ALGEBRAIC NUMBER THEORY W4043

1. Homework, week 2, due February 3, 2022

Part I: To be handed in

1. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic extension with $d \equiv 3 \pmod 4$, and let $\mathcal{O}_K$ be the integer ring.

(a) If $x = \alpha + \beta\sqrt{d}$, with $\alpha, \beta \in \mathbb{Q}$, show that

$$Tr_{K/\mathbb{Q}}(x) = 2\alpha.$$

(Remember that $Tr_{K/\mathbb{Q}}(x)$ is the trace of the matrix of multiplication by x in K.)

(b) Let $\Lambda \subset K$ be the subset of $\lambda \in K$ such that

$$Tr_{K/\mathbb{Q}}(\lambda \cdot a) \in \mathbb{Z} \ \forall a \in \mathcal{O}_K.$$

Show that Λ is a subgroup of K that strictly contains $\mathcal{O}_K$.

2. Show that an element $(a, b) \in \mathbb{Z}^2$ can be completed to a basis for $\mathbb{Z}^2$ if and only if a and b are relatively prime.

3. Let K be a quadratic extension of $\mathbb{Q}$ with integer ring $\mathcal{O}_K$. Show that not every subgroup of $\mathcal{O}_K$ containing $2\mathcal{O}_K$ is an ideal.

4. Let p be a prime number. Give a complete classification of all rings R that are vector spaces of dimension 2 over $\mathbb{F}_p$. (Use the fact that a finite integral domain is a field.)

5. Let $n, m \in \mathbb{Z}$. Let $I \subset \mathbb{Z}$ be the smallest ideal containing the principal ideals (n) and (m). (This is written $I = (n)+(m)$.) Show that I is generated by the g.c.d of n and m.

Part II: Review of modules and Noetherian rings

This is background for next week; exercises are not to be handed in!

1. Read and do all (or most of) the exercises on modules over a PID at `http://www.imsc.res.in/~knr/14mayafs/Notes/ps.pdf`

2. Study the notes on Noetherian rings and do all (or most of) the exercises at
`http://www.math.columbia.edu/~harris/W40432017/Harvardnotes.pdf`
(These notes were copied from an anonymous Harvard Mathematics Department website two years ago, but are no longer accessible).

3. Let A be a Noetherian ring and let $f : A \to A$ be a ring homomorphism. Prove that f is an isomorphism if and only if f is surjective.

(Hint. Assume that f is surjective and denote by I_j the kernel of $f^{(j)} = f \circ f \circ \cdots \circ f$ (j times). Show that $\{I_j\}$ forms an increasing sequence of ideals of A and therefore $I_j = I_{j+1}$ for some j Deduce that $f(f^{(j)}(a)) = 0 \Rightarrow f^{(j)}(a) = 0$ for any $a \in A$, and use the surjectivity of f to complete the proof.)