Week 3

Logicism and Formalism

But in opposition to the lived space in which the indefiniteness of the adumbrations is a transcendence that essentially can never be mastered, the idealized space of mathematics allows us to go immediately to the infinite limit of what is in fact an unfinished movement. Thus, the transcendence of every lived future can be absolutely appropriated and reduced in the very gesture which frees that future for an infinite development. Mathematical space no longer knows what Sartre calls "transphenomenality." The developments of mathematical space will never *de jure* escape us; that is why it might seem more reassuring, more *our own*. But is that not also because it has become more foreign to us?

(Derrida, § 10, p. 136)

mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.

(Russell, 1901)

But it cannot be foreign to the machine…

The present work has two main objects. One of these, the proof that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles, is undertaken in Parts II–VII of this Volume, and will be established by strict symbolic reasoning in Volume II. The demonstration of this thesis has, if I am not mistaken, all the certainty and precision of which mathematical demonstrations are capable. As the thesis is very recent among mathematicians, and is almost universally denied by philosophers, I have undertaken, in this volume, to defend its various parts, as occasion arose, against such adverse theories as appeared most widely held or most difficult to disprove. I have also endeav-

(Russell, *Principles of Mathematics*)

We should be concerned only with those objects regarding which our minds seem capable of obtaining certain and indubitable knowledge.

… it is to be concluded, not that arithmetic and geometry are the only subjects to be studied, but only that in seeking the correct path to truth we should be concerned with nothing about which we cannot have a certainty equal to that of the demonstrations of arithmetic and geometry.

(*Rules for the Direction of the Mind*, Rule 2)

(For future discussion: how did machines become the arbiters of certainty and indubitability?)

EUCLID’S AXIOMS

Things which are equal to the same thing are also equal to one another.
If equals be added to equals, the wholes are equal.
If equals be subtracted from equals, the remainders are equal.
Things which coincide with one another are equal to one another.
The whole is greater than the part.

τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα. (Euclid’s Greek, no word for “thing” nor any noun)

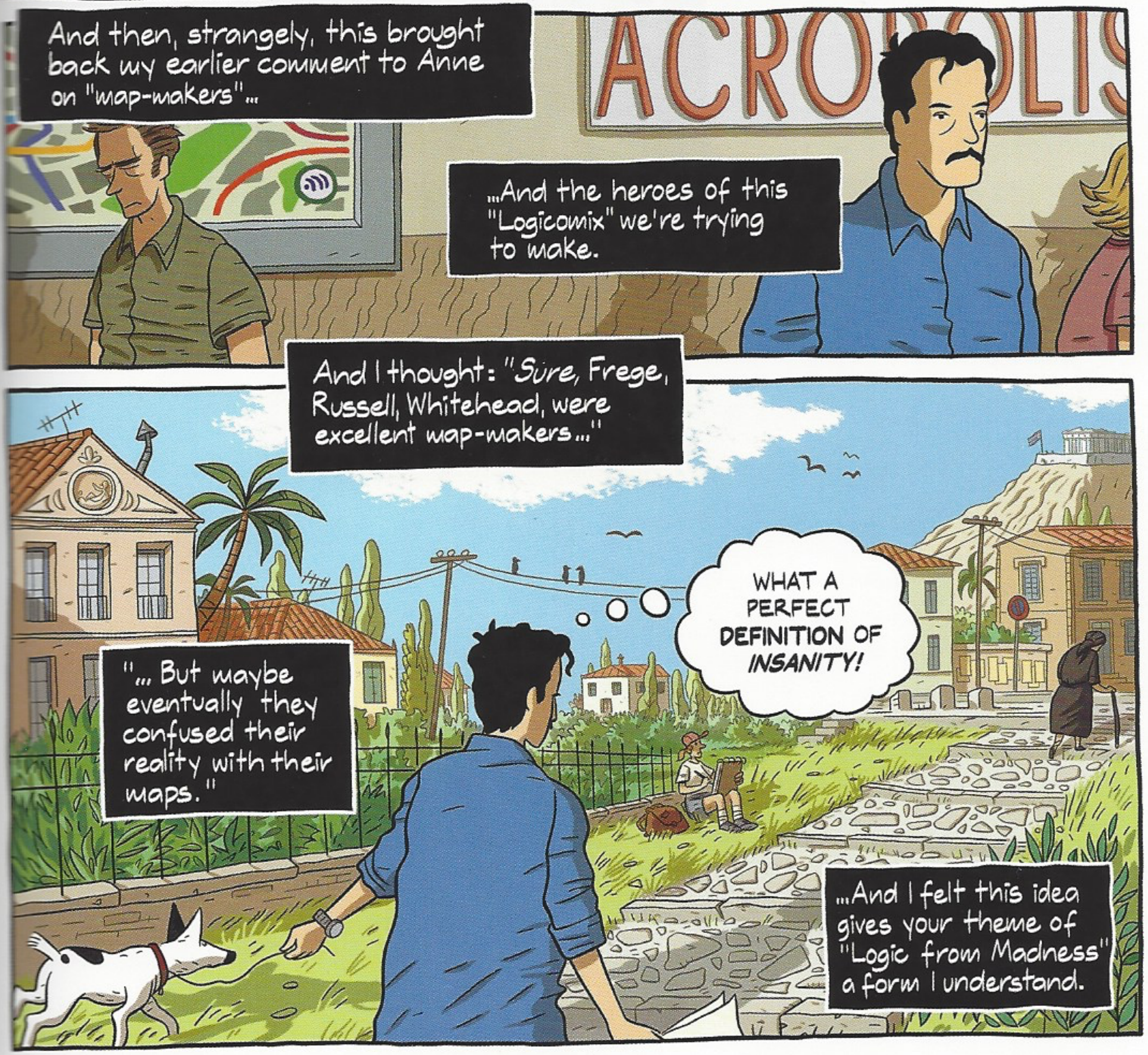
EUCLID’S POSTULATES (based on definitions)

