Problem set #11 solutions

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First we claim the following:

Claim: Let $H$ be a group of order $p^k$, where $k > 1$. Then $H$ is not simple.

Proof: Consider $Z(H)$, the center of $H$; we know $Z(H)$ is a normal subgroup of $H$. By the class equation,

$$|H| = |Z(H)| + \sum_A |H : C_H(A)|$$

where $A$ runs over the non-singleton conjugacy classes of $H$, and $C_H(A)$ is the centralizer in $H$ of $A$. For each $A$, we have $|H : C_H(A)|$ divides $|H| = p^k$, so $p$ divides $|H : C_H(A)|$. Since $p$ divides $|H|$, we see from the class equation that $p$ must divide $|Z(H)|$; that is, $Z(H) \neq \{1\}$.

This leaves two possibilities: either $\{1\} \subsetneq Z(H) \subsetneq H$, or $Z(H) = H$.

In the former case, $Z(H)$ is a normal subgroup of $H$ which is neither $\{1\}$ nor $H$, so $H$ is not simple. In the latter case, $H$ is abelian. The fact that $|H| = p^k$ with $k > 1$ guarantees $H$ has a subgroup which is neither $\{1\}$ nor $H$. In an abelian group, any subgroup is normal; so $H$ is not simple. Hence in either case $H$ is not simple. \(/\)

Now, for a group $G$ of order $p^n$, let $G = H_n \supseteq H_{n-1} \supseteq \cdots \supseteq H_1 \supseteq H_0 = \{1\}$ be any composition series.

For any $i = 1, \ldots, n$, we know $|H_i|, |H_{i-1}|$ divide $|G| = p^n$ and are thus powers of $p$; then so is $[H_i : H_{i-1}]$. Then $H_i/H_{i-1}$ is a simple group of order $p^k$ for some $k > 0$, so by the Claim, we have $k = 1$; so $H_i/H_{i-1}$ has order $p$, and is therefore isomorphic to $Z_p$.

Hence every composition factor of $G$ is isomorphic to $Z_p$. There must be $r$ such factors. \(\blacksquare\)

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§13.3, exercise 4

a) A composition series for $\mathbb{Z}_n$ corresponds naturally to the prime factorization of $n$.

For $\mathbb{Z}_{12}$, the composition series are

$$\mathbb{Z}_{12} \supseteq \langle 2 \rangle \supseteq \langle 4 \rangle \supseteq \{0\}$$
$$\mathbb{Z}_{12} \supseteq \langle 2 \rangle \supseteq \langle 6 \rangle \supseteq \{0\}$$
$$\mathbb{Z}_{12} \supseteq \langle 3 \rangle \supseteq \langle 6 \rangle \supseteq \{0\}$$

b) For $\mathbb{Z}_{48}$, the composition series are

$$\mathbb{Z}_{48} \supseteq \langle 2 \rangle \supseteq \langle 4 \rangle \supseteq \langle 8 \rangle \supseteq \langle 16 \rangle \supseteq \{0\}$$
$$\mathbb{Z}_{48} \supseteq \langle 2 \rangle \supseteq \langle 4 \rangle \supseteq \langle 8 \rangle \supseteq \langle 24 \rangle \supseteq \{0\}$$
$$\mathbb{Z}_{48} \supseteq \langle 2 \rangle \supseteq \langle 4 \rangle \supseteq \langle 12 \rangle \supseteq \langle 24 \rangle \supseteq \{0\}$$
$$\mathbb{Z}_{48} \supseteq \langle 2 \rangle \supseteq \langle 6 \rangle \supseteq \langle 12 \rangle \supseteq \langle 24 \rangle \supseteq \{0\}$$
$$\mathbb{Z}_{48} \supseteq \langle 3 \rangle \supseteq \langle 6 \rangle \supseteq \langle 12 \rangle \supseteq \langle 24 \rangle \supseteq \{0\}$$
c) The subgroups of $Q_8$ are $\{1\}, (-1), (i), (j), (k), Q_8$, all of which are normal in $Q_8$. For each of these we need to check whether the quotient group is simple and, if so, build the composition series from there.

The composition series are

\[
Q_8 \supseteq \langle i \rangle \supseteq \langle -1 \rangle \supseteq \{1\} \\
Q_8 \supseteq \langle j \rangle \supseteq \langle -1 \rangle \supseteq \{1\} \\
Q_8 \supseteq \langle k \rangle \supseteq \langle -1 \rangle \supseteq \{1\} 
\]

d) The normal subgroups of $D_4 := \langle r, s \mid r^4 = s^2 = id, srs^{-1} = r^{-1} \rangle$ are $\{1\}, \langle r^2 \rangle, \langle r \rangle, \langle s, r^2 \rangle, \langle sr, sr^3 \rangle, D_4$.

