1. True or False? If false, give a counterexample; if true, provide an explanation. The explanation can be brief but it is not enough to say that the statement was explained in the course.

(a) A group of order 392 has either 1 or 8 Sylow 7-subgroups.

(b) For any \( n \) let \( A(n) \) denote the number of distinct non-isomorphic abelian groups of order \( n \). Then \( A(65) > A(64) \).

(c) Let \( G \) be a group of even order. Then it has at least one conjugacy class, not including the identity element, with an odd number of elements.

(d) Let \( H \) be a subgroup of the alternating group \( A_5 \). Suppose \( H \) contains every 3-cycle. Then \( H = A_5 \).

2. (a) Determine the centralizer of the product \( (12)(34)(56)(78) \) of four 2-cycles in \( S_8 \). Use this to determine the number of all elements of \( S_8 \) that can be written as products of four 2-cycles.

(b) Determine the centralizer of the product \( (12)(34)(56)(78) \) of four 2-cycles in \( S_{12} \). Use this to determine the number of all elements of \( S_{12} \) that can be written as products of four 2-cycles.

3. How many elements of order 5 are there in \( S_5 \times \mathbb{Z}_{25} \)?

4. (a) What is the number of conjugacy classes of the dihedral group \( D_{2n} \)? Prove your answer, and note that it depends on whether \( n \) is odd or even.

(b) Write down the class equation for \( D_{2n} \) and identify the centralizer of each element.

5. Let \( G \) be a finite group, \( N \subseteq G \) a normal subgroup. Let \( H = G/N \) be the quotient group, and let \( \pi : G \to H \) denote the quotient map.

Let \( X \) denote the set of conjugacy classes in the group \( N \). In other words, two elements \( n_1, n_2 \in N \) are in the same conjugacy class if there is an element \( n \in N \) such that \( n \cdot n_1 \cdot n^{-1} = n_2 \). The conjugacy class of an element \( n \in N \) is denoted \([n]\).

(a) Show that \( G \) acts on the set \( X \) by conjugation: if \( n \in N \), and \( g \in G \), then \( g([n]) \) is the conjugacy class \([gn^{-1}]\). Show that this action is well-defined: in other words, if \([n_1] = [n_2]\) then \( g([n_1]) = g([n_2]) \). (Warning: do not confuse conjugacy in \( G \) with conjugacy in \( N \).)
(b) Write down the class equation for the action of $G$ on $X$.

(c) Suppose $N$ is abelian. Show that there is an action of $H$ on $X$ such that, for all $n \in N$, $g \in G$,

$$g([n]) = \pi(g)([n]).$$

6. (15 points) Construct two non isomorphic non-abelian groups of order 168, each of which contains a normal abelian subgroup of order 8. (Hint: try to use direct products of smaller groups.)

7. Show that there are no simple groups of order 38 and 40.

8. Let $p$ be a prime number and let $G$ be a finite $p$-group. Write down the steps of the proof that $G$ is solvable.

9. Write down the class equation for the groups $K_4$, $Q_8$, and $S_4$.

10. Let $G$ be a group, $H \subseteq G$, $K \trianglelefteq G$ two subgroups, with $K$ normal. Suppose the derived subgroup $D(H) \subseteq H$ is strictly smaller than $H$ and $H \cap K = \{e\}$.

Prove that $H \cdot K$ has a normal subgroup $J$ such that $H \cdot K/J$ is abelian and $|H \cdot K/J| > 1$. 