A straight line segment may be drawn from any given point to any other.
A straight line may be extended to any finite length.
A circle may be described with any given point as its center and any distance as its radius.
All right angles are congruent.
If a straight line intersects two other straight lines, and so makes the two interior angles on one side of it together less than two right angles, then the other straight lines will meet at a point if extended far enough on the side on which the angles are less than two right angles.

Horizon is the always-already-there of a future which keeps the indetermination of its infinite openness intact (even though this future was announced to consciousness). As the structural determination of every material indeterminacy, a horizon is always virtually present in every experience; for it is at once the unity and the incompletion for that experience-the anticipated unity in every incompletion. The notion of horizon converts critical philosophy's state of abstract possibility into the concrete infinite potentiality secretly presupposed therein. The notion of horizon thus makes the a priori and the teleological coincide.

(Derrida, *Introduction to Husserl*…, p. 117)

Does the "infinite potentiality" authorized by the axioms of set theory translate Husserl's "horizon"?



And then, strangely, this brought back my earlier comment to Anne on "map-makers"...
ACROPOLIS
...And the heroes of this "Logicomix" we're trying to make.
And I thought: "Sure, Frege, Russell, Whitehead, were excellent map-makers..."
"... But maybe eventually they confused their reality with their maps."
WHAT A PERFECT DEFINITION OF *INSANITY!*
...And I felt this idea gives your theme of "Logic from Madness" a form I understand.

...THE FAMOUS PRONOUNCEMENT OF **EUBOULIDES!**
WHO?
THE MAN WHO SAID...
"MY FELLOW CITIZENS... I AM NOW ***LYING*** TO YOU!"
!
THINK OF IT: IF HE IS ***LYING***
FACT **TELLING *THE TRUTH***
TELLING THE
?

Liar's paradox: *All Cretans are liars.* (spoken by Epimenides of Crete)

More compact version: *This sentence is false.*

Formalized version: $\vdash$ **Fliar**: **Fliar** is false. (from the **very long** article on the Liar's paradox on Stanford Encyclopedia of Philosophy)

Russell's paradox: The set *S* of all sets that do not contain themselves.

$$\phi(x): \; x \in x; \mathrm{S} = \{\mathrm{x} : \sim\phi(x)\}$$

This formula is valid in Frege's version of the *Naïve comprehension axiom*

$$(\mathrm{NC})\ \exists S\ \forall x\ (x \in S \equiv \phi),$$

and allows the definition of *S* as a set.

