THE JORDAN-HÖLDER THEOREM

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We have seen examples of chains of normal subgroups:

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \ldots G_r = \{e\} \tag{1.1}$$

in which each group G_{i+1} is normal in the preceding group G_i (though not necessarily normal in G). Such a series is often called *subnormal*, and this is the terminology we use. For example, there is the sequence of derived subgroups

$$G \supseteq D(G) = [G, G] \supseteq D^2(G) = [D(G), D(G)] \ldots$$

which ends with $D^r(G) = \{e\}$ if G is a solvable group, in which $D^i(G)/D^{i+1}(G)$ is abelian.

At the other extreme, the group G is *simple* if it contains no proper normal subgroups other than $\{e\}$. A subnormal series such as (**??**) is called a *composition series* if each of the quotient groups G_i/G_{i+1} is simple; in particular, $G_i \neq G_{i+1}$ for all i.

Lemma 1.2. *Let G be a finite group. Then G has a composition series.*

Proof. We induct on the order of G. We know that a group of order 1 has a composition series. Suppose every group of order less than $|G|$ has a composition series. If G is simple, then we are done. If not, then G has a non-trivial proper normal subgroup N. By induction, N and G/N both have composition series. Say

$$G/N = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_r = \{e\}.$$

is a composition series. By the correspondence principle, each H_i corresponds to a subgroup G_i containing N, with $H_i = G_i/N$ for all i. By the Third Isomorphism Theorem,

$$G_i/G_{i+1} \xrightarrow{\sim} (G_i/N)/(G_{i+1}/N) = H_i/H_{i+1}$$

which is simple. On the other hand, $H_r = N$ has a composition series

$$N = G_r \supseteq G_{r+1} \cdots \supseteq G_N = \{e\}.$$

Then

$$G = G_0 \supseteq G_1 \cdots \supseteq N = G_r \supseteq G_{r+1} \cdots \supseteq G_N = \{e\}$$

is a composition series for G. □

We write the collection of simple factors (J_α, m_α) where J_α is a simple group and m_α is the number of time it appears as a quotient G_i/G_{i+1}. We call m_α the *multiplicity* of the simple factor J_α. We call it a collection rather than a set, because the same element can appear more than once; sometimes this is called a *multiset.*

Example 1.3 (Cyclic groups of prime power order)**.** *The cyclic group $\mathbb{Z}_{p^a}$ has a composition series:*

$$\mathbb{Z}_{p^a} \supseteq (p) \supseteq (p^2) \supseteq \cdots \supseteq (p^{a-1}) \supseteq \{0\}$$

where (p^i) denotes the multiples of p^i modulo p^a, for any $i \leq a$. We can use the Third Isomorphism Theorem: if $\langle p^i \rangle \subseteq \mathbb{Z}$ is the subgroup of multiples of p^i for each i, then the subgroups of $\mathbb{Z}_{p^a}$ correspond to subgroups of $\mathbb{Z}$ containing $\langle p^a \rangle$. In particular

$$(p^i) = \langle p^i \rangle / \langle p^a \rangle \subseteq \mathbb{Z}_{p^a}.$$

Then by the Third Isomorphism Theorem

$$(p^i)/(p^{i+1}) = (\langle p^i \rangle / \langle p^a \rangle)/(\langle p^{i+1} \rangle / \langle p^a \rangle) \xrightarrow{\sim} \langle p^i \rangle / \langle p^{i+1}$$

and multiplication by p^i is an isomorphism

$$\mathbb{Z}/\langle p \rangle \xrightarrow{\sim} \langle p^i \rangle / \langle p^{i+1} \rangle.$$

So the collection of simple factors of $\mathbb{Z}_{p^a}$ is $(\mathbb{Z}_p, a)$ (multiplicity a).

Example 1.4 (Cyclic groups)**.** *Let $n \in \mathbb{Z}$. Write $n = \prod_i p_i^{a_i}$ as a product of prime factors. Then the cyclic group $\mathbb{Z}_n$ is isomorphic to a product of cyclic groups $\mathbb{Z}_{p_i^{a_i}}$ and the collection of simple factors of $\mathbb{Z}_n$ is the union of the simple factors of all the $\mathbb{Z}_{p_i^{a_i}}$:*

$$(\mathbb{Z}_{p^i}, a_i).$$

Example 1.5 (Abelian groups)**.** *We know that any abelian group is isomorphic to a direct product of cyclic groups:*

$$\prod_i \prod_j \mathbb{Z}_{p_i^{a_{ij}}}$$

where the p_i are distinct prime numbers and the $a_{ij}j$ are positive integers. The only simple abelian groups are the cyclic groups of prime order. So the collection of simple factors is

$$\{(\mathbb{Z}_{p_i}, m_i = \sum_j a_{ij})\}.$$

In other words, $\mathbb{Z}_{p_i}$ occurs as a simple factor a_{ij} times in the cyclic group $\mathbb{Z}_{p_i^{a_{ij}}}$, and the total multiplicity is the sum of the multiplicities in the simple factors.

