)]$$

where A runs over the non-singleton conjugacy classes of H, and $\mathcal{C}_H(A)$ is the centralizer in H of A. For each A, we have $[H : \mathcal{C}_H(A)]$ divides $|H| = p^k$, so p divides $[H : \mathcal{C}_H(A)]$. Since p divides $|H|$, we see from the class equation that p must divide $|\mathcal{Z}(H)|$; that is, $\mathcal{Z}(H) \neq \{1\}$.

This leaves two possibilities: either $\{1\} \subsetneq \mathcal{Z}(H) \subsetneq H$, or $\mathcal{Z}(H) = H$.

In the former case, $\mathcal{Z}(H)$ is a normal subgroup of H which is neither $\{1\}$ nor H, so H is not simple.

In the latter case, H is abelian. The fact that $|H| = p^k$ with $k > 1$ guarantees H has a subgroup which is neither $\{1\}$ nor H. In an abelian group, any subgroup is normal; so H is not simple.

Hence in either case H is not simple. //

Now, for a group G of order p^r, let $G = H_n \supsetneq H_{n-1} \supsetneq \cdots \supsetneq H_1 \supsetneq H_0 = \{1\}$ be any composition series.

For any $i = 1, \ldots, n$, we know $|H_i|$, $|H_{i-1}|$ divide $|G| = p^r$ and are thus powers of p; then so is $[H_i : H_{i-1}]$. Then H_i/H_{i-1} is a simple group of order p^k for some $k > 0$, so by the Claim, we have $k = 1$; so H_i/H_{i-1} has order p, and is therefore isomorphic to $\mathbb{Z}_p$.

Hence every composition factor of G is isomorphic to $\mathbb{Z}_p$. There must be r such factors. ■

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§13.3, exercise 4

a) A composition series for $\mathbb{Z}_n$ corresponds naturally to the prime factorization of n.

For $\mathbb{Z}_{12}$, the composition series are

$$\mathbb{Z}_{12} \supsetneq \langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \{0\}$$
$$\mathbb{Z}_{12} \supsetneq \langle 2\rangle \supsetneq \langle 6\rangle \supsetneq \{0\}$$
$$\mathbb{Z}_{12} \supsetneq \langle 3\rangle \supsetneq \langle 6\rangle \supsetneq \{0\}$$

b) For $\mathbb{Z}_{48}$, the composition series are

$$\mathbb{Z}_{48} \supsetneq \langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \langle 8\rangle \supsetneq \langle 16\rangle \supsetneq \{0\}$$
$$\mathbb{Z}_{48} \supsetneq \langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \langle 8\rangle \supsetneq \langle 24\rangle \supsetneq \{0\}$$
$$\mathbb{Z}_{48} \supsetneq \langle 2\rangle \supsetneq \langle 4\rangle \supsetneq \langle 12\rangle \supsetneq \langle 24\rangle \supsetneq \{0\}$$
$$\mathbb{Z}_{48} \supsetneq \langle 2\rangle \supsetneq \langle 6\rangle \supsetneq \langle 12\rangle \supsetneq \langle 24\rangle \supsetneq \{0\}$$
$$\mathbb{Z}_{48} \supsetneq \langle 3\rangle \supsetneq \langle 6\rangle \supsetneq \langle 12\rangle \supsetneq \langle 24\rangle \supsetneq \{0\}$$

c) The subgroups of Q_8 are $\{1\}$, $\langle -1\rangle$, $\langle i\rangle$, $\langle j\rangle$, $\langle k\rangle$, Q_8, all of which are normal in Q_8. For each of these we need to check whether the quotient group is simple and, if so, build the composition series from there.

The composition series are

$$\begin{aligned} &Q_8 \supsetneq \langle i\rangle \supsetneq \langle -1\rangle \supsetneq \{1\}\\ &Q_8 \supsetneq \langle j\rangle \supsetneq \langle -1\rangle \supsetneq \{1\}\\ &Q_8 \supsetneq \langle k\rangle \supsetneq \langle -1\rangle \supsetneq \{1\}\end{aligned}$$

d) The normal subgroups of $D_4 := \langle r,\, s \mid r^4 = s^2 = \mathrm{id},\ srs^{-1} = r^{-1}\rangle$ are $\{1\}$, $\langle r^2\rangle$, $\langle r\rangle$, $\langle s,\, r^2\rangle$, $\langle sr,\, sr^3\rangle$, D_4.

