1. (a) Let \( q(X,Y) = aX^2 + bXY + cY^2 \) be a positive-definite binary quadratic form with integer coefficients. Assume it has discriminant \( \Delta = -7 \) and is reduced. Recall that a reduced quadratic form has the property that \( a \leq \sqrt{|\Delta|}/3 \approx 1.53 \). Give the possible values for \( (a,b,c) \).

(b) Use the result of (a) to determine the class number of \( K = \mathbb{Q}(\sqrt{-7}) \).

(c) For each \( q \) as in (a), determine the set of primes \( p \) represented by \( q \). What is their relation to the set of primes that split in \( K \)?

**Dirichlet characters**

Let \( n \) be a positive integer. A Dirichlet character modulo \( n \) is a function \( \chi : \mathbb{Z} \to \mathbb{C} \) with the following properties:

1. \( \chi(ab) = \chi(a)\chi(b) \).
2. \( \chi(a) \) depends only on the residue class of \( a \) modulo \( n \).
3. \( \chi(a) = 0 \) if and only if \( a \) and \( n \) have a non-trivial common factor.

It follows that a Dirichlet character modulo \( n \) can also be considered a function \( \chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C} \).

Let \( X(n) \) denote the set of distinct Dirichlet characters modulo \( n \). We consider \( X(p) \) when \( p \) is prime and show it forms a cyclic group with identity element \( \chi_0 \) defined by \( \chi_0(a) = 1 \) if \( (a,p) = 1 \), \( \chi_0(a) = 0 \) if \( p \mid a \).

2. Show that for any \( \chi \in X(p) \), \( \chi(1) = 1 \), and \( \chi(a) \) is a \((p-1)\)st root of \( 1 \) for all \( a \in (\mathbb{Z}/p\mathbb{Z})^\times \).

3. For all \( a \in (\mathbb{Z}/p\mathbb{Z})^\times \), show that \( \chi(a^{-1}) = \overline{\chi(a)} \) where \( \overline{\chi} \) is the complex conjugate function.

4. Show that \( \sum_{a \in \mathbb{Z}/p\mathbb{Z}} \chi(a) = 0 \) if \( \chi \neq \chi_0 \).

5. Show that the Legendre symbol \( a \mapsto \left( \frac{a}{p} \right) \) for \( (a,p) = 1 \), extended to take the value 0 at integers divisible by \( p \), defines a Dirichlet character modulo \( p \) that is different from \( \chi_0 \).