## ALGEBRAIC NUMBER THEORY W4043

### Homework, week 4, due February 17

1. Let $d > 0$ be a square-free positive integer congruent to 2 (mod 4).

(a) Every unit $u \in \mathbb{Z}[\sqrt{d}]$ is of the form $a - b\sqrt{d}$ where $a^2 - db^2 = \pm 1$, and the group $\Gamma$ of units is the product of an infinite cyclic group with $\{\pm 1\}$. Consider the subset $\Sigma$ of $\Gamma$ consisting of $u_i = a_i - b_i\sqrt{d}$ with $a_i > 0, b_i > 0$, ordered so that $b_1 \leq b_2 \leq b_3 \ldots$. Show that $u_1$ and $-1$ are generators of $\Gamma$. The element $u_1$ is called the *fundamental unit* of $\mathbb{Z}[\sqrt{d}]$.

(b) Show that the following algorithm finds $u_1$: Letting $b = 1, 2, 3, \ldots$, consider the quantities $q^{\pm}(b) = db^2 \pm 1$. Let $b_1$ be the smallest positive integer such that either $q^+(b_1)$ or $q^-(b_1)$ is a perfect square. Let $a_1$ be the positive square root of $q(b_1)$; then $u_1 = a_1 - b_1\sqrt{d}$.

(c) Use this algorithm to find the fundamental units $u_1$ of $\mathbb{Z}[\sqrt{6}]$, $\mathbb{Z}[\sqrt{10}]$, $\mathbb{Z}[\sqrt{14}]$. In each case determine $N_{K/\mathbb{Q}}(u_1)$, where $K = \mathbb{Q}(\sqrt{d})$ in each case.

2. Exercises 6.7 and 6.22 in Hindry's book, pp. 115, 123.

3. As Hindry shows on p. 99, the ring $R = \mathbb{Z}[\sqrt{10}]$, which is equal to the ring of integers in $\mathbb{Q}(\sqrt{10})$, is not a principal ideal domain. Indeed, the integer 9 has two inequivalent factorizations:

$$9 = 3^2 = (\sqrt{10} - 1)(\sqrt{10} + 1).$$

(a) Use the computation in 2 (c) to confirm that the two factorizations are indeed inequivalent.

(b) The integer 10 is definitely a square modulo 3. What is the prime factorization of the ideal $(3) \subset R$?

4. Let $R$ be a Dedekind ring with only finitely many prime ideals. Show that $R$ is a PID. (Hint: say $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r$ are the prime ideals. Find an element $x_i \in \mathfrak{p}_i$ that is not in any of the $\mathfrak{p}_j$ with $j \neq i$, and factor the ideal $(x_i)$. Another piece of information is necessary.)

5. List the squares modulo 8.

(1) Show by testing all possibilities that the equation $x^2 - 11y^2 = 7$ has no solution modulo 8.

(2) Use this to show that every unit in the ring of integers of $\mathbb{Q}(\sqrt{11})$ has norm 1.