The composition series are

$$\begin{aligned} &D_4 \supsetneq \langle r\rangle \supsetneq \langle r^2\rangle \supsetneq \{1\}\\ &D_4 \supsetneq \langle s,\, r^2\rangle \supsetneq \langle s\rangle \supsetneq \{1\}\\ &D_4 \supsetneq \langle s,\, r^2\rangle \supsetneq \langle r^2\rangle \supsetneq \{1\}\\ &D_4 \supsetneq \langle s,\, r^2\rangle \supsetneq \langle sr^2\rangle \supsetneq \{1\}\\ &D_4 \supsetneq \langle sr,\, sr^3\rangle \supsetneq \langle sr\rangle \supsetneq \{1\}\\ &D_4 \supsetneq \langle sr,\, sr^3\rangle \supsetneq \langle sr^3\rangle \supsetneq \{1\}\\ &D_4 \supsetneq \langle sr,\, sr^3\rangle \supsetneq \langle r^2\rangle \supsetneq \{1\}\end{aligned}$$

e) The normal subgroups of S_3 are $\{\mathrm{id}\}$, A_3, S_3, and the normal subgroups of $\mathbb{Z}_4$ are $\{0\}$, $\langle 2\rangle$, $\mathbb{Z}_4$.

The composition series are

$$\begin{aligned} &S_3\times\mathbb{Z}_4 \supsetneq S_3\times\langle 2\rangle \supsetneq S_3\times\{0\} \supsetneq A_3\times\{0\} \supsetneq \{\mathrm{id}\}\times\{0\}\\ &S_3\times\mathbb{Z}_4 \supsetneq S_3\times\langle 2\rangle \supsetneq A_3\times\langle 2\rangle \supsetneq A_3\times\{0\} \supsetneq \{\mathrm{id}\}\times\{0\}\\ &S_3\times\mathbb{Z}_4 \supsetneq S_3\times\langle 2\rangle \supsetneq A_3\times\langle 2\rangle \supsetneq \{\mathrm{id}\}\times\langle 2\rangle \supsetneq \{\mathrm{id}\}\times\{0\}\\ &S_3\times\mathbb{Z}_4 \supsetneq A_3\times\mathbb{Z}_4 \supsetneq A_3\times\langle 2\rangle \supsetneq A_3\times\{0\} \supsetneq \{\mathrm{id}\}\times\{0\}\\ &S_3\times\mathbb{Z}_4 \supsetneq A_3\times\mathbb{Z}_4 \supsetneq A_3\times\langle 2\rangle \supsetneq \{\mathrm{id}\}\times\langle 2\rangle \supsetneq \{\mathrm{id}\}\times\{0\}\\ &S_3\times\mathbb{Z}_4 \supsetneq A_3\times\mathbb{Z}_4 \supsetneq \{\mathrm{id}\}\times\mathbb{Z}_4 \supsetneq \{\mathrm{id}\}\times\langle 2\rangle \supsetneq \{\mathrm{id}\}\times\{0\}\end{aligned}$$

f) The normal subgroups of S_4 are $\{\mathrm{id}\}$, K (the subgroup of all products of two disjoint 2-cycles), A_4, S_4.

The composition series are

$$\begin{aligned} &S_4 \supsetneq A_4 \supsetneq K \supsetneq \langle (12)(34)\rangle \supsetneq \{\mathrm{id}\}\\ &S_4 \supsetneq A_4 \supsetneq K \supsetneq \langle (13)(24)\rangle \supsetneq \{\mathrm{id}\}\\ &S_4 \supsetneq A_4 \supsetneq K \supsetneq \langle (14)(23)\rangle \supsetneq \{\mathrm{id}\}\end{aligned}$$

[*Remark*: All of the factor groups are abelian, which means S_4 is solvable. In Galois theory one uses this to show that every degree-4 polynomial equation with rational coefficients is solvable by radicals.]

g) Knowing that A_n is simple for $n \geq 5$, we see that the only composition series of S_n is

$$S_n \supsetneq A_n \supsetneq \{\mathrm{id}\}$$

[*Remark*: This means S_n is not solvable for $n \geq 5$; and in general, degree-n polynomial equations are *not* solvable by radicals.]

