Problem set #5 solutions

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1

§5.3, exercise 1

(a) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}\) = (12453)

(b) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}\) = (14)(53)

(c) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 1 & 4 & 2 \end{pmatrix}\) = (13)(25)

(d) \(\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 2 & 5 \end{pmatrix}\) = (24)

§5.3, exercise 2

(a) \((1345)(234) = (135)(24)\)

(b) \((12)(1253) = (253)\)

(c) \((143)(23)(24) = (14)(23)\)

(d) \((1423)(34)(56)(1324) = (12)(56)\)

(e) \((1254)(13)(25) = (1324)\)

(f) \((1254)(13)(25)^2 = (13254)\)

(j) \((1254)^{100} = [(1254)^4]^{25} = \text{id}^{25} = \text{id}\), since the kth power of a k-cycle is the identity.

2

The permutation \((1234)\) has order 4 in \(\Sigma_4\). Note that the elements of \(\Sigma_4\), when decomposed into cycles, are:

- the identity (which has order 1)
- 2-cycles (all of which have order 2)
- 3-cycles (all of which have order 3)
- 4-cycles (all of which have order 4)
- products of disjoint 2-cycles (all of which have order 2)

so the maximal order in \(\Sigma_4\) is 4. Finally, \((1234)\) is already a cycle, so it is the product of (one) disjoint cycle.

[Note: The maximal order in \(\Sigma_n\) is not \(n\), in general. For instance, \((123)(45) \in \Sigma_5\) has order 6.]

3

Consider \(\sigma = (12) \in \Sigma_4\) and \(\tau = (23) \in \Sigma_4\).

Since \(\sigma\) and \(\tau\) are both 2-cycles, we have \(\sigma^2 = \tau^2 = \text{id}\). But \(\sigma \tau = (12)(23) = (123) \neq (132) = (23)(12) = \tau \sigma\).
Observe that if $\sigma$ is a permutation in $\Sigma_4$ which ‘preserves the square’, then (since vertices 1 and 2 are adjacent) the vertices $\sigma(1)$ and $\sigma(2)$ must be adjacent, whereupon the vertices $\sigma(3)$ and $\sigma(4)$ are uniquely determined. Therefore, either $\sigma(1 + i) - \sigma(1) \equiv i \pmod{4}$ for all $i$, or $\sigma(1 + i) - \sigma(1) \equiv -i \pmod{4}$ for all $i$.

Now we will show that the subset $D := \{ \sigma : \sigma(1 + i) - \sigma(1) \equiv \pm i \} \subset \Sigma_4$ is a subgroup. [In this problem, the symbol ‘$\equiv$’ will denote congruence modulo 4.]

- $D$ is closed under multiplication: if $\sigma, \tau \in D$, with $\sigma(1 + i) - \sigma(1) \equiv u_\sigma i$ and $\tau(1 + i) - \tau(1) \equiv u_\tau i$, where $u_\sigma, u_\tau = \pm 1$, then

$$
\tau(\sigma(1 + i)) - \tau(\sigma(1)) = (\tau(\sigma(1 + i)) - \tau(1)) - (\tau(\sigma(1)) - \tau(1))
\equiv u_\tau (\sigma(1 + i) - 1) - u_\tau (\sigma(1) - 1)
\equiv u_\tau u_\sigma i
$$

so $(\tau \sigma)(1 + i) - (\tau \sigma)(1) \equiv u_\tau u_\sigma i$, which (since $u_\tau u_\sigma = \pm 1$) means $\tau \sigma \in D$.

- $D$ contains the identity: we have $id(1 + i) - id(1) = (1 + i) - 1 = +i$ for all $i$, so $id \in D$.

- $D$ contains inverses: if $\sigma \in D$, with $\sigma(1 + i) - \sigma(1) \equiv u_\sigma i$ where $u_\sigma = \pm 1$, then

$$
i = (1 + i) - 1 = \sigma(\sigma^{-1}(1 + i)) - \sigma(\sigma^{-1}(1))
\equiv (\sigma(\sigma^{-1}(1 + i)) - \sigma(1)) - (\sigma(\sigma^{-1}(1)) - \sigma(1))
\equiv u_\sigma (\sigma^{-1}(1 + i) - 1) - u_\sigma (\sigma^{-1}(1) - 1)
\equiv u_\sigma (\sigma^{-1}(1 + i) - \sigma^{-1}(1))
$$

so $\sigma^{-1}(1 + i) - \sigma^{-1}(1) \equiv u_\sigma^{-1} i$, which (since $u_\sigma^{-1} = u_\sigma = \pm 1$) means $\sigma^{-1} \in D$.

So $D \subset \Sigma_4$ is a subgroup.

Finally, we see that choosing an element $\sigma \in D$ entails choosing one of four possibilities for $\sigma(1)$ and choosing whether $\sigma$ is order-preserving or order-reversing (i.e., whether $\sigma(1 + i) - \sigma(1) \equiv i$ or $\sigma(1 + i) - \sigma(1) \equiv -i$). Hence there are $4 \cdot 2 = 8$ different possibilities for $\sigma \in D$, so the order of $D \subset \Sigma_4$ is 8.

Indeed, one verifies that $D = \{ id, (1234), (13)(24), (14235), (13), (12)(34), (24), (14)(23) \} \subset \Sigma_4$. [Note: This proof can be easily modified to show that the subset $D_n \subset \Sigma_n$ of permutations which preserve the regular $n$-gon is a subgroup of order $2n$. The geometric view is that the order-preserving permutations in $D_n$ are rotations of the $n$-gon, while the order-reversing permutations in $D_n$ are reflections of the $n$-gon.]

5

(a) The 3-cycle $(132) \in \Sigma_3$ has order 3.

(b) The 6-cycle $(125364) \in \Sigma_6$ generates a cyclic subgroup of order 6, which means $(125364)^2 = (156)(234)$ has order $\frac{6}{\gcd(2, 6)} = 3$ in $\Sigma_6$.

[Alternatively, we know the product of disjoint cycles in $\Sigma_n$ of lengths $\ell_1, \ldots, \ell_m$ has order lcm ($\ell_1, \ldots, \ell_m$), so $(156)(234)$ has order lcm $(3, 3) = 3$ in $\Sigma_6$.]

(c) The 5-cycle $(14235) \in \Sigma_5$ generates a cyclic subgroup of order 5, so $(14235)^2$ has order $\frac{5}{\gcd(2, 5)} = 5$ in $\Sigma_5$.

[Alternatively, $(14235)^2 = (12543)$ is also a 5-cycle and thus has order 5.]