Naive set: any collection of anything that corresponds to a concept (represented by a function)

Contemporary set theory does not admit (NC); in ZFC it is replaced by

$$\text{(ZA)}\ \forall A\ \exists S\ \forall x\ (x \in S \equiv (x \in A \land \phi)).$$

If we try to let A be the set of all sets, we seem to return to Russell's paradox; but the solution is that there is no such set A…

Axiomatic set ("the paradise that Cantor created for us"): an undefined notion about which we only know that it satisfies some axioms, usually ZFC (Zermelo-Fraenkel + axiom of choice), which do not imply the existence of the set of all sets.

Hilbert's optimism

Let us admit that the situation in which we presently find ourselves with respect to the paradoxes is in the long run intolerable. Just think: in mathematics, this paragon of reliability and truth, the very notions and inferences, as everyone learns, teaches, and uses them, lead to absurdities. And where else would reliability and truth be found if even mathematical thinking fails?…

But there is a completely satisfactory way of escaping the paradoxes without committing treason against our science.…

(1) We shall carefully investigate those ways of forming notions and those modes of inference that are fruitful; we shall nurse them support them, and make them usable, wherever there is the slightest promise of success. ***No one shall be able to drive us from the paradise that Cantor created for us****.*

(2) It is necessary to make inferences everywhere as reliable as they are in ordinary elementary number theory, which no one questions…

(in van Heijenoort, pp. 375-6; my emphasis)

PROOF THAT THE SQUARE ROOT OF 2 IS IRRATIONAL (in Post's notation)

P: ∃ a rational r such that $r^2 = 2$
Q: ∀ rational r, ∃p and q that are not both even and r = p/q.
So then
~Q: ∃ a rational r such that ∀ p, q with r = p/q, p and q are both even.

We take Q as an axiom: $\vdash Q$.
The structure is: $\vdash P \wedge Q \Rightarrow \sim Q$ which is a contradiction, but to analyze it more closely, we want to conclude
$\vdash \; S(P,Q) = [Q \wedge [P \wedge Q \Rightarrow \sim Q]] \Rightarrow \sim P$

$S(P,Q) = [Q \wedge [P \wedge Q \Rightarrow \sim Q]] \Rightarrow \sim P$ is a *tautology*.

We substitute any truth values A, B for P and Q and $S(A,B) = T$

Basic rules of calculation:

$$P \Rightarrow Q = Q \vee \sim P$$

$$T \wedge T = T,\ T \wedge F = F \wedge F = F,\ T \Rightarrow F = F$$

$$S(T,T) = [T \wedge [T \wedge T \Rightarrow \sim T]] \Rightarrow \sim T$$

$$= [T \wedge [T \Rightarrow F]\ ] \Rightarrow F\ = [T \wedge (F \vee \sim T)] \Rightarrow F$$

$$= F \vee \sim[T \wedge (F \vee \sim T)] = F \vee \sim T \vee \sim(F \vee \sim T)$$

$$= F \vee F \vee T \vee T$$

$$= T$$

Questions about mathematics within logic

How to define $\sqrt{-15}$ within pure logic? Within set theory?

$$\exists r: \; r^2 + 15 = 0$$

How do we know it means what we intended? The formalist answer is now framed within algebra:

- **(Algebra of rings)** A commutative ring is a set with addition and multiplication that satisfy the usual rules of arithmetic, and specifically the distributive law.

- **(Abstract algebra)** Is there a commutative ring R containing the integers **Z**, and an element $r \in R$, such that $r^2 + 15 = 0$?

  (This is an existential question about rings and can be formulated in a logic with existential quantifiers, which I have not introduced.)

Formalist solution for $\sqrt{-15}$

Let R = {(a,b), a, b $\in$ **Z**}, with the rule (a,b)(a',b') = (aa' - 15bb',ab' + a'b), and include **Z** as the elements (a,0). Then the element (0,1) satisfies

$$(0,1)(0,1) = (-15,0).$$

Formalist solution for Weil conjectures

Weil: Is there a *cohomology theory of algebraic varieties* with the following long list of properties…?

Grothendieck and associates, completed by Deligne: Yes (after a few thousand pages).

The formalist approach displays the parallel between these two problems, although one is much more elaborate than the other. Is this merely a difference of complexity?

