MODERN ALGEBRA I GU4041

HOMEWORK 4, DUE FEBRUARY 20: EXAMPLES OF GROUPS

1. Show that the set $\mathbb{Z}_5^*$ of non-zero residue classes modulo 5 forms a group under multiplication. Is it a cyclic group?

2. (a) List all the subgroups of the cyclic group $\mathbb{Z}_{49}$.

(b) Does the cyclic group $\mathbb{Z}_{48}$ have a subgroup of order 12? Of order 14? Justify your answer.

3. (a) Let G be a group in which each element is its own inverse: $g = g^{-1}$ for all g. Show that G is abelian.

(b) Let G be a finite group with identity e and an even number of elements. Show that there is an element $g \neq e$ such that $g^2 = e$.

4. Let G be any group and $g \in G$. Consider the set $H =< g >$ of all powers g^n where $n \in \mathbb{Z}$. Here we let $g^0 = e$ and if $n = -m$ then we set $g^n = (g^{-1})^m$.

(a) Show that H is a cyclic subgroup of G.

(b) Show that every cyclic subgroup of G is of the form $< g >$ for some $g \in G$.

5. List all the cyclic subgroups of the quaternion group Q_8.

6. Recall that the direct product $G \times H$ of two groups G and H is the set of ordered pairs (g, h) with $g \in G$ and $h \in H$. The identity in $G \times H$ is the element (e_G, e_H), where e_G is the identity in G and e_H is the identity in H. Multiplication in $G \times H$ is given by

$$(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2).$$

(a) Show that $G \times H$ is a group with respect to this operation. In particular, this means defining the inverse of an element (g, h).

(b) Show that the direct product $\mathbb{Z}_2 \times \mathbb{Z}_5$ is isomorphic to a cyclic group by finding a cyclic generator. Show that the direct product $\mathbb{Z}_3 \times \mathbb{Z}_3$ is *not* a cyclic group. How many cyclic subgroups does $\mathbb{Z}_5 \times \mathbb{Z}_5$ contain?

RECOMMENDED READING

Judson book, Section 4.1, 4.2 (read quickly); Gallagher's notes, Chapter 9.