Week 4

Wittgenstein and Quine

Why the obsession with infinity?

Of course there are many reasons… but the primary source of the (ontological?) frustration is the literal *unthinkability* of the set **R** of real numbers as a *totality* [*Ingesamt*, in the language of Hilbert 1904].

Some background is necessary. A real number between 0 and 1 can be written as an infinite (possibly repeating) decimal:

0.142857142857… (= 1/7)

0.33333333… (= 1/3)

0.2000000… (= 1/5)

.14159265… (= π - 3) NOT REPEATING

At least π has a description in terms of circles, or trigonometry, that makes it possible to compute to any degree of accuracy (to 50 trillion decimal places on January 29, 2020).

For most real numbers, NO finite description is possible.

This is one way of explaining Cantor's discovery that there are more real numbers than whole numbers.

We can also write a real number as an infinite binary "decimal" — only 0's and 1's.

So in binary

$$.010101\ldots = 1/4 + 1/16 + 1/64 + \ldots \quad (= 1/3)$$

How many infinite binary "decimals" are there?
Let **N** = {1, 2, 3, …} be the set of positive integers.

Index the places in a "decimal" by **N**:

$$x = .\underset{1,2,3,4,5,6,}{010101}\ldots \qquad \text{has entry 1 in places } 2,4,6,\ldots$$

So the information in x is the same as the information in the **subset** {2, 4, 6, …} of **N,** in other words the set of even positive integers.

There is an (almost) 1-1 correspondence

$$\{\text{binary "decimals"}\} \leftrightarrow \{\text{subsets of } \mathbf{N}\}$$

To any subset Z in **N**, write down the "decimal" with 1 in the n'th place if n is in Z and 0 otherwise.

The empty set is .00000…, the set **N** is .1111111…, and so on.

Cantor's diagonalization argument (yet another precursor of Gödel's theorem) shows: the set P(**N**) of subsets of **N** is "bigger" than **N**.
In fact, for any set X, the set P(X) (*power set*) of subsets of X has more elements than X.

We have seen that P(**N**) is "the same size" as **R**. So there is no *rational* way to list the elements of **R** in order, which means that **R** exceeds our faculty of reason (as well as intuition).

But mathematicians need (or think we need) to work with **R**.

Thus from one perspective there is a huge lacuna in the middle of mathematics.

Write $|\mathbf{N}|$ for the "size" (cardinality) of **N**, usually written $\aleph_0$, $|\mathbf{R}|$ for the cardinality of **R**.
Cantor showed $|\mathbf{R}| > |\mathbf{N}|$.

Cantor's continuum hypothesis (Hilbert's first problem): Let z be a cardinality such that $|\mathbf{R}| \geq z \geq |\mathbf{N}|$. Then either $z = |\mathbf{R}|$ or $z = |\mathbf{N}|$.

(David Foster Wallace wrote a rather confused book about this.)

Gödel and Paul Cohen showed, in two stages, that this is *undecidable*: either the continuum hypothesis or its negation is compatible with the axioms of set theory (ZFC). (There may even be infinitely many distinct cardinalities between $|\mathbf{R}|$ and $|\mathbf{N}|$).

In other words, *there is no truth of the matter*! It's up to mathematicians to decide which version of set theory to choose.

Quine's *On What There Is*.

Before we can ask whether there is a description of the set of subsets of **R**, we may want to ask: does **R** exist? The first chapter of Quine's *From a Logical Point of View* (*FLPV*) is one of the most influential treatments of *existence* in Anglo-American philosophy. (If Pegasus didn't exist, how could we speak of Pegasus? And "how many possible men are there in that doorway"?)

Quine's nominalism (1947)

how, if we regard the sentences of mathematics merely as strings of marks without meaning, we can account for the fact that mathematicians can proceed with such remarkable agreement as to methods and results. Our answer is that such intelligibility as mathematics possesses derives from the syntactical or metamathematical rules governing those marks.

(Goodman and Quine, "Steps Toward a Constructive Nominalism")

From the first paragraph of that article:

We do not believe in abstract objects…. Any system that countenances abstract entities we deem unsatisfactory as a final philosophy.

Their solution is to use the syntax of symbolic logic… for x's and y's, etc. Does this really escape abstraction?