[*Note*: The Jordan-Hölder theorem states that for any finite group, the composition series are isomorphic, in the sense that the set of composition factors is the same up to ordering. One checks that this is true for the groups above.]

h) We claim that there is no composition series for $\mathbb{Q}$.

Suppose $\mathbb{Q} \supsetneq H_1 \supsetneq \cdots \supsetneq H_n \supsetneq \{0\}$ is a composition series. To derive a contradiction, we construct a subgroup H_{n+1} of H_n which is neither $\{0\}$ or H_n; since $\mathbb{Q}$ is abelian, any subgroup is a normal subgroup. To do this, choose some non-zero $x \in H_n$ and consider two cases:

If $x/m \in H_n$ for all non-zero $m \in \mathbb{Z}$, then H_n must be the entire group $\mathbb{Q}$, in which case we let $H_{n+1} := \mathbb{Z}$.

Otherwise, there exists a non-zero $m \in \mathbb{Z}$ with $x/m \notin H_n$. Define $H_{n+1} := \{mh : h \in H_n\}$. This is a subgroup of H_n since $mh \in H_n$ for any $h \in H_n$; and $H_{n+1} \neq H_n$, since $x \in H_n$ but $x \notin H_{n+1}$.

Thus, in either case we have constructed a subgroup H_{n+1} with $H_n \supsetneq H_{n+1} \supsetneq \{0\}$, and we have the desired contradiction. Hence our supposition is false and $\mathbb{Q}$ has no composition series. ■

§13.3, exercise 12

Given a group G and a normal subgroup N, the correspondence theorem states that there is a one-to-one correspondence between subgroups $H \subseteq N$ and subgroups $N \subseteq H' \subseteq G$.

Moreover, H_1, $H_2 \subseteq N$ satisfy $H_1 \trianglelefteq H_2$ if and only if the corresponding $N \subseteq H_1'$, $H_2' \subseteq G$ satisfy $H_1' \trianglelefteq H_2'$, in which case $H_2/H_1 \cong H_2'/H_1'$.

Now suppose $N \supseteq H_1 \supseteq \cdots \supseteq H_{r-1} \supseteq \{1\}$ and $G/N \supseteq G_1 \supseteq \cdots \supseteq G_{s-1} \supseteq \{1\}$ are composition series for N and G/N, respectively. Then we can use the correspondence theorem to lift the composition series for N to get $G \supseteq H_1' \supseteq \cdots \supseteq H_{r-1}' \supseteq G/N$. Then a composition series for G is

$$G \supseteq H_1' \supseteq \cdots \supseteq H_{r-1}' \supseteq G/N \supseteq G_1 \supseteq \cdots \supseteq G_{s-1} \supseteq \{1\}$$

The first r composition factors are isomorphic to the composition factors of N; the remaining s composition factors are the composition factors of G/N. Since these are all abelian (as N and G/N are solvable), we conclude that G is solvable. ■

3

Let G be a solvable group, with $G \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \{1\}$ be a subnormal series for G, and let $H \subseteq G$ be any subgroup.

Define $H_i := (H \cap G_i)$ for $i = 1, \ldots, n$; then $H_i \subseteq G_i$ is a subgroup. Also, $G_{i+1} \trianglelefteq G_i$ is a normal subgroup, so by the second isomorphism theorem, $(H_i \cap G_{i+1}) = H_{i+1}$ is a normal subgroup of H_i, and G_{i+1} is a normal subgroup of $H_i G_{i+1}$ (which is a subgroup of G_i); and $H_i/H_{i+1} \cong H_i G_{i+1}/G_{i+1} \subseteq G_i/G_{i+1}$. Since G is solvable, G_i/G_{i+1} is an abelian group, so H_i/H_{i+1} is also an abelian group.