The composition series are

\[
D_4 \supseteq \langle r \rangle \supseteq \langle r^2 \rangle \supseteq \{1\} \\
D_4 \supseteq \langle s, r^2 \rangle \supseteq \langle s \rangle \supseteq \{1\} \\
D_4 \supseteq \langle s, r^2 \rangle \supseteq \langle r^2 \rangle \supseteq \{1\} \\
D_4 \supseteq \langle s, r^2 \rangle \supseteq \langle sr^2 \rangle \supseteq \{1\} \\
D_4 \supseteq \langle sr, sr^3 \rangle \supseteq \langle sr \rangle \supseteq \{1\} \\
D_4 \supseteq \langle sr, sr^3 \rangle \supseteq \langle sr^3 \rangle \supseteq \{1\} \\
D_4 \supseteq \langle sr, sr^3 \rangle \supseteq \langle r^2 \rangle \supseteq \{1\} 
\]

e) The normal subgroups of $S_3$ are $\{id\}, A_3, S_3$, and the normal subgroups of $Z_4$ are $\{0\}, \langle 2 \rangle, Z_4$.

The composition series are

\[
S_3 \times Z_4 \supseteq S_3 \times \langle 2 \rangle \supseteq S_3 \times \{0\} \supseteq A_3 \times \{0\} \supseteq \{id\} \times \{0\} \\
S_3 \times Z_4 \supseteq S_3 \times \langle 2 \rangle \supseteq A_3 \times \{0\} \supseteq \{id\} \times \{0\} \\
S_3 \times Z_4 \supseteq S_3 \times \langle 2 \rangle \supseteq A_3 \times \{0\} \supseteq \{id\} \times \{0\} \\
S_3 \times Z_4 \supseteq A_3 \times Z_4 \supseteq A_3 \times \langle 2 \rangle \supseteq A_3 \times \{0\} \supseteq \{id\} \times \{0\} \\
S_3 \times Z_4 \supseteq A_3 \times Z_4 \supseteq A_3 \times \langle 2 \rangle \supseteq \{id\} \times \{0\} \supseteq \{id\} \times \{0\} \\
S_3 \times Z_4 \supseteq A_3 \times Z_4 \supseteq \{id\} \times Z_4 \supseteq \{id\} \times \{2\} \supseteq \{id\} \times \{0\} 
\]

f) The normal subgroups of $S_4$ are $\{id\}, K$ (the subgroup of all products of two disjoint 2-cycles), $A_4, S_4$.

The composition series are

\[
S_4 \supseteq A_4 \supseteq K \supseteq \langle (12)(34) \rangle \supseteq \{id\} \\
S_4 \supseteq A_4 \supseteq K \supseteq \langle (13)(24) \rangle \supseteq \{id\} \\
S_4 \supseteq A_4 \supseteq K \supseteq \langle (14)(23) \rangle \supseteq \{id\} 
\]

[Remark: All of the factor groups are abelian, which means $S_4$ is solvable. In Galois theory one uses this to show that every degree-4 polynomial equation with rational coefficients is solvable by radicals.]

g) Knowing that $A_n$ is simple for $n \geq 5$, we see that the only composition series of $S_n$ is

\[
S_n \supseteq A_n \supseteq \{id\} 
\]

[Remark: This means $S_n$ is not solvable for $n \geq 5$; and in general, degree-$n$ polynomial equations are not solvable by radicals.]

[Note: The Jordan-Hölder theorem states that for any finite group, the composition series are isomorphic, in the sense that the set of composition factors is the same up to ordering. One checks that this is true for the groups above.]
Given a group \( G \) and a normal subgroup \( N \), the correspondence theorem states that there is a one-to-one correspondence between subgroups \( H \subseteq N \) and subgroups \( N \subseteq H' \subseteq G \).

Moreover, \( H_1, H_2 \subseteq N \) satisfy \( H_1 \subseteq H_2 \) if and only if the corresponding \( N \subseteq H_1', H_2' \subseteq G \) satisfy \( H_1' \subseteq H_2' \), in which case \( H_2/H_1 \cong H_2'/H_1' \).

Now suppose \( N \supseteq H_1 \supseteq \cdots \supseteq H_{r-1} \supseteq \{1\} \) and \( G/N \supseteq G_1 \supseteq \cdots \supseteq G_{s-1} \supseteq \{1\} \) are composition series for \( N \) and \( G/N \), respectively. Then we can use the correspondence theorem to lift the composition series for \( N \) to get \( G \supseteq H_1' \supseteq \cdots \supseteq H_{r-1}' \supseteq G/N \supseteq G_1 \supseteq \cdots \supseteq G_{s-1} \supseteq \{1\} \).

The first \( r \) composition factors are isomorphic to the composition factors of \( N \); the remaining \( s \) composition factors are the composition factors of \( G/N \). Since these are all abelian (as \( N \) and \( G/N \) are solvable), we conclude that \( G \) is solvable.

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Let \( G \) be a solvable group, with \( G \supseteq G_1 \supseteq \cdots \supseteq G_n \supseteq \{1\} \) be a subnormal series for \( G \), and let \( H \subseteq G \) be any subgroup.