A given finite group G can have more than one composition series. Nevertheless, there is a uniqueness theorem that is analogous to the uniqueness of prime factorization of an integer. First we prove a lemma

Lemma 1.6. *Let G be a group with two normal subgroups H and J, $H \neq J$. Suppose G/H and G/J are both simple. Then*

$$G/H \xrightarrow{\sim} J/H \cap J; G/J \xrightarrow{\sim} H/H \cap J.$$

Proof. If $H \subseteq J$ then $G/H \supseteq J/H$, and since G/H is simple this implies $J = G$ or $J = H$, both of which are impossible. Thus $G \supseteq H \cdot J \supseteq J$ and since $H \cdot J \neq J$ we must have $G = H \cdot J$.

Now we apply the Second Isomorphism Theorem:

$$G/H = H \cdot J/H \xrightarrow{\sim} J/H \cap J.$$

The same proof works for G/J. □

Theorem 1.7 (Jordan-Hölder Theorem)**.** *Let G be a finite group. Suppose G has two composition series:*

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_i \supseteq G_{i+1} \cdots \supseteq G_{r+1} = \{e\}$$

$$G = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{s+1} = \{e\}.$$

Then $r = s$ and the two collections of quotients

$$\{G_i/G_{i+1}\},\ \{H_j/H_{j+1}\}$$

are equal (not taking order into account).

Proof. This is of course an induction proof. The case $|G| = 1$ is trivial. If $r = 1$ then G is simple so again we must have $H_1 = G_1$. Now suppose the theorem is known for groups of order $|G|$. We assume r is the minimal length of a composition series for G. Suppose $G_1 = H_1$. Then by induction on $|G|$ the composition series for G_1 and H_1 are equivalent, and so we are done. Thus we must assume $G_1 \neq H_1$. Now G/G_1 is simple, so the only subgroups of G containing G_1 are G_1 and G. Since $G_1 \cdot H_1$ is normal in G and contains but is not equal to G_1, we have $G = G_1 \cdot H_1$. Let $K_1 = G_1 \cap H_1$. By the lemma,

$$G/G_1 \xrightarrow{\sim} H_1/K_1;\ G/H_1 \xrightarrow{\sim} G_1/K_1.$$

Also $H_1 \trianglelefteq H_0 = G_0$, the intersection $K_i = H_1 \cap G_i$ is normal in each G_i as well. Also, $K_{i+1} \trianglelefteq K_i$ for each i. So we have a new subnormal series

$$G = G_0 \supseteq G_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}.$$

Note that this is of length $r + 1$. The image of K_i in G_i/G_{i+1} is normal and is isomorphic to

$$K_i/K_i \cap G_{i+1} = K_i/(H_1 \cap G_i) \cap G_{i+1} = K_i/H_1 \cap G_{i+1} = K_i/K_{i+1}$$

because $G_{i+1} \subset G_i$. Since G_i/G_{i+1} is simple, we have either $K_i/K_{i+1} = G_i/G_{i+1}$ or $K_i = K_{i+1}$. In particular, every non-trivial term in

$$G_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}$$

is simple, and thus it must be a composition series when the non-trivial terms are removed. But by induction, every composition series for G_1 has length $r-1$ and any two are equivalent. So exactly one quotient K_j/K_{j+1} is trivial, and with that removed we have a composition series equivalent to

$$G_1 \supseteq G_2 \supseteq \cdots \supseteq G_r \supseteq G_{r+1} = \{e\}.$$

In particular the two collections

$$\{K_i/K_{i+1}, i \neq j, i \geq 1\}$$

and

$$\{G_i/G_{i+1}, i \geq 1\}$$

are the same.

On the other hand, we also have two composition series for H_1:

$$H_1 \supseteq H_2 \supseteq \cdots \supseteq H_{s+1} = \{e\}$$

and

$$H_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}$$

where we remove the term $K_j = K_{j+1}$. Again by induction, these are equivalent, but the first is of length $s-1$ and the second of length $r-1$. Thus $r = s$. And again the two collections

$$\{K_i/K_{i+1}, i \neq j, i \geq 1\}$$

and

$$\{H_i/H_{i+1}, i \geq 1\}$$

are the same. So for $i \geq 1$, we have

$$\{G_i/G_{i+1}, i \geq 1\} = \{H_i/H_{i+1}, i \geq 1\}$$

Finally, we return to the two series

$$G \supseteq H_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}$$
$$G \supseteq G_1 \supseteq K_1 \supseteq K_2 \supseteq \cdots \supseteq K_r \supseteq K_{r+1} = \{e\}$$

(with K_j omitted). By the lemma, as we have said, the two unordered pairs

$$(G/H_1, H_1/K_1); (G/G_1, G_1/K_1)$$

are equal. Moreover, the remaining terms are equal, so we are done.

□