The Cauchy-Frobenius Lemma

Also known as *Burnside's Counting Theorem*, or *The Lemma that is not Burnside's*

GU4041

Columbia University

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Outline

Consider a finite group G acting on a finite set X. How many orbits does it have?

The answer is: the number of orbits equals the **average number of fixed points of elements of** G Let $|G\backslash X|$ denote the number of orbits. Then

Theorem (Cauchy-Frobenius, Burnside)

$$|G\backslash X| = \frac{1}{|G|}\sum_{g\in G} X^g,$$

where

$$X^g = \{x \in X, g(x) = x\}.$$

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A lemma

If $x \in X$, let $G_x = \{g \in G \mid g(x) = x\}$. This is clearly (!) a subgroup of G, called the *stabilizer* or *isotropy* group of x. Recall that $\mathcal{O}_x \subset X$ is the orbit of G containing x.

Lemma

Suppose $y \in \mathcal{O}_x$. Then there is an isomorphism $G_x \xrightarrow{\sim} G_y$ (usually more than one).

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Since $y \in \mathcal{O}_x$, there is $\gamma \in G$ with $\gamma(x) = y$; $\gamma^{-1}(y) = x$. Claim that

$$\gamma G_x \gamma^{-1} = G_y.$$

Indeed, if $g \in G_x$, then

$$\gamma g \gamma^{-1}(y) = \gamma g(x) = \gamma(x) = y.$$

This proves the inclusion $\gamma G_x \gamma^{-1} \subseteq G_y$; but since $\gamma^{-1}(y) = x$ we also have

$$G_x = \gamma^{-1}(\gamma G_x \gamma^{-1})\gamma \subseteq \gamma^{-1} G_y \gamma \subseteq G_x$$

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Proof of the theorem

Consider the subset $Z = \{(g, x) \mid g(x) = x\} \subset G \times X$. Then

$$|Z| = \sum_{g \in G} |X^g| = \sum_{x \in X} |G_x|.$$

Here we first counted $|Z|$ by the partition according to $g \in G$, then according to the partition according to $x \in X$. Count the right-hand side by orbits:

$$\sum_{x \in X} |G_x| = \sum_{\mathcal{O}} \sum_{x \in \mathcal{O}} |G_x|,$$

where the sum is over all orbits.

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Proof of the theorem

$$\sum_{x\in X}|G_x| = \sum_{\mathcal{O}}\sum_{x\in\mathcal{O}}|G_x|,$$

But by the lemma, $|G_x| = |G_y|$ if $x, y \in \mathcal{O}$, and $|\mathcal{O}| = \frac{|G|}{|G_x|}$ for any $x \in \mathcal{O}$. So

$$\sum_{g\in G}|X^g| = \sum_{\mathcal{O}}\frac{|G|}{|G_x|}\cdot|G_x| = \sum_{\mathcal{O}}|G| = |G\backslash X||G|.$$

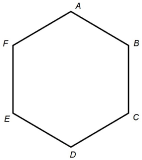
The theorem follows when we divide both sides by $|G|$:

$$\frac{1}{|G|}\sum_{g\in G}|X^g| = \frac{1}{|G|}|G\backslash X||G| = |G\backslash X|.$$

Burnside's Lemma is familiar in computer science. Here are detailed notes from Columbia Computer Science course CS W4205 `http://www.cs.columbia.edu/~cs4205/files/CM9.pdf`

The notes include an introduction to permutation groups, a proof of the Cauchy-Frobenius theorem, and numerous applications.

We work out an example: how many ways are there to color the vertices of a regular hexagon with two colors, up to rotation?



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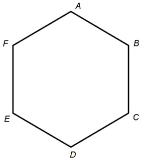

The hexagon

If we let V be the set of vertices and C the set of colors, then the set X of colorings is $|Functions(V, C)| = 2^6$. The group G of rotations has order 6, generated by rotation r through $60°$. We list the fixed points:

- ($g = e$) $|X^e| = |X| = 64$.
- ($g = r$) $|X^r| = 2$ (for the 2 colors)
- ($g = r^2$) $|X^{r^2}| = 2^2 = 4$ (because each of the 2 orbits of $\langle r^2 \rangle$ has 2 possibilities)
- ($g = r^3$) $|X^{r^3}| = 2^3 = 8$ (because each of the 3 orbits of $\langle r^3 \rangle$ has 2 possibilities)
- ($g = r^4$) $|X^{r^4}| = 2^2 = 4$ (same as r^2)
- ($g = r^5$) $|X^{r^5}| = 2$ (same as r)

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Burnside's counting theorem for the hexagon

$$|G\backslash X| = \frac{1}{|G|}[64+2+4+8+4+2] = \frac{84}{6} = 14$$

And here they are (b for blue, y for yellow):

$$bbbbbb, yyyyyy$$

$$bbbbby, yyyyyb$$

$$bbbbyy, yyyybb.bbbyby, yyybyb, bbybby, yybyyb$$

$$bybyby, bbbyyy, bbybyy, bybbyy$$