MODERN ALGEBRA I GU4041

Homework 3, due February 13: Basic properties of groups

1. Let X be a set with two elements e, f.

(a) Can you define a binary operation

$$\star : X \times X \to X$$

that is not associative?

(b) Suppose e is a two-sided identity for $\star$, in other words

$$e \star e = e, \ e \star f = f \star e = f.$$

(Here we write $e \star e$ instead of $\star(e,e)$, as usual.) How many such operations are there? Are they all necessarily commutative? Associative?

2. (a) Let $(X, \star)$ and $(Y, \circ)$ be two sets with binary operations. Suppose

$$f : X \to Y$$

is a bijection that defines an isomorphism of binary structures, i.e.

$$f(x_1 \star x_2) = f(x_1) \circ f(x_2).$$

Show that $f^{-1} : Y \to X$ is also an isomorphism of binary structures.

(b) In the notation of (a), if $X = Y$ and $\star = \circ$, show that the identity map from X to itself defines an isomorphism of binary structures.

3. Let $n \geq 3$ be an integer. Let Δ_n be a regular polygon with n sides in the complex plane, with one vertex at the point 1 and the other vertices on the circle $x^2 + y^2 = 1$. Let μ_n denote the set of vertices of Δ_n.

(a) Use either the exponential function or trigonometric functions to list the coordinates of the points in μ_n.

(b) Show that the subset $\mu_n \subset \mathbb{C}$ is a group under multiplication.

(c) Define an isomorphism of groups $f : \mathbb{Z}/n\mathbb{Z} \to \mu_n$.

(d) How many solutions does part (c) have? Explain.

4. List all subgroups of the Klein 4 group and of the cyclic group $\mathbb{Z}/4\mathbb{Z}$. How many subgroups contain 3 elements in each case?

5. Let X be a set with 3 elements. How many distinct binary operations

$$X \times X \to X$$

are there?

6. A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is *idempotent* if $A^2 = A$.

(a) Check that the matrices $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are both idempotents (you don't need to write this down). Find an idempotent matrix that is equal to neither of these.

(b) Suppose A is idempotent and invertible. Show that $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Recommended reading

Howie book, Chapter 1; you should do as many exercises as you can.