Mathematics plays a central role for Quine, specifically because the question has traditionally been tied up with the existence of *universals*, and also of *counterfactuals*. He sees the competing medieval doctrines regarding universals — *realism, conceptualism, nominalism*, reappearing in 20th century philosophy of mathematics as *logicism, intuitionism, formalism*. With regard to logicism his analysis is more by analogy to realism: Frege et al "condone[] the use of bound variables to refer to abstract entities… indiscriminately." The pairing of formalism with nominalism is more in keeping with Hilbert's position, as Quine interprets it:

…the formalist keeps classical mathematics as a play of insignificant notations, [which]… *can still be of utility... But utility need not imply significance.* (*FLPV*, p. 15)

Trigger warning

The next page is a quotation from an article published in Germany in 1936, a reminder that mathematics was not immune to Nazi propaganda.

Some Nazis defended realism

> Pure mathematics too has real objects—whoever wishes to deny this is a representative of Jewish-liberal thought, like philosophical sophisticates.... Every theory of pure mathematics has the right to exist if it is really in a position to answer concrete questions which concern real objects like whole numbers or geometric figures—or if at least it serves for the construction of things which happen there. Otherwise it is incomplete, or else a document of Jewish-liberal confusion, born from the brains of rootless artists who by juggling with object-less definitions mislead themselves and their thoughtless public.... In the future, we will have German mathematics.

(Edward Tornier. *Deutsche Mathematik*, 1936, vol. 1,page 2)

... one ought to say something to deflate the following banal argument: Are there infinitely many primes? Yes! And prime numbers are numbers? - Well, uh, what does that mean? Of course prime numbers are numbers, but what's being said here? Well, if prime numbers are numbers and there are infinitely many primes, then numbers exist!

(Ian Hacking, *Why is There Philosophy of Mathematics At All?*)

Quine's analysis of a mathematical sentence (the existence of a prime number greater than 1000000) leads him to the conclusion that "classical mathematics requires universals as values of its bound variables" — this is in keeping with Quine's ontological slogan:

to be is to be the value of a variable (or less flippantly)

"To be assumed as an entity is, purely and simply, to be reckoned as the value of a variable"

(*FLPV*, p. 15, p. 13, p. 103)

And thus Quine writes that "the *only* way we can involve ourselves in ontological commitments" is through bound variables, which "range over our whole ontology."

In particular, although Quine's "ontological commitment" to mathematics is indirect (see next page), he has what we might call a *methodological commitment* to formal logic as a standard against which doctrines might be tested.

Aristotle defined *first philosophy* as "being as such" or "being as being". For Quine, as for Russell, *first philosophy* is systematically subordinated to formal logic.

Quine is not a platonist

From the point of view of the conceptual scheme of the elementary arithmetic of rational numbers alone, the broader arithmetic of rational and irrational numbers would have the status of a convenient myth, simpler than the literal truth (namely the arithmetic of rationals) and yet containing that literal truth as a scattered part….

Consider, for example, the crisis which was precipitated in the foundations of mathematics, at the turn of the century, by the discovery of Russell's paradox and other antinomies of set theory. These contradictions had to be obviated by unintuitive, ad hoc devices; our mathematical mythmaking became deliberate and evident to all.

(*FLPV*, p. 18)

Quine nevertheless has an *ontological commitment* to mathematical objects

[The] question what ontology actually to adopt still stands open, and the obvious counsel is tolerance and an experimental spirit.

(*FLPV*, p. 19)

Quine calls the "conceptual scheme of physical objects" — including universals — a "convenient myth" that is

a good and useful one... in so far as it simplifies our account of physics.... Since mathematics is an integral part of this [physical] higher myth, the utility of this myth for physical science is evident enough.

(*FLPV*, p. 18)

Quine-Putnam indispensability thesis (formulated most explicitly by Hilary Putnam): one needs to admit the existence of mathematical objects for the purposes of natural science.

quantification over mathematical entities is indispensable for science, both formal and physical; therefore we should accept such quantification; but this commits us to accepting the existence of the mathematical entities in question. This type of argument stems, of course, from Quine, who has for years stressed both the indispensability of quantification over mathematical entities and the intellectual dishonesty of denying the existence of what one daily presupposes. (Putnam, *Philosophy of Logic*, Chapter 8)

Putnam uses the continuum hypothesis to reject realism

Gödel's proof that the continuum hypothesis is consistent with ZFC is based on the principle called $V = L$.