Define \( H_i := (H \cap G_i) \) for \( i = 1, \ldots, n \); then \( H_i \subseteq G_i \) is a subgroup. Also, \( G_{i+1} \subseteq G_i \) is a normal subgroup, so by the second isomorphism theorem, \( (H_i \cap G_{i+1}) = H_{i+1} \) is a normal subgroup of \( H_i \), and \( G_{i+1} \) is a normal subgroup of \( H_i/G_i+1 \) (which is a subgroup of \( G_i \)); and \( H_i/H_{i+1} \cong H_iG_{i+1}/G_i+1 \subseteq G_i/G_{i+1} \). Since \( G \) is solvable, \( G_i/G_{i+1} \) is an abelian group, so \( H_i/H_{i+1} \) is also an abelian group.

Hence \( H \supseteq H_1 \supseteq \cdots \supseteq H_n \supseteq \{1\} \) is a subnormal series for \( H \) that shows that \( H \) is solvable.

4

An example is \( S_3 \). The factor groups in the subnormal series \( S_3 \supseteq A_3 \supseteq \{id\} \) are, respectively, isomorphic to \( Z_2 \) and \( Z_3 \), which are abelian, so \( S_3 \) is solvable; but the center of \( S_3 \) is trivial.

5

a) To show that \( H \subseteq GL_3(\mathbb{R}) \) is a subgroup, we verify that \( H \) is closed under multiplication and contains the identity and inverses:

- \( H \) is closed under multiplication, since \( u(x_1, y_1, z_1)u(x_2, y_2, z_2) = u(x_1 + x_2, y_1 + y_2, x_1y_2 + z_1 + z_2) \).
- \( H \) contains \( id \), since the identity of \( GL_3(\mathbb{R}) \) is \( u(0, 0, 0) \).
- \( H \) contains inverses, since (using the above) the inverse of \( u(x, y, z) \) is \( u(-x, -y, -z + xy) \).

Hence \( H \subseteq GL_3(\mathbb{R}) \) is a subgroup.
b) Suppose \( u(x_0, y_0, z_0) \in Z(H) \), and let \( u(x, y, z) \in H \) be any element. Then
\[
\begin{align*}
  u(x_0, y_0, z_0) u(x, y, z) &= u(x_0 + x, y_0 + y, x_0 y + z_0 + z) \\
  \text{and} \\
  u(x, y, z) u(x_0, y_0, z_0) &= u(x + x_0, y + y_0, x y_0 + z + z_0)
\end{align*}
\]
are equal if and only if \( x_0 y = x y_0 \). Since \( x, y \in \mathbb{R} \) are arbitrary, we must have \( x_0 = y_0 = 0 \).

Hence the center of \( H \) is \( Z(H) = \{ u(0, 0, z) : z \in \mathbb{R} \} \).

c) The descending central series is given by \( H = H_0 \supseteq H_1 \supseteq H_2 \supseteq \cdots \), where \( H_i = [H_{i-1}, H] \).

Thus, \( H_1 = [H, H] \). The commutator of \( u(x_1, y_1, z_1) \) and \( u(x_2, y_2, z_2) \) is
\[
\begin{align*}
  u(x_1, y_1, z_1) u(x_2, y_2, z_2) u(x_1, y_1, z_1)^{-1} u(x_2, y_2, z_2)^{-1} \\
  &= u(x_1, y_1, z_1) u(x_2, y_2, z_2) u(-x_1, -y_1, -z_1 + x_1 y_1) u(-x_2, -y_2, -z_2 + x_2 y_2) \\
  &= u(x_1 + x_2, y_1 + y_2, z_1 + z_2 + x_1 y_2) u(-x_1 - x_2, -y_1 - y_2, -z_1 - z_2 + x_1 y_1 + x_1 y_2 + x_2 y_2) \\
  &= u(0, 0, 2 x_1 y_2 + x_1 y_1 + x_2 y_2 + (x_1 + x_2)(-y_1 - y_2)) \\
  &= u(0, 0, x_1 y_2 - x_2 y_1)
\end{align*}
\]
so the commutator is always in \( Z(H) \). Moreover, any element of \( Z(H) \) can be realized as a commutator, for example, by setting \( x_1 = 1 \) and \( y_1 = 0 \). Since \( Z(H) \) is a group, this means \( [H, H] = Z(H) \).

Now, since any element of \( Z(H) \) commutes with any element of \( H \), the commutator subgroup \([Z(H), H]\) is trivial. Hence \( H \) is nilpotent and the descending central series is \( H \supseteq Z(H) \supseteq \{ \text{id} \} \).

d) Possible answers include \( H_0 = \{ \text{id} \}, H_1 = \{ u(0, 0, q) : q \in \mathbb{Q} \}, H_2 = \{ u(x, z) : x, z \in \mathbb{R} \} \). ■