(Logically speaking these are questions of very different orders.)

Frege's "conceptual content"

I take the following example from geometry. Assume that on the circumference of a circle there is a fixed point A about which a ray revolves. When this ray passes through the center of the circle, we call the other point at which it intersects the circle the point B associated with this position of the ray. The point of intersection, other than A, of the ray and the circumference will then be called the point B associated with the position of the ray at any time; this point is such that continuous variations in its position must always correspond to continuous variations in the position of the ray. Hence the name B denotes something indeterminate so long as the corresponding position of the ray has not been specified. We can now ask: what point is associated with the position of the ray when it is perpendicular to the diameter? The answer will be: the point A. In this case, therefore, the name B has the same content as has the name A; and yet we could not have used only one name from the beginning, since the justification for that is given only by the answer. One point is determined in two ways: (1) immediately through intuition and (2) as a point B associated with the ray perpendicular to the diameter.

(Begriffsschrift, §8)

Logicism: Wittgenstein's *Tractatus* on mathematics

6.1. *The propositions of logic are tautologies.*

6.1261. *In logic process and result are equivalent. (Therefore no surprises.)*

6.1262. *Proof in logic is merely a mechanical expedient to facilitate the recognition of tautologies in complicated cases.*

6.2. *Mathematics is a logical method.*

6.21. *A proposition in mathematics does not express a thought.*

6.211. *In life it is never a mathematical proposition which we need, but we use mathematical propositions only in order to infer from propositions which do not belong to mathematics to others which equally do not belong to mathematics.*

Lakatos: *According to logical positivism, a statement is meaningful only if it is either 'tautological' or empirical. Since informal mathematics is neither 'tautological' nor empirical, it must be meaningless, sheer nonsense.*

Logicism: Russell derives numbers from logic

I hold—and it is an important part of my purpose to prove—that all Pure Mathematics (including Geometry and even rational Dynamics) contains only one set of indefinables, namely the fundamental logical concepts discussed in Part I.

*...to define as the number of a class the class of all classes similar to the given class. Membership of this class of classes (considered as a predicate) is a common property of all the similar classes and of no others; moreover every class of the set of similar classes has to the set a relation which it has to nothing else, and which every class has to its own set. Thus the conditions are completely fulfilled by this class of classes, and it has the merit of being determinate when a class is given, and of being different for two classes which are not similar. This, then, is an irreproachable definition of the number of a class in purely logical terms.... Mathematically, **a number is nothing but a class of similar classes**: this definition allows the deduction of all the usual properties of numbers, whether finite or infinite, and is the only one (so far as I know) which is possible in terms of the fundamental concepts of general logic.*

(from *Principles of Mathematics* (1903), pp. 163, 168, 170, my emphasis)

Formalism: Derrida on the self-evidence of form

Whether it is a question of [Husserl's] determining *eidos* in opposition to "Platonism," or form (*Form*) . . . or *morphe* . . . in opposition to Aristotle, the force, vigilance, and efficacity of [Husserl's] critique remain intrametaphysical. . . . Only a form is *self-evident*, only a form has or is an *essence*, only a form *presents itself* as such. . . . Form is presence itself. Formality is whatever aspect of the thing in general presents itself, lets itself be seen, gives itself to be thought. . . . [M]etaphysical thought . . . is a thought of Being as form. (*Margins*, pp. 157–58)

Formalism: a Euclidean-style proof

Theorem: if g and h in G commute gh = hg, then for all n > 1,

$$g^n h = h g^n.$$

Proof:	P: $gh = hg$	(hypothesis)
	P': $g^n h = g(g^{n-1})h = g(g^{n-1}h)$	(axioms of groups)
	P'': By induction	
	$g(g^{n-1}h) = g(hg^{n-1})$	(Peano axioms)
	P''' $g(hg^{n-1}) = (gh)g^{n-1}$	(axioms of groups)
	P^4 $(gh)g^{n-1} = (hg)g^{n-1} = h(gg^{n-1}) = hg^n$	(hypothesis + axioms)
	p^5 $g^n h = hg^n$	(combine P' through P^5)

Formalism: A Turing logical computing machine

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$	None	$P0, R$	$\mathfrak{c}$
$\mathfrak{c}$	None	R	$\mathfrak{e}$
$\mathfrak{e}$	None	$P1, R$	$\mathfrak{f}$
$\mathfrak{f}$	None	R	$\mathfrak{b}$

(from Turing, 1937)   This machine just prints .01010101… forever. It doesn't *halt*.