Cohen's proof that the negation of the continuum hypothesis is also consistent with ZFC entails $V \neq L$.

the realist standpoint is that there is a fact of the matter—a fact independent of our legislation—as to whether $V = L$ or not.

(Putnam, "Models and Reality," 1980)

Putnam uses this and similar examples to reject "metaphysical realism" and thus to subordinate Aristotle's *first philosophy* to *science*:

The program of realism in the philosophy of science-of ***empirical*** *realism, not metaphysical realism-is to show that scientific theories can be regarded as better and better representations of an objective world with which we are interacting…* (*Ibid.*)

Question: In view of these competing attitudes to First Philosophy, what dialogue is possible between philosophy in the Quine-Putnam style and that exemplified by Husserl in Derrida's reading?

Ian Hacking on Wittgenstein's notion of cartesian proof

It is tempting to co-opt the apt words used by Wittgenstein's translators: 'perspicuous' and 'surveyable', and say that Descartes wanted proof to be both. Here is Wittgenstein's key sentence of the late 1930s.

*Perspicuity (**Übersichtlichkeit**) is part of proof. If the process by which I get a result were not surveyable (**übersehbar**) I might indeed make a note that this numbers is what comes out—but what fact is that supposed to confirm for me? I don't know what is **supposed** to come out. (RFM I §153, p. 45).*

(Hacking, *Why Is There Philosophy of Mathematics At All?*, p. 26)

A cartesian (surveyable, perspicuous, synoptic) proof

I see it and I believe it (Plato's *Meno*)

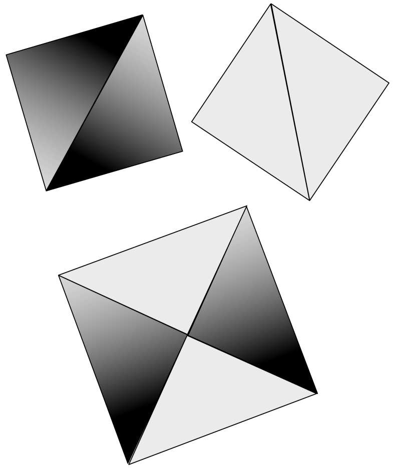

Figure: Dividing a square into two squares

Hacking on Wittgenstein, continued

If you are inclined to use Wittgenstein's words, you may find it useful to observe that he ***introduced*** *them, in connection with maths, in the quotation above. Both* ***Übersichtlichkeit*** *and* ***übersehbar*** *are used in §54. Thereafter he* ***quoted*** *those sentences, marked in quotation marks, and commented upon the words. There is a sense (Quine's) in which he hardly ever* ***used*** *the words in connection with mathematics after their first usage; rather he elucidated what he had meant.* (Hacking, *Ibid.*)

Hacking contrasts *cartesian* proofs, like the one in *Meno*, with *leibnizian* proofs obtained by systematic calculation on the basis of rules, not necessarily guided by an idea. The terminology is due to Hacking, who suspects that most proofs are leibnizian. Wittgenstein is identified as a cartesian on the basis of the quotation from *RFM I*, where he talks about what is *supposed* to come out.

Where does "surveyability" fit in Wittgenstein's philosophy?

Elsewhere in *RFM* Wittgenstein speaks of "the hardness of the logical **must**" (an expression found in other collections of his comments). Then there is this image:

> What is the characteristic mark of 'internal properties'? That they persist always, unalterably, in the whole they constitute; as it were independently of any outside happenings. As the construction of a machine on paper does not break when the machine itself succumbs to external forces. — Or again, I should like to say that they are not subject to wind and weather like physical features of things; rather they are unassailable, like shadows. (RFM 74)

(From *RFM*, I, §102)

Although Wittgenstein, like Quine, stresses the social consensus necessary for mathematics, the undeniable experience of logical compulsion pervades *RFM*.

Are mathematics and logical compulsion identical?

The question arises: can the machine that does not break, the unassailable shadow, be imagined without an experience of mathematics? Or is the compelling quality of mathematics inherited from our ability to conceive of these metaphors?

After all, the Egyptians had a resurrection myth, with a separation of the *idea* of Osiris, who persisted through repeated cycles of death and rebirth, from the *body* of Osiris, which was cut into pieces.