
2. Let \( d > 0 \) be a square-free positive integer congruent to 3 (mod 4).
   (a) Every unit \( u \in \mathbb{Z}[\sqrt{d}] \) is of the form \( a - b\sqrt{d} \) where \( a^2 - db^2 = \pm 1 \), and the group \( \Gamma \) of units is the product of an infinite cyclic group with \( \{\pm 1\} \).
   Consider the subset \( \Sigma \) of \( \Gamma \) consisting of \( u = a_i - b_i\sqrt{d} \) with \( a_i > 0, b_i > 0 \), ordered so that \( b_1 \leq b_2 \leq b_3 \ldots \). Show that \( u_1 \) and \(-1\) are generators of \( \Gamma \).
   The element \( u_1 \) is called the fundamental unit of \( \mathbb{Z}[\sqrt{d}] \).
   (b) Show that the following algorithm finds \( u_1 \): Letting \( b = 1, 2, 3, \ldots \), consider the quantities \( q^\pm(b) = db^2 \pm 1 \). Let \( b_1 \) be the smallest positive integer such that either \( q^+(b_1) \) or \( q^-(b_1) \) is a perfect square. Let \( a_1 \) be the positive square root of \( q(b_1) \); then \( u_1 = a_1 - b_1\sqrt{d} \).
   (c) Use this algorithm to find the fundamental units \( u_1 \) of \( \mathbb{Z}[\sqrt{7}] \), \( \mathbb{Z}[\sqrt{11}] \), \( \mathbb{Z}[\sqrt{15}] \). In each case determine \( N_{K/\mathbb{Q}}(u_1) \), where \( K = \mathbb{Q}(\sqrt{d}) \) in each case.

3. As Hindry shows on p. 99, the ring \( R = \mathbb{Z}[\sqrt{10}] \), which is equal to the ring of integers in \( \mathbb{Q}(\sqrt{10}) \), is not a principal ideal domain. Indeed, the integer 9 has two inequivalent factorizations:
   \[ 9 = 3^2 = (\sqrt{10} - 1)(\sqrt{10} + 1). \]
   (a) Show that \( 3 + \sqrt{10} \) is a unit in \( R \). Use this fact to confirm that the two factorizations are indeed inequivalent.
   (b) The integer 10 is definitely a square modulo 3. What is the prime factorization of the ideal \((3) \subset R\)?

4. Let \( R \) be a Dedekind ring with only finitely many prime ideals. Show that \( R \) is a PID. (Hint: say \( p_1, p_2, \ldots, p_r \) are the prime ideals. Find an element \( x_i \in p_i \) that is not in any of the \( p_j \) with \( j \neq i \), and factor the ideal \((x_i) \). Another piece of information is necessary.)