Formalism: The *Halting Problem* is undecidable

- The *Halting Problem* is the following: is there a Turing machine P with the property that it can examine any Turing machine Q and an input (symbol) I and compute
- P(Q,I) = 1 if Q halts when given I
- P(Q,I) = 0 if Q runs infinitely when given I.

- Suppose such a P exists. Then define P':
- P(P',Q) = 0 if P(Q,Q) = 1
- P(P',Q) = 1 if P(Q,Q) = 0.

- Then P(P',P') = 1 if and only if P(P',P') = 0, contradiction. (**Unless 1 = 0**.)

Formalism: Sketch of Gödel's First Incompleteness Theorem

Suppose there exists a proof system P such that for any statement F about integers, P(F) = 1 if F has a proof, P(F) = 0 if it can be disproved.

Now consider Q to be a Turing machine. This is represented as before by an integer. The property (Q,I): "Q halts with input I" is a statement about integers (this is the key point, and the one that required Gödel's insight and actual calculations).

Thus P((Q,I)) = 1 if there is a proof that Q halts with input I, whereas P((Q,I)) = 0 if there is no such proof — which means that Q doesn't halt with input I, because if it did halt then its halting would be a (finite) proof that it halts.

Then P((Q,I)) is decidable. But this contradicts Turing's result.

Human or Computer Poet?

1. is beauty itself
   that they were walking there. All along the new world naked,
   cold, familiar wind -

2. Pink confused with white
   flowers and flowers reversed
   take and spill the shaded flame
   darting it back
   into the lamp's horn

3. The winds of the oozy woods which wear
   the ocean, with azure moss and flowers
   So sweet, the purple even
   I sleep in the arrows
   Of the dome of death.

4. O thou,
   Who moved among some fierce Maenad, even among noise
   and blue
   Between the bones sang, scattered and the silent seas.

5. She eyes me with an ingrown eye,
   in the rhythm of teacup tapping
   thinks of sweeping away crumbs

Human or Computer Poet?

6. At six I cannot pray:
   Pray for lovers,
   through narrow streets
   And pray to fly
   But the Virgin in their dark wintry bed

7. What seas what shores what granite islands towards my timbers
   and woodthrush calling through the fog
   My daughter.

8. Imagine now a tree in white sails still whirled
   About the leaves
   will be of silences
   Calm and angels

9. -and the sun, dipping into the avenues
   streaking the tops of
   the irregular red houselets,and
   the gay shadows dropping and dropping.

10. The morning and already
    a perfect if slightly paled
    old park turned with young women
    seized in amber

Answers (1-5)

1. is beauty itself
   that they were walking there. All along the new world naked,
   cold, familiar wind -

   Computer

2. Pink confused with white
   flowers and flowers reversed
   take and spill the shaded flame
   darting it back
   into the lamp's horn

   William Carlos Williams

3. The winds of the oozy woods which wear
   the ocean, with azure moss and flowers
   So sweet, the purple even
   I sleep in the arrows
   Of the dome of death.

   Computer

4. O thou,
   Who moved among some fierce Maenad, even among noise
   and blue
   Between the bones sang, scattered and the silent seas.

   Computer

5. She eyes me with an ingrown eye,
   in the rhythm of teacup tapping
   thinks of sweeping away crumbs

   Raymond Kurzweil

Answers (6-10)

6. At six I cannot pray:
   Pray for lovers,
   through narrow streets
   And pray to fly
   But the Virgin in their dark wintry bed

   Computer

7. What seas what shores what granite islands towards my timbers
   and woodthrush calling through the fog
   My daughter.

   T.S. Eliot

8. Imagine now a tree in white sails still whirled
   About the leaves
   will be of silences
   Calm and angels

   Computer

9. -and the sun, dipping into the avenues
   streaking the tops of
   the irregular red houselets,
   and
   the gay shadows dropping and dropping.

   William Carlos Williams

10. The morning and already
    a perfect if slightly paled
    old park turned with young women
    seized in amber

    Raymond Kurzweil