MODERN ALGEBRA I GU4041

Homework 8, due March 26: Isomorphism theorems, composition series

1. Let n and m be two positive integers. Denote by $f : \mathbb{Z} \to \mathbb{Z}_n$ and $g : \mathbb{Z} \to \mathbb{Z}_m$ the two natural maps, and define

$$f \times g : \mathbb{Z} \to \mathbb{Z}_n \times \mathbb{Z}_m$$

by $(f \times g)(x) = (f(x), g(x))$.

(a) Suppose $(n, m) = 1$. Show that the kernel of $f \times g$ is the subgroup of multiples of nm in $\mathbb{Z}$.

(b) Still supposing $(n, m) = 1$, use the First Isomorphism Theorem to show that

$$\mathbb{Z}_{nm} \xrightarrow{\sim} \mathbb{Z}_n \times \mathbb{Z}_m.$$

(c) If $n = m$, determine the image and kernel of $f \times g$, and show that this is consistent with the First Isomorphism Theorem.

2. Let G be a group and $N \triangleleft G$ be a normal subgroup. Suppose N is of prime index p in G. Let $H \subset G$ be any subgroup. Prove that exactly one of the following is true:

(i) $H \subset N$; or
(ii) $G = HN$ and $[H : H \cap N] = p$.

3. Let G be a group and $N \triangleleft G$ and $M \triangleleft G$ be normal subgroups. Suppose also that $G = NM$.

(a) Prove that there is an isomorphism

$$G/N \cap M \xrightarrow{\sim} G/N \times G/M.$$

(Hint: Use the First Isomorphism Theorem.)

(b) Use (a) and the Second Isomorphism Theorem to deduce that, if G is the product of two normal subgroups N and M such that $N \cap M = (\mathbf{e})$ then

$$G \xrightarrow{\sim} M \times N.$$

4. Judson book, section 11.3, exercises 14, 17.

Recommended reading

Howie notes, sections 6.4, 6.5; Judson, Chapter 11.