Hence $H \supseteq H_1 \supseteq \cdots \supseteq H_n \supseteq \{1\}$ is a subnormal series for H that shows that H is solvable. ■

4

An example is S_3. The factor groups in the subnormal series $S_3 \supsetneq A_3 \supsetneq \{\mathrm{id}\}$ are, respectively, isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_3$, which are abelian, so S_3 is solvable; but the center of S_3 is trivial. ■

5

a) To show that $H \subseteq \mathrm{GL}_3(\mathbb{R})$ is a subgroup, we verify that H is closed under multiplication and contains the identity and inverses:

- H is closed under multiplication, since $u(x_1, y_1, z_1)u(x_2, y_2, z_2) = u(x_1+x_2, y_1+y_2, x_1y_2+z_1+z_2)$.
- H contains id, since the identity of $\mathrm{GL}_3(\mathbb{R})$ is $u(0, 0, 0)$.
- H contains inverses, since (using the above) the inverse of $u(x, y, z)$ is $u(-x, -y, -z+xy)$.

Hence $H \subseteq \mathrm{GL}_3(\mathbb{R})$ is a subgroup.

b) Suppose $u(x_0, y_0, z_0) \in \mathcal{Z}(H)$, and let $u(x, y, z) \in H$ be any element. Then

$$\begin{aligned} u(x_0, y_0, z_0)\,u(x, y, z) &= u(x_0 + x, y_0 + y, x_0 y + z_0 + z) \\ \text{and} \quad u(x, y, z)\,u(x_0, y_0, z_0) &= u(x + x_0, y + y_0, x y_0 + z + z_0) \end{aligned}$$

are equal if and only if $x_0 y = x y_0$. Since $x, y \in \mathbb{R}$ are arbitrary, we must have $x_0 = y_0 = 0$. Hence the center of H is $\mathcal{Z}(H) = \{u(0, 0, z) : z \in \mathbb{R}\}$.

c) The descending central series is given by $H =: H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots$, where $H_i = [H_{i-1}, H]$. Thus, $H_1 = [H, H]$. The commutator of $u(x_1, y_1, z_1)$ and $u(x_2, y_2, z_2)$ is

$$\begin{aligned} &u(x_1, y_1, z_1)\,u(x_2, y_2, z_2)\,u(x_1, y_1, z_1)^{-1}u(x_2, y_2, z_2)^{-1} \\ &\quad = u(x_1, y_1, z_1)\,u(x_2, y_2, z_2)\,u(-x_1, -y_1, -z_1 + x_1 y_1)\,u(-x_2, -y_2, -z_2 + x_2 y_2) \\ &\quad = u(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2)\,u(-x_1 - x_2, -y_1 - y_2, -z_1 - z_2 + x_1 y_1 + x_1 y_2 + x_2 y_2) \\ &\quad = u(0, 0, 2x_1 y_2 + x_1 y_1 + x_2 y_2 + (x_1 + x_2)(-y_1 - y_2)) \\ &\quad = u(0, 0, x_1 y_2 - x_2 y_1) \end{aligned}$$

so the commutator is always in $\mathcal{Z}(H)$. Moreover, any element of $\mathcal{Z}(H)$ can be realized as a commutator, for example, by setting $x_1 = 1$ and $y_1 = 0$. Since $\mathcal{Z}(H)$ is a group, this means $[H, H] = \mathcal{Z}(H)$.

Now, since any element of $\mathcal{Z}(H)$ commutes with any element of H, the commutator subgroup $[\mathcal{Z}(H), H]$ is trivial. Hence H is nilpotent and the descending central series is $H \supseteq \mathcal{Z}(H) \supseteq \{\mathrm{id}\}$.

d) Possible answers include $H_0 = \{\mathrm{id}\}$, $H_1 = \{u(0, 0, q) : q \in \mathbb{Q}\}$, $H_2 = \{u(x, x, z) : x, z \in \mathbb{R}\}$. ∎