I. The first part of this assignment establishes some of the basic properties of quadratic forms attached to ideals in imaginary quadratic fields. A quadratic space of rank \( n \) over \( \mathbb{Z} \) is a pair \((M, q)\), where \( M \) is a free rank \( n \) \( \mathbb{Z} \)-module (free abelian group on \( n \) generators) and \( q : M \to \mathbb{Z} \) is a quadratic form, i.e. a function satisfying

1. \( q(am) = a^2 q(m) \), \( a \in \mathbb{Z}, m \in M \);
2. The function \( B_q : M \times M \to \mathbb{Z} \), defined by \( B_q(m, m') = q(m + m') - q(m) - q(m') \) is a bilinear form, i.e.
3. \( B_q(m, m') = B_q(m', m) \);
4. \( B_q(am + bm', m'') = aB_q(m, m'') + bB_q(m', m'') \).

We only consider the case \( n = 2 \) and identify \( M \) with \( \mathbb{Z}^2 \). If \( \{e_1, e_2\} \) is the standard \( \mathbb{Z} \)-basis of \( \mathbb{Z}^2 \), \( B_q \) is determined by the \( 2 \times 2 \) symmetric matrix \( (b_{ij}) \) where \( B_q(e_i, e_j) = b_{ij} \) (and you can check that this in turn determines \( q(m) = \frac{1}{2} B_q(m, m) \)). We identify \( q \) with a polynomial in two variables \((X, Y)\) by setting

\[
q(X, Y) = q(Xe_1 + Ye_2).
\]

A (binary) quadratic form \( q(X, Y) = aX^2 + bXY + cY^2 \)

Say \((M, q)\) and \((M', q')\) are isomorphic if there is an isomorphism \( f : M \to M' \) of abelian groups such that \( q' \circ f = q \). Define the discriminant of the quadratic form \( q \) by \( \Delta(q) = -\det(b_{ij}) \) and check for yourselves (without writing it down) that two isomorphic quadratic spaces have the same discriminant.

1. Consider \( q_1(X, Y) = X^2 + 15Y^2 \), \( q_2(X, Y) = 3X^2 + 5Y^2 \). Show that \( q_1 \) and \( q_2 \) have the same discriminant but don’t define isomorphic quadratic spaces.

2. Let \( d \) be a positive squarefree integer. Let \( K = \mathbb{Q}(\sqrt{-d}) \), with integer ring \( \mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{-d}}{2}] \) if \( d \equiv 3 \) (mod 4) and \( \mathcal{O}_K = \mathbb{Z}[\sqrt{-d}] \) if \( d \equiv 1, 2 \) (mod 4). We write \( \Delta_d = -d \) if \( d \equiv 3 \) (mod 4) and \( \Delta_d = -4d \) if \( d \equiv 1, 2 \) (mod 4) (this is the discriminant of the field \( K \)).

(a) Show that the quadratic form \( q = q_{\mathcal{O}_K} \) on the rank 2 \( \mathbb{Z} \)-module \( \mathcal{O}_K \), defined by \( q(x) = N_{K/\mathbb{Q}}(x) \), has discriminant \( \Delta_d \). Moreover, \( q \) is positive definite: \( q(x) > 0 \) for all \( x \neq 0 \).

(b) Show that the bilinear form \( B_q \) associated to \( q \) is given by

\[
B_q(x, y) = Tr_{K/\mathbb{Q}}(x\sigma(y)) = x\sigma(y) + \sigma(x)y
\]

where \( \sigma \in Gal(K/\mathbb{Q}) \) is the non-trivial element.
(c) In general, let \( I \subset \mathcal{O}_K \) be an ideal, \( N(I) = [\mathcal{O}_K : I] = |\mathcal{O}_K / I| \). Define \( q_I : I \to \mathbb{Q} \) by \( q_I(x) = N_{K/\mathbb{Q}}(x)/N(I) \). Show that \( q_I \) takes values in \( \mathbb{Z} \) and the pair \((I, q_I)\) is a quadratic space over \( \mathbb{Z} \).

(d) Show that \((I, q_I)\) is of discriminant \( \Delta_d \).

II. 1. Do exercise 6.15, p. 120 from Hindry’s book.

2. Let \( v_1, \ldots, v_n \in \mathbb{R}^n \) be \( n \) linearly independent vectors. Let

\[
G = \left\{ \sum_{i=1}^{n} a_i v_i, a_i \in \mathbb{Z} \right\}
\]

be the subgroup of \( \mathbb{R}^n \) generated by the set of \( v_i \). Define the fundamental domain \( D \subset \mathbb{R}^n \) to be the set

\[
D = \sum_{i=1}^{n} d_i v_i, 0 \leq d_i < 1 \}\]

(a) Show that every element \( v \in \mathbb{R}^n \) can be written uniquely as a sum \( d + g \) where \( d \in D \) and \( g \in G \).

(b) For any \( r > 0 \), let \( B(r) \) be the ball of radius \( r \) around 0:

\[
B(r) = \{ v \in \mathbb{R}^n \mid ||v|| \leq r \}.
\]

For any \( h \in G \), let \( D_h = h + D = \{ h + d \mid d \in D \} \) (in other words, \( h \) is fixed but \( d \) varies in \( D \)). Show that the set of \( h \in G \) such that \( B(r) \cap D_h \neq \emptyset \) is finite.