ALGEBRAIC NUMBER THEORY W4043

Take home final, due December 12 at 3 PM

1. (a) Let $K = \mathbb{Q}(\sqrt{-13})$. Determine the integer ring $\mathcal{O}_K$ and the discriminant Δ_K.

(b) Show that $V = \mathcal{O}_K/2\mathcal{O}_K$ is a 2-dimensional vector space over the field $\mathbb{F}_2$ of 2-elements. For any $v_1, v_2 \in V$, we define an element $B(v_1, v_2) \in \mathbb{F}_2$ as follows. Choose $\tilde{v}_1 \in \mathcal{O}_K$, $\tilde{v}_2 \in \mathcal{O}_K$ such that

$$\tilde{v}_i \equiv v_i \pmod 2$$

(in other words, lift v_1 and v_2 to elements of $\mathcal{O}_K$.) Let

$$\tilde{B}(v_1, v_2) = Tr_{K/\mathbb{Q}} v_1 \cdot v_2 \in \mathbb{Z}$$

and define $B(v_1, v_2)$ to be the reduction of $\tilde{B}(v_1, v_2)$ modulo 2. Show that $B(v_1, v_2)$ depends only on v_1 and v_2 and not on the choice of $\tilde{v}_1$ and $\tilde{v}_2$. Show that $B(v_1, v_2)$ is a symmetric bilinear form, i.e.

$$B(v_1, v_2) = B(v_2, v_1); B(\lambda v + \mu v', v_2) = \lambda B(v, v_2) + \mu B(v', v_2)$$

whenever $v_1, v, v', v_2 \in V$ and $\lambda, \mu \in \mathbb{F}_2$.

(c) Show that there exists $v \in V$ such that $B(v, w) = 0$ for all $w \in V$.

(d) Choose a basis $\{e_1, e_2\}$ of V over $\mathbb{F}_2$ and write down the matrix

$$A = \begin{pmatrix} B(e_1, e_1) & B(e_1, e_2) \\ B(e_2, e_1) & B(e_2, e_2) \end{pmatrix}.$$

Show that $\det(A) = 0$. (Note: the vanishing of the determinant should be independent of the choice of basis.)

2. (a) Show that $\varprojlim_n \mathbb{Z}/15^n\mathbb{Z}$ is a ring isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_5$. Is it an integral domain?

(b) Suppose we replace 15 with 30: what is the result?

3. Let $N > 0$ be an integer and let χ be a Dirichlet character modulo N. Define χ^{-1} to be the Dirichlet character such that $\chi^{-1}(a)\chi(a) = 1$ if $(a, N) = 1$, and $\chi^{-1}(a) = 0$ otherwise. We assume χ is not the trivial character; in other words, there is a prime to N with $\chi(a) \neq 1$.

(a) Let $D(s, \chi) = L(s, \chi) \cdot L(s, \chi^{-1})$. Show that this is a Dirichlet series with Euler product

$$D(s, \chi) = \prod_p D_p(s, \chi)$$

and compute $D_p(s, \chi)$ for all p.

(b) Let p be a prime that does not divide N. Show that

$$D_p(s,\chi) = (1 - a_p p^{-s} + b_p p^{-2s})^{-1}$$

where a_p and b_p are real numbers.

(c) Show that

$$D(s,\chi) = \sum_{n \geq 1} \frac{a_n}{n^{-s}}$$

where the a_n are all real, and that when $n = p$ is a prime not dividing N, then the a_p are the same as in (b). Find the set of absolute convergence of $D(s,\chi)$.

(d) Show that $D(s,\chi)$ extends to an entire function of $\mathbb{C}$. Is this consistent with (b) in view of Landau's Lemma? Explain.

4. Let $p \neq q$ be two odd primes. Let $K_p = \mathbb{Q}(\zeta_p)$, $K_q = \mathbb{Q}(\zeta_q)$ $K_{pq} = \mathbb{Q}(\zeta_p, \zeta_q)$, where $\zeta_p = e^{2\pi i/p}$ and $\zeta_q = e^{2\pi i/q}$.

(a) Show that $[K_{pq} : K_p] = q - 1$.

(b) Let $x = \zeta_p - \zeta_q \in K_{pq}$. Show that

$$N_{K_{pq}/K_p}(x) = 1 + \zeta_p + \zeta_p^2 + \cdots + \zeta_p^{q-1}.$$

(c) Show that

$$N_{K_{pq}/K_p}(x) = \frac{1 - \zeta_p^q}{1 - \zeta_p}.$$

(d) Conclude that x is a unit in the ring of integers of K_{pq}.