
2. Let $d > 0$ be a square-free positive integer congruent to 3 (mod 4).
   (a) Every unit $u \in \mathbb{Z}[\sqrt{d}]$ is of the form $a - b\sqrt{d}$ where $a^2 - db^2 = \pm 1$, and
   the group $\Gamma$ of units is the product of an infinite cyclic group with $\{\pm 1\}$.
   Consider the subset $\Sigma$ of $\Gamma$ consisting of
   
   $u_i = a_i - b_i\sqrt{d}$
   
   such that $a_i > 0, b_i > 0$, ordered so that $b_1 \leq b_2 \leq b_3 \ldots$.
   Show that $u_1$ and $-1$ are generators of $\Gamma$.

   The element $u_1$ is called the fundamental unit of $\mathbb{Z}[\sqrt{d}]$.

   (b) Show that the following algorithm finds $u_1$: Letting $b = 1, 2, 3, \ldots$, consider the quantities $q^\pm(b) = db^2 \pm 1$. Let $b_1$ be the smallest positive integer such that either $q^+(b_1)$ or $q^-(b_1)$ is a perfect square. Let $a_1$ be the positive square root of $q(b_1)$; then $u_1 = a_1 - b_1\sqrt{d}$.

   (c) Use this algorithm to find the fundamental units $u_1$ of $\mathbb{Z}[\sqrt{7}], \mathbb{Z}[\sqrt{11}], \mathbb{Z}[\sqrt{15}]$. In each case determine $N_{K/\mathbb{Q}}(u_1)$, where $K = \mathbb{Q}(\sqrt{d})$ in each case.

3. As Hindry shows on p. 99, the ring $R = \mathbb{Z}[\sqrt{10}]$, which is equal to the ring of integers in $\mathbb{Q}(\sqrt{10})$, is not a principal ideal domain. Indeed, the integer 9 has two inequivalent factorizations:

   $9 = 3^2 = (\sqrt{10} - 1)(\sqrt{10} + 1)$.

   (a) Show that $3 + \sqrt{10}$ is a unit in $R$. Use this fact to confirm that the two factorizations are indeed inequivalent.

   (b) The integer 10 is definitely a square modulo 3. What is the prime factorization of the ideal $(3) \subset R$?

   (c) Use the Minkowski bound to show that the class number of $\mathbb{Q}(\sqrt{10})$ is 2.

4. Let $R$ be a Dedekind ring with only finitely many prime ideals. Show that $R$ is a PID. (Hint: say $p_1, p_2, \ldots, p_e$ are the prime ideals. Find an element $x_i \in p_i$ that is not in any of the $p_j$ with $j \neq i$, and factor the ideal $(x_i)$. Another piece of